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MODELLING CONTACT PROBLEMS WITH FRICTION IN FAULT MECHANICS

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The aim of this contribution is two-fold. First, we review the friction models applied in geophysics. These models cover: state- and rate-dependent friction, rate-dependent friction and slip-dependent friction.
Second, we propose a new description of friction in the spirit of modern contact mechanics, introducing sliding rules which interrelate the contact stresses with the slip velocity. Sliding rules are formulated in a subdifferential form. Initial-boundary value problems are formulated in the strong and variational forms. By applying Green's function, the variational formulation for finding normal and tangential contact stresses is proposed.

Key words: fault mechanics, friction, contact laws, state variables

1. Introduction

Frictional behaviour of rocks plays an important role in earthquake processes and their prediction. Once a fault has been formed, its further motion is controlled by friction, which has a contact property rather than the bulk property. In rock friction studies two aspects are crucial: the stability of engineering structures and the mechanics of earthquakes.

The modern seismology claims that the earthquakes are processes of creation of discontinuity of displacement fields in upper mantle of the Earth at different depths. These processes take place on old and new systems of tectonic faults.

As a rule, many factors play an essential role during processes occuring on tectonic faults: friction, temperature, chemical reactions, phase transitions, porosity, etc., cf Teisseyre (1985, 1995) and Teisseyre and Majewski (2001).

Though each of these factors is important, the frictional behaviour of rocks has been investigated in many papers. A program of such investigations can be summarized as follows, cf Rice (1983):

(i) characterization of complete behaviour of the slip surface, i.e., finding the distribution of contact (tangent) stresses as a function of normal stress, temperature, slip rate, slip distance and history of slip;

(ii) description of mechanical interaction between contact surfaces and the surrounding elastic bodies.

Friction is an important phenomenon which has obviously to be taken into account in seismological modelling of material behaviour in a neighborhood of tectonic faults, see Ben-Zion and Rice (1995), Dieterich and Kilgore (1994), Rice (1993), Zheng and Rice (1998), Sleep (1997, 1998), Segal and Rice (1995), Rudnicki and Wu (1995), Ranjith and Rice (1999), Senatorski (2000a,b). Contact problems with friction, including unilateral problems, have been frequently studied in the contact mechanics, cf Bielski and Telega (1994), Brogliato (1999), Telega (1988) and the references cited therein.

The investigation of friction laws on geological faults emerges as a key issue for earthquake modelling. Two types of processes have to be taken into account: quasi-static one as a long term process of slow loading and duration of about a few scores of years, and the second one, the rapid unloading, i.e. a dynamic process occuring during a few seconds. Both the dynamic and quasi-static processes are necessary to complete the description of the phenomenon of friction on tectonic faults.

It is convenient to distinguish three models of frictional sliding studied in geophysical literature. The first one is the model of rate- and state-dependent friction. In this theory, the so-called state variables are applied and the idea is due to Rabinowicz (1965). Next it has been developed by Dieterich (1978, 1979), Ruina (1980, 1983), Rice (1983), Scholz (1994, 1996, 1998), cf also the references therein.

The second model is the *rate-dependent* friction, cf Madariaga et al. (1998), Cochard and Madariaga (1994).

The third model considered in seismology is the *slip-dependent* friction, cf Favreau et al. (1999), Campillo and Ionescu (1997), Descalu et al. (2000), Ionescu and Campillo (1999), Ionescu and Paumier (1997).

In this contribution we shall first briefly describe general aspects of tectonic plates and provide classification of tectonic faults sufficient for this study. Next, we review the role of state variables in modelling the friction on tectonic faults. Following our earlier contribution (Bielski and Telega, 2000), a more elaborate description of the sliding process with friction which includes not only the friction condition but also the sliding rule is introduced. The sliding rules are formulated in the subdifferential form, convenient for variational formulations. In the general case both the friction condition and the sliding rule depend on state variables. Finally, variational formulation will be proposed for a quasi-static problem modelling geological faults. For the sake of simplicity, it is assumed that contacting tectonic plates are made of anisotropic, linear elastic materials. Physically more involved material behaviour can likewise be considered; for instance, in Rowshandel and Nemat-Nasser (1986), the foundation is viscoelastic. The problem of anisotropic fault region was also considered by Rybicki (1992).

This paper is confined to quasi-static problems for contacting tectonic plates. However, a more general than the usually used rate- and state-dependent model of friction is taken into account. Also, anisotropy and inhomogeneity of tectonic plates are included in our approach in a natural manner.

2. General aspects of tectonic plates

Plate tectonics is a model in which the outer shell of the earth is divided into a number of thin, rigid plates that are in relative motion with respect to one another. The relative velocities of the plates are of the order of a few scores of milimeters per year, for instance for the San Andreas fault the average velocity is 0.5-4.5 mm/year. A large number of all earthquakes, volcanic eruptions, and mountain forming occurs at plate boundaries. Figure 1 sketches the distribution of the major surface plates. The plates are made up of relatively cool rocks and have an average thickness of about 100 km. The plates are being continually created and absorbed.

At ocean ridges, the adjacent plates diverge from each other in a process known as seafloor spreading. As the adjacent plates diverge, hot mantle rock ascends to fill the gap. The hot, solid mantle rock behaves like a fluid because of solid-state creep process. As the hot mantle rock cools, it becomes rigid and accretes to the plates, creating new plate area. For this reason, ocean ridges are also known as accreting plate boundaries.

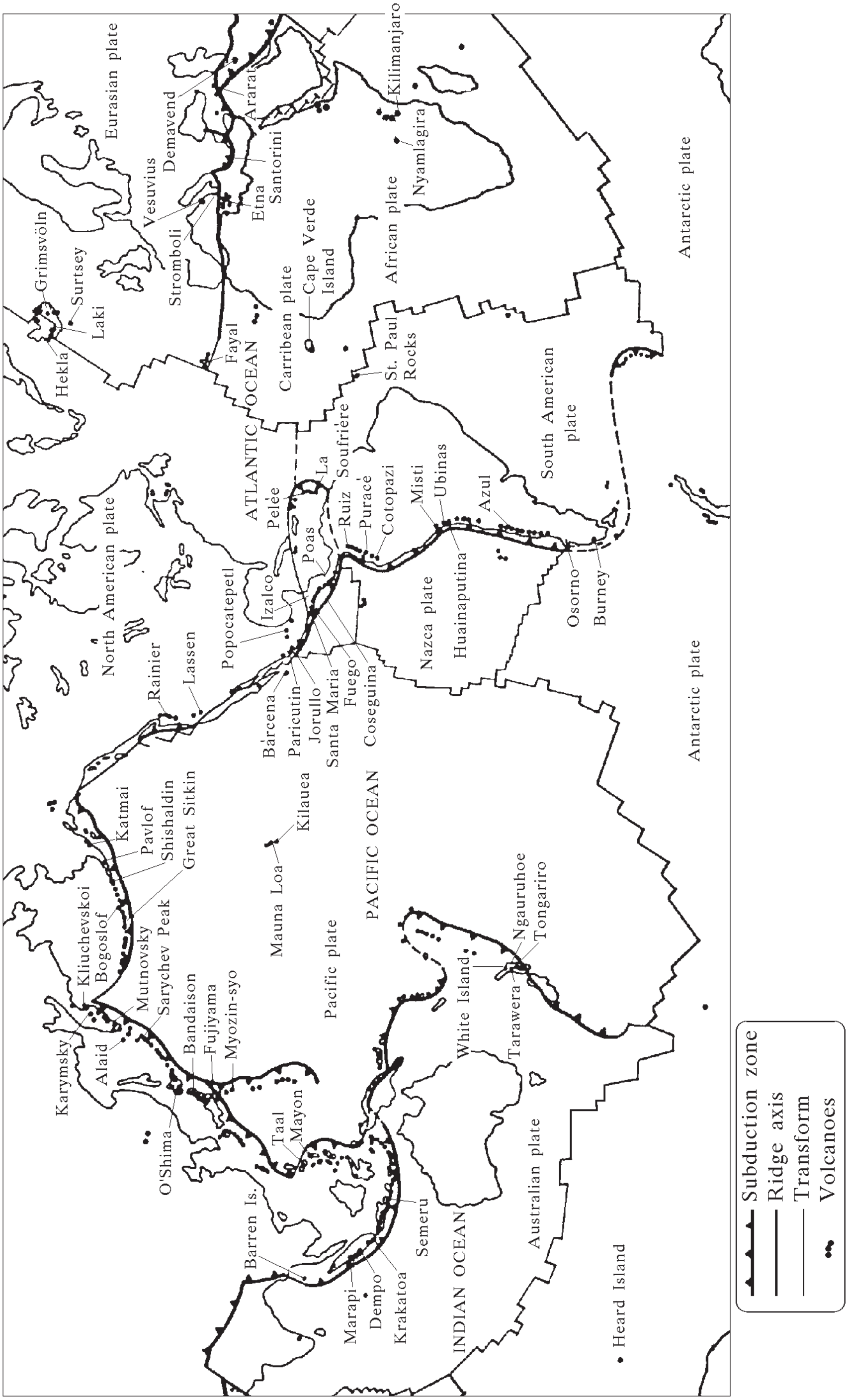

Fig. 1. The map of main tectonic plates and their boundaries, after Turcotte and Schubert (1982)

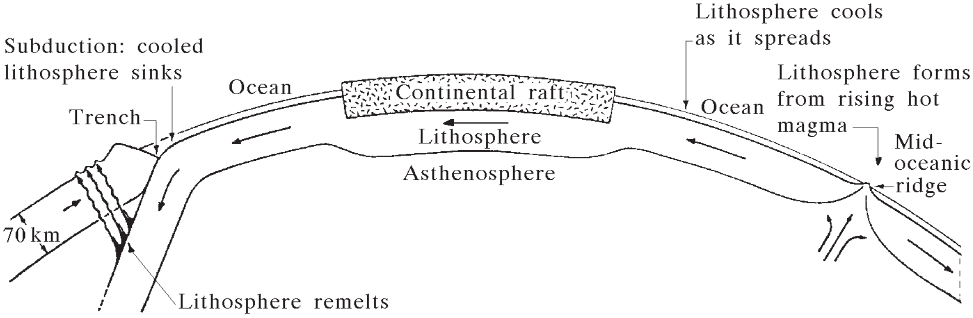

Fig. 2. Schematic presentation of subduction processes, after Press and Siever (1986)

The main oceanic ridges are depicted in Fig. 1. Because the surface area of the earth is essentially constant, there must be a complementary process of plate consumption. This occurs at ocean trenches. The surface plates bend and descend into the interior of the earth in a process known as subduction. At an ocean trench, the two adjacent plates converge, and one descends beneath the other. For this reason ocean trenches are also known as convergent plate boundaries. The distribution of the trenches is depicted in Fig. 1 by triangular symbols, which indicate the direction of subduction. A cross-sectional view of the creation and consumption of a typical plate is illustrated in Fig. 2. The part of earth's interior that comprises the plates is referred to as the lithosphere. The rocks that make up the lithosphere are relatively cool and rigid; as a result, the interiors of the plates do not deform significantly as they move about the surface of the earth. When the plates move away from ocean ridges, they cool and thicken. The solid rocks beneath the lithosphere are sufficiently hot to be able to deform freely; these rocks comprise the asthenosphere, which lies below the lithosphere. The lithosphere slides over the asthenosphere with relatively little resistance. As the rocks of the lithosphere becomes cooler, their density increases because of thermal contraction. As a result, the lithosphere becomes gravitationally unstable with respect to the hot asthenosphere beneath. At the ocean trench the lithosphere bends and sinks into the interior of the earth because of this negative buoyancy. Major faults separate the descending lithospheres from the adjacent overlying lithospheres. These faults are the sites of a large number of the great earthquakes. Examples are the Chilean, Alaskian, San Francisco (San Andreas Fault), Anatolian Fault earthquakes, as well as Chinese and Japan faults. The location of the descending lithospheres can be accurately determined by the earthquakes occuring in the cold, brittle rocks of the lithosphere. Earthquake source dynamics provides the key elements for the prediction of strong ground motion and to understand the physics of earthquake initiation, propagation and healing. Recent studies indicate the fundamental role of friction in earthquakes, cf Cotton and Campillo (1995), Beroza and Mikumo (1996), Ide and Takeo (1997), Fukuyama and Madariaga (1998), Iio (1997), Lockner (1998), King and Cocco (2001), Lapusta et al. (2000), Ben-Zion and Rice (1997), Cochard and Rice (2000), Brown (1998), Beeler et al. (1996), Boatwright and Cocco (1996), Place and Mora (1999), Oglesby et al. (2000), Nielsen et al. (2000), Richardson and Marone (1999), Roy and Marone (1996), Sleep (1995, 1999).

3. Classification of faults

Threre is an abundant classification of tectonic faults from the geological point of view. For our purposes are sufficient the following classifications of faults, followed by Turcotte and Schubert (1982).

One may distinguish three main types of faults. Every other fault can be treated as a combination of the three main types. In general, a certain characteristic type dominates in each fault.

(i) As the first type consider the *thrust faulting*. Thrust faultings occur when the oceanic lithosphere is thrust under the adjacent continental (or oceanic) lithosphere at an oceanic trench. Thrust faults also play an important role in the compression of the lithosphere during continental collisions. Idealized thrust fault is depicted in Fig. 3. The elevating block is known as the hanging wall, and the depressed block is called the foot wall. The upward movement of the hanging wall is also referred to as reverse faulting.

Let the stresses in the x, y, and z directions be the principal stresses (x, z are the horizontal coordinates, y is the vertical coordinate). The vertical component of the stress σ_{yy} is the overburden or lithostatic pressure $\sigma_{yy} = \varrho g y$. The vertical deviatoric stress σ^D_{yy} is zero. To produce the thrust faults, a compressive deviatoric stress applied in the x direction σ^D_{xx} is required, $\sigma^D_{xx} > 0$. The horizontal compressive stress is $\sigma_{xx} = \varrho g y + \sigma^D_{xx}$, therefore it exceeds the vertical lithostatic stress or $\sigma_{xx} > \sigma_{yy}$. For the fault geometry shown in Fig. 3 it is appropriate to assume that there is no strain in the z direction. In this particular situation we can write $\sigma^D_{zz} = \nu\sigma^D_{xx}$. The deviatoric stress in the z direction is also compressive, but its magnitude is a factor of ν times less than the deviatoric applied stress. Therefore the horizontal

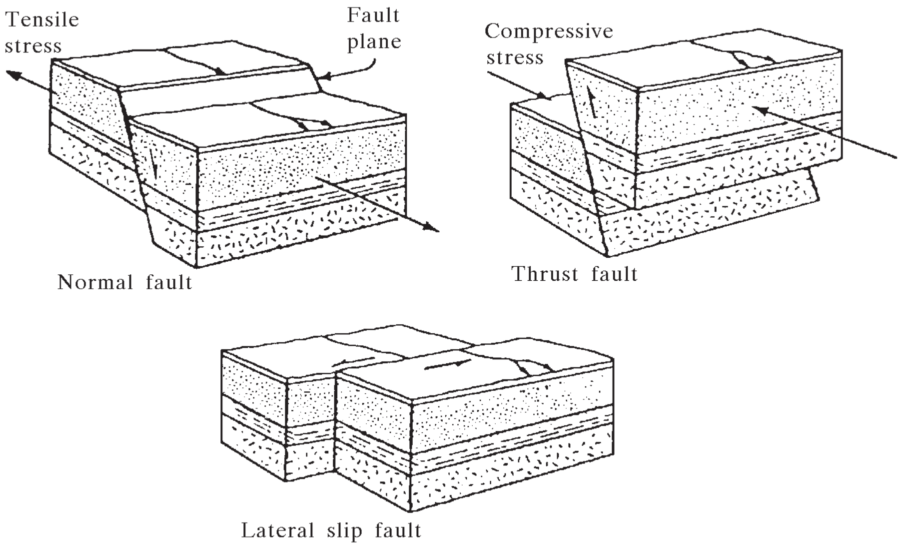

Fig. 3. Schemat of three main types of tectonic faults, after Press and Siever (1986)

compressive stress $\sigma_{zz} = \varrho gy + \sigma_{zz}^D = \varrho gy + \nu\sigma_{xx}^D$ exceeds the vertical stress σ_{yy}, but it is smaller than the horizontal stress σ_{xx}. Thrust faults satisfy the condition $\sigma_{xx} > \sigma_{zz} > \sigma_{yy}$. The vertical stress is the least compressive stress.

(ii) *Normal faulting* accommodates horizontal extensional strain. It occurs on the flanks of oceanic ridges where new lithosphere is being created. Normal fault also occurs in continental rift valleys where the lithosphere is being stretched. Applied tensile stresses can produce normal faults as shown in Fig. 3. The displacements on the fault planes dipping at an angle to the horizontal lead to horizontal tensile strain. *Normal faulting* is associated with a state of stress in which the vertical component of stress is the lithostatic pressure $\sigma_{yy} = \varrho gy$ and the applied deviatoric horizontal stress σ_{xx}^D is tensile $\sigma_{xx}^D < 0$. The horizontal stress $\sigma_{xx} = \varrho gy + \nu\sigma_{xx}^D$ is therefore smaller than the vertical stress σ_{yy}

$$\sigma_{yy} > \sigma_{xx}$$

Consequently, deviatoric stress in the z direction σ_{zz}^D is also tensile, but its magnitude is a factor of ν smaller than the deviatoric stresses applied. The total stress $\sigma_{zz} = \varrho gy + \nu\sigma_{xx}^D$ is smaller than σ_{yy} but is larger than σ_{xx}. *Normal faults* satisfy the condition $\sigma_{yy} > \sigma_{zz} > \sigma_{xx}$, thus the vertical stress is the maximum compressive stress. Both thrust faults and normal faults are also known as dip-slip faults because the displacement along the fault takes place on a dipping plane.

(iii) A *strike-slip* fault is a fault along which the displacement is strictly horizontal. Thus there is no strain in the y direction.

The state of stress in the strike-slip faulting consists of a vertical lithostatic stress $\sigma_{yy} = \varrho g y$ and horizontal deviatoric principal stress that are compressive in one direction and tensile in the other.

One horizontal stress will thus be larger than σ_{yy} while the other will be smaller, so we have

$$\sigma_{xx} > \sigma_{yy} > \sigma_{zz} \qquad \text{or} \qquad \sigma_{zz} > \sigma_{yy} > \sigma_{xx}$$

For the strike-slip faulting the vertical stress is always the intermediate stress.

4. Friction laws involving internal parameters

Brace and Byerlee (1966) hypothesized that stick-slip instabilities in the observed laboratory friction experiments might stand for a good model to earthquake rupture. Consequently, laboratory experiments are thought as models of possible fault motion in the earth.

Experiments have been performed with many rock types, with and without various fault gouge layers, at a range of slip rates, confining pressure, pore pressures, temperatures, and in machines with different geometry and compliances, cf Blanpied et al. (1998), Dieterich (1978, 1979), Dieterich and Conrad (1984), Jaeger and Cook (1976), Mair and Marone (1999), Morrow et al. (2000), Olsen et al. (1998), Savage et al. (1996), Sleep (1999), Weeks and Tullis (1985).

A different approach to friction experiments consists in postulating a constitutive description of a surface slip from which earthquake or laboratory experiments can be predicted through modelling, cf Ruina (1980, 1983), Rice (1983,1993), Scholz (1994, 1996, 1998), Segal and Rice (1995), Sleep (1995, 1997, 1998), Zheng and Rice (1998), Ben-Zion and Rice (1995, 1997), Cao and Aki (1986). Such modeling of elastic systems reveals that instabilities in frictional slip depend on a reduction of the friction force during some part of the sliding, i.e. on slip weakening. For this reason, the role of slip weakening has been investigated in many papers. Particularly, Byerlee (1970) suggested that the friction coefficient varies from point to point on slip surfaces and that instabilities are associated with decrease in the friction force from its peak values as sliding proceeds.

Dieterich (1972) claims that slip weakening occurs after a time-dependent healing during stationary contact. Similar mechanisms have been earlier proposed as a basis for slip instabilities, primarily in metals, Rabinowicz (1965). Basing on the ideas of Rabinowicz (1965), Dieterich (1978,1979) and Ruina (1980) studied and developed a class of friction laws based on using the state variables.

Ruina (1983) exploited the experimental data by Dieterich (1979) and proposed a model of friction involving state variables. This author provided examples to characterize the state variables and to study the stability of steady sliding, neglecting the inertia forces. Also a friction law based on one state variable was used.

Let us pass to a brief presentation of the Ruina (1983) model. This model comprises basic experimentally observed features, especially the following ones: fading memory and steady-state, positive instantaneous slip rate-dependence, and negative dependence on the recent slip rates.

These ideas and observations led to the following description of friction. Let τ be the shear stress and σ let denote the normal stress. After Ruina (1983) we write

$$\tau = \sigma F(\vartheta, V) \tag{4.1}$$

where ϑ is a state variable (or a collection of such variables, $\vartheta = (\vartheta_i)$, $i = 1, \dots, n$), V is the rate of the slip. The evolution equation for ϑ has the form

$$\frac{d\vartheta_i}{dt} = G_i(\sigma, V, \vartheta_i) \tag{4.2}$$

From the practical viewpoint, the number of state variables ϑ_i should be small. The variables ϑ_i then represent some kind of average of an undoubtedly complicated surface state. The temperature of the surface can be taken as a single state variable if the heat flow is idealized as being dependent only on the temperature of the surface and the temperature of an external constant temperature reservoir.

Detailed analysis of experiments made on different types of rocks lead to the following description of friction provided that one state variable is used

$$\begin{aligned} \tau &= \sigma\Big(\mu_0 + \vartheta + A \ln \frac{V}{V_c}\Big) \\ \dot{\vartheta} &= -\frac{V}{d_c}\Big(\vartheta + B \ln \frac{V}{V_c}\Big) \end{aligned} \tag{4.3}$$

Here μ_0 is the coefficient of static friction, A and B are constitutive parameters to be determined by experiments, and d_c is the characteristic slip distance depending on the surface. This law is valid for a large range of slip rates and shares the apparent defect of no-healing (no change of ϑ) for a zero slip rate. This law can be illustrated graphically as in Fig. 4.

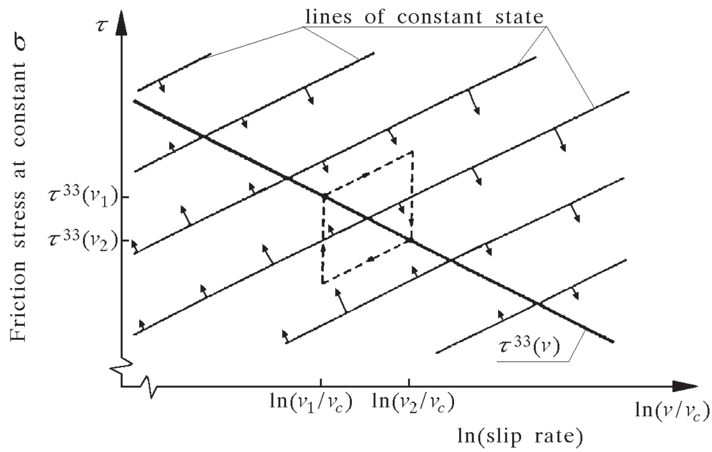

Fig. 4. Friction stress at constant normal stress versus slip rate (of ln), after Ruina (1983)

In Figure 4 the lines of constant state, ϑ, are light solid lines and show the instantaneous positive dependence of τ on the slip rate V. The heavy line is the steady state friction law and is a decreasing function of the slip rate in the example of Fig. 4 $(B > A)$. As governed by Eq. $(4.3)_2$, ϑ decreases above the steady state line, below it ϑ increases. Any slip corresponds to a pen motion on the plot of Fig. 4 and is the simultaneous solution of the friction law and any constraints imposed by the loading mechanism. The arrows indicate the component of this motion perpendicular to the lines of constant ϑ.

Ruina (1980) derived an experiment which cannot be described by a single state variable law of the form (4.1), because of violation of condition (4.2). He showed that his experiment is well described by a friction law involving two state variables

$$
\begin{aligned}
\tau &= \sigma\Big(\mu_0 + \vartheta_1 + \vartheta_2 + A \ln \frac{V}{V_c}\Big) \\
\vartheta_i &= -\frac{V}{d_i}\Big(\vartheta_i + B \ln \frac{V}{V_c}\Big) \qquad\qquad i = 1, 2
\end{aligned}
\tag{4.4}
$$

This model is also called the Ruina-Dieterich model, cf Perrin et al. (1995).

The model is referred to as the slip model since the state evolves only when there $V \neq 0$. As previously, V is the one-dimensional slip velocity.

The quantities A, B, V_c, and d_i $(i = 1, 2)$ are constants to be determined by experiment, d_i is the slip length scale for state evolution; A and B, both positive, account for the short-time velocity strengthening and for the steady-state velocity weakening, respectively. The model specified by Eqs (4.4) can be extended to arbitrary number of state variables, ϑ_i, $i = 1, 2, \ldots, n$, each of them having specific weakening constant B_i and length scale d_i. However, due to its simplicity, most frequently used is the model with one or two state variables, cf Ruina (1980, 1983), Weeks and Tullis (1985).

Now we describe the Dieterich-Ruina slowness model, cf Perrin et al. (1995). This model has the following form

$$
\begin{aligned}
\tau(t) &= \sigma\Big[\mu_0 - A\ln\Big(1 + \frac{V_\infty}{V(t)}\Big) + B\ln\Big(1 + \frac{\vartheta(t)}{\vartheta_0}\Big)\Big] \\
\frac{d\vartheta(t)}{dt} &= 1 - \vartheta\frac{V(t)}{L}
\end{aligned}
\tag{4.5}
$$

One might think of the state variable ϑ here in an abstract way, cf Ruina (1980, 1983) and (4.1). Dieterich (1979) and Dieterich and Conrad (1984) interpret it as the average age of the load supporting the contacts between the sliding surface. In that case the constitutive law of the form (4.5) is more sensible than the one of the form $(4.3)_2$, since it yields

$$
\frac{d\vartheta}{dt} = 1 \qquad \text{for } V = 0
$$

That contact time interpretation led Dieterich to use extensively equations (4.5), although equation $(4.5)_2$ seems to have been written first by Ruina (1980). The quantities τ_0, A, B, V_∞, and ϑ_0 are cut-offs for high velocity and short contact duration.

Perrin et al. (1995) used a regularized version of Dieterich-Ruina model (4.5) to study the self-healing slip pulse on a frictional surface. Dieterich (1992) pointed out that the model presented by (4.3) leads to non-physical behaviour for extremely low slip velocities. The same happens for long "contact times" ϑ. To remedy these drawbacks, Perrin et al. (1995) introduced two cut-off velocities V_0 and V_1 and modified the model (4.5) as follows

$$
\begin{aligned}
\tau(t) &= \sigma\Big[\mu_0 + A\ln\Big(\frac{V_0 + V(t)}{V_\infty + V(t)}\Big) + B\ln\Big(1 + \frac{\vartheta(t)}{\vartheta_0}\Big)\Big] \\
\frac{d\vartheta(t)}{dt} &= 1 - \vartheta\frac{V_1 + V(t)}{L}
\end{aligned}
\tag{4.6}
$$

Notice that the state variable ϑ is contained in $[0, L/V_1]$. This might be illogical if ϑ had to be interpreted as the contact time. However, considering a cut-off precisely means that we are getting outside the measurable range and it is by nature artificial.

Chester (1994) extended Ruina's friction law and included the temperature. In the case of one state variable, the friction coefficient μ is then expressed by

$$\mu = \mu_0 + A\Big[\ln\frac{V}{V_c} + \frac{Q_A}{R}\Big(\frac{1}{T} - \frac{1}{T_*}\Big)\Big] + B\vartheta \tag{4.7}$$

Here Q_A is the apparent activation enthalpy, T is the absolute temperature, and R denotes the gas constant. Obviously T_* is a reference temperature, such that μ evolves toward μ_0 when $V = V_c$ and $T = T_*$. The evolution equation for the state variable is modified to the form

$$\dot{\vartheta} = -\frac{V}{d_c}\Big[\vartheta + \ln\Big(\frac{V}{V_c}\Big) + \frac{Q_B}{R}\Big(\frac{1}{T} - \frac{1}{T_*}\Big)\Big] \tag{4.8}$$

The apparent activation enthalpies, Q_A and Q_B, presumably reflect the rate-limiting steps in processes responsible for the direct and evolution effects, respectively. For the steady state, i.e. if $\dot{\vartheta} = 0$, then

$$\mu_{ss} = \mu_0 + (A - B)\ln\frac{V}{V_c} + \frac{AQ_A - BQ_B}{R}\Big(\frac{1}{T} - \frac{1}{T_*}\Big) \tag{4.9}$$

Here μ_{ss} denotes the coefficient of friction for the steady state. We observe that for $T_* = T$, Eqs (4.7) and (4.8) reduce to Ruina's equations Eqs (4.3). Some results concerning the temperature-dependent friction are depicted in Fig. 5-Fig. 8.

4.1. Slip-dependent friction

Up to now we dealt with sliding in one direction only. Let us pass to the general case.

Let $\varOmega \subset \mathbb{R}^3$ be a sufficiently regular domain and $\varGamma = \partial\varOmega$ its boundary. $\varGamma$ consists of three nonoverlapping parts: $\varGamma_0$, $\varGamma_1$, and $\varGamma_c$, such that $\varGamma = \overline{\varGamma}_0 \cup \overline{\varGamma}_1 \cup \overline{\varGamma}_c$ and the surface measure of $\varGamma_c$ is positive. The bar over a set denotes its closure. $\varGamma_c$ is the surface of possible contact, for instance the fault surface. By $\boldsymbol{N} = (N_i)$ we denote a unit exterior vector normal to $\varGamma_c$. Latin indices run from 1 to 3 and the summation convention is used throughout the paper. A vector $\boldsymbol{v} = (v_i)$ defined on $\varGamma$ may be decomposed as follows

$$\boldsymbol{v} = v_N\boldsymbol{N} + \boldsymbol{v}_T \tag{4.10}$$

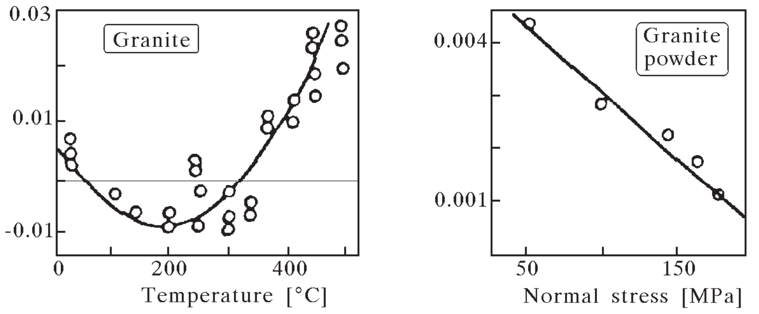

Fig. 5. Systematic of the friction parameters (A-B). (a) Dependence of (A-B) on temperature for granite. (b) Dependence of (A-B) on pressure for granular granite. This effect, due tu lithification, should be augmented with temperature, after Scholz (1998)

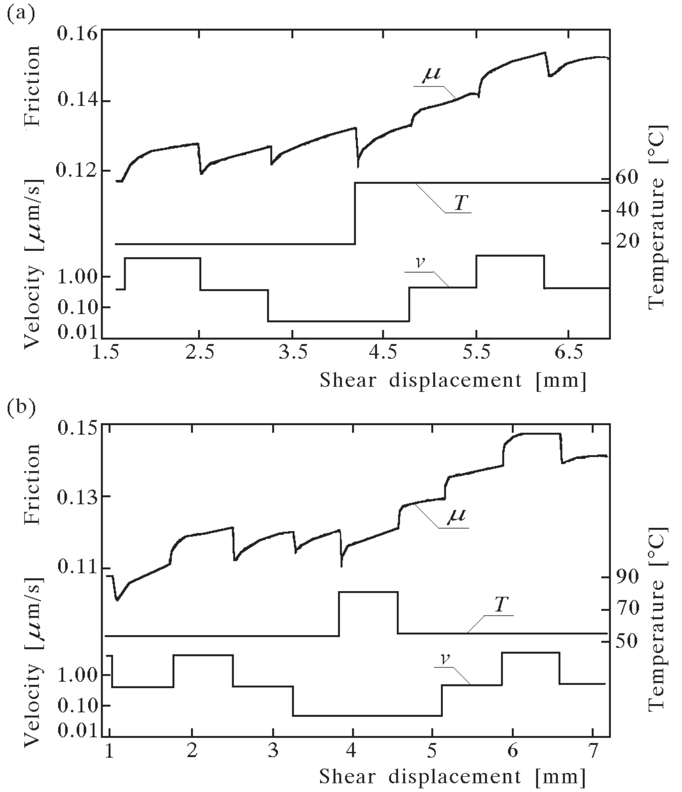

Fig. 6. Friction of graphite powder along the inclined interface at a constant confining pressure of 60MPa during velocity and temperature stopping. Velocity and temperature stepping sequence is shown, simplified after Chester (1994)

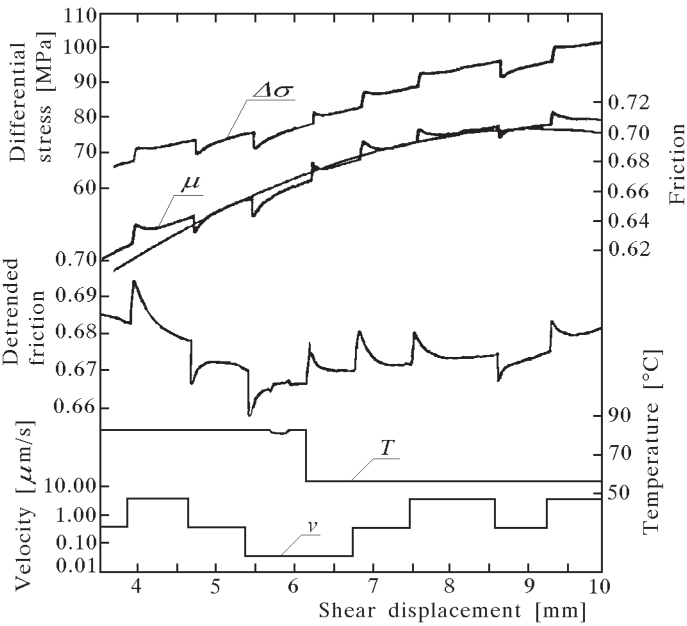

Fig. 7. Representative results of velocity and temperature stepping experiment on quartz gouge. Velocity and temperature stepping is shown, simplified after Chester (1994)

where $v_N = v_i N_i$ denotes the normal component of $\boldsymbol{v}$, while $v_{Ti} = v_i - v_N N_i$ are its tangential components. If $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor, a similar decomposition holds for the stress vector $(\sigma_{ij}N_j)$ defined on the boundary Γ. Thus we write

$$\sigma_{ij}N_j = \sigma_N N_i + \sigma_{Ti} \tag{4.11}$$

where $\sigma_N = \sigma_{ij}N_iN_j$ and $\sigma_{Ti} = \sigma_{ij}N_j - \sigma_N N_i$.

The slip-dependent friction in quasi-static and dynamic cases was considered in a series of papers by Ionescu and Paumier (1997), Ionescu and Campillo (1996), Favreau et al. (1999), Campillo et al. (1996). These authors considered the contact problems with friction between a linear elastic body and a rigid foundation. The elastic body is an infinite elastic strip bounded by two planes. Such a strip is in contact with the rigid foundation and submitted to shearing, or the half-spaces being in contact. Quasi-static and dynamic stick-slip motions are related to the earthquake instabilities. On the contact interface the friction, the Coulomb law with a slip-dependent friction coefficient was used provided that normal pressure was prescribed, see Ionescu and Paumier

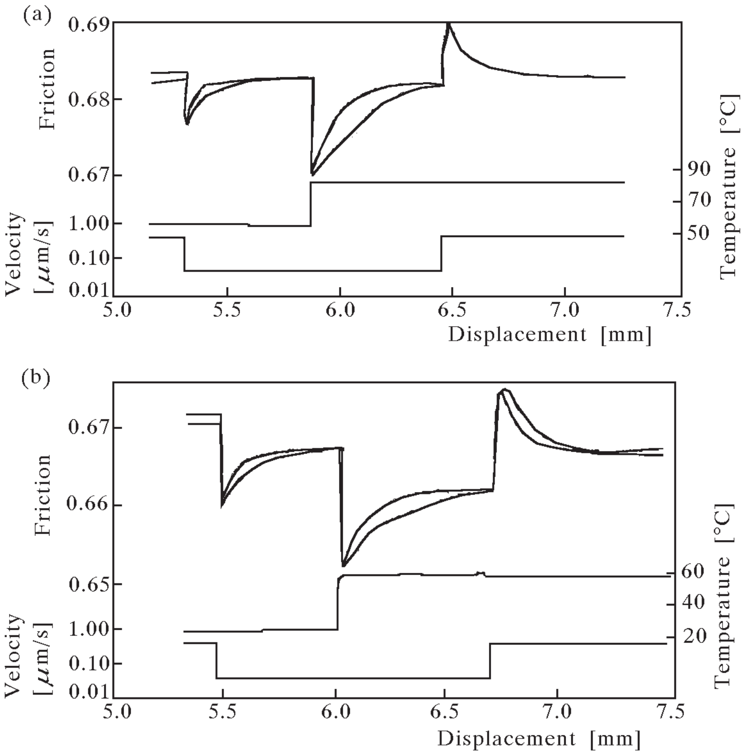

Fig. 8. Comparison of model simulations and detrended friction records from velocity and temperature stepping experiments on quartz gouge under (a) dry and (b) water saturated conditions. Friction versus shear displacement fromsimulations shown by heavy line is superposedwith friction recordf from experiment. The velocity and temperature stepping sequence is shown, after Chester (1994)

(1997). Only the anti-plane problem was studied, both the static and dynamic cases.

Let us consider the shearing of an infinite elastic slab bounded by two planes: $x_1 = l$, $x_1 = 0$, and $x_2 = h$, $x_2 = 0$.

On the contact surface $\Gamma_c = [0, l] \times \{h\} \times R$, the slab is in contact with friction with the rigid body which pushes it with the constant normal force

$$\sigma = \sigma_{22} = -S \qquad \text{or} \qquad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{N} \cdot \boldsymbol{N} = -S \qquad \text{on } \Gamma_1$$

where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u})$ is the stress tensor and $\boldsymbol{N}$ is the outward unit normal vector. Along $\Gamma_0 = [0, l] \times \{0\} \times R$ the displacement is prescribed

$$u_1 = 0 \qquad u_2 = 0 \qquad u_3 = B$$

and on $\Gamma_c = \{0, l\} \times [0, h] \times R$

$$u_1 = 0 \qquad\qquad \sigma_{12} = \sigma_{13} = 0$$

Let $\boldsymbol{u}_A = \mathbf{0}$ and

$$u_2 = u_2(x_2) \qquad\qquad \frac{\partial u_3}{\partial x_3} = 0$$

Since no perturbation of the equilibrium in the x_1-direction is considered, we get

$$u_2(x_2) = -\frac{S}{\lambda + 2G} x_2 \tag{4.12}$$

where $\lambda, G > 0$ are the Lamé constants. Let us denote by Ω the rectangle $\Omega := (0, l) \times (0, h)$; moreover we set

$$w := u_3 - B\Big(1 - \frac{x_2}{h}\Big)$$

First, we describe the static case. In this case the slip-dependent friction law on Γ_c is described by

$$\sigma_T(u) = -S\mu(|u_T|)\frac{u_T}{|u_T|} \qquad \text{if} \qquad u_T \neq 0 \quad \text{on } \Gamma_c \tag{4.13}$$

and

$$|\sigma_T(u)| \leqslant \mu(0)S \qquad \text{if} \qquad u_T = 0 \quad \text{on } \Gamma_c \tag{4.14}$$

Here u_T and σ_T are the tangential displacement and tangential stress, respectively. The equlibrium equation

$$\operatorname{div} \sigma = 0 \tag{4.15}$$

and the boundary conditions lead to the following problem:

Find $w : \Omega \to R$ such that

$$\begin{aligned}
&\Delta w = 0 && \text{in } \Omega\\
&\frac{\partial w(x_1, x_2)}{\partial x_1} = 0 && \text{for } x_1 = l \text{ and } x_1 = 0 \qquad \forall x_2 \in (0, h)\\
&w(x_1, 0) = 0 && \forall x_1 \in (0, l)\\
&G\frac{\partial w(x_1, h)}{\partial x_2} + \mu(|w(x_1, h)|)S \operatorname{sgn} w(x, h) = q && \text{if } w(x_1, h) \neq 0\\
&\Big|G\frac{\partial w(x_1, h)}{\partial x_1} - q\Big| \leqslant \mu(0)S && \text{if } w(x_1, h) = 0
\end{aligned} \tag{4.16}$$

where q is the tangential stress, which corresponds to the stick case, i.e., $q = GB/h$.

In the dynamic case, the slip-dependent friction law on the contact surface is described by the following system

$$\begin{aligned} \sigma_T(u) &= -S\mu(|u_T|)\frac{\partial u}{\partial t}\Big|\frac{\partial u}{\partial t}\Big|^{-1} && \text{if} \quad \dot{u}_T \neq 0 \quad \text{on} \quad \Gamma_c \\ |\sigma_T(u)| &= -S\mu(|u_T|) && \text{if} \quad \dot{u}_T = 0 \quad \text{on} \quad \Gamma_c \end{aligned}$$

The momentum balance law $\operatorname{div}\sigma = \varrho\ddot{u}$ and the boundary conditions lead to the following dynamic problem:

Find $w: \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} & \ddot{w}(t) = c^2 \Delta w(t) && \text{in} \quad \Omega \\ & \frac{\partial w(t,l,x_2)}{\partial x_1} = \frac{\partial w(t,0,x_2)}{\partial x_2} = w(t,x_1,0) = 0 \\ & G\frac{\partial w(t,x_1,h)}{\partial x_2} + \mu(|w(t,x_1,h)|)S\operatorname{sgn}\big(\dot{w}(t,x_1,h)\big) = q && \text{if} \quad \dot{w} \neq 0 \\ & \Big|G\frac{\partial w(t,x_1,h)}{\partial x_2} - q\Big| \leqslant \mu(|w(t,x_1,h)|)S && \text{if} \quad \dot{w}(t,x_1,h) \neq 0 \\ & w(0) = w_0 \qquad \dot{w}(0) = w_1 && \text{in} \quad \Omega \end{aligned}$$

Here $c = \sqrt{G/\varrho}$ is the shear velocity and w_0, w_1 are the initial conditions.

The static analysis of the first of the formulated problems was performed by Ionescu and Paumier (1997) using variational methods.

5. Friction conditions and sliding rules

The descriptions of friction on geological faults discussed previously are confined to one-dimensional modelling of the change of the friction coefficient. In this section we propose an alternative and rather general approach to modelling the friction condition and sliding rule in the spirit of modern contact mechanics, cf Telega (1988). In the the comprehensive rewiev paper (Shillor et al., 2002), the available variational and numerical methods of solving quasistatic contact problems are discussed. Let Ω^a $(a = 1, 2)$ be a domain in the three-dimensional physical space occupied by a linear-elastic body in its undeformed state. Γ_c denotes the contact surface (the fault surface) of the two contacting bodies. Unbounded domains are not excluded. Let $\boldsymbol{N}^a = (N_i^a)$

denote the outer unit normal vector to $\partial\Omega^a$. We set $\boldsymbol{N} = \boldsymbol{N}^2 = -\boldsymbol{N}^1$. Let $\boldsymbol{\sigma}^a = (\sigma^a_{ij})$ $(i,j = 1,2,3)$ be the stress tensor in the body Ω^a; $\boldsymbol{u}^a$ stands for the displacement vector. Moreover, by $\sigma^a_N = \sigma^a_{ij} N^a_i N^a_j$ we denote the normal component of the stress vector and $\boldsymbol{\sigma}^a_T = (\sigma^a_{ij} N^a_j) - \sigma^a_N N^a$ is the tangent stress vector, while $[\![\boldsymbol{u}_T]\!] = \boldsymbol{u}^1_T - \boldsymbol{u}^2_T$ denotes the jump of the tangent displacement across the fault surface Γ_c. Throughout this paper the summation convention is consequently applied, except that $a = 1,2$. According to the action-reaction principle we set $\boldsymbol{\sigma}_T = \boldsymbol{\sigma}^1_T = -\boldsymbol{\sigma}^2_T$. In the absence of state variables, the friction condition is assumed to be given by $f(\sigma_N, \boldsymbol{\sigma}_T) \leqslant 0$, where f is a continuous function. Anisotropic friction is not precluded. For a fixed σ_N we introduce a set $K(\sigma_N)$ of admissible tangential stresses as follows

$$K(\sigma_N) = \{\boldsymbol{\tau} |\ f(\sigma_N, \boldsymbol{\tau}) \leqslant 0, \quad \boldsymbol{\tau} \cdot \boldsymbol{N} = 0 \ \text{ on } \ \Gamma_c\}$$

Prior to the formulation of the friction law we recall the definition of a subdifferential of a convex function. If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then its subdifferential at x_0 is a subset of $\mathbb{R}^n$ such that

$$\partial f(x_0) = \{y \in \mathbb{R}^n : \ f(x) - f(x_0) \geqslant \langle y, x - x_0 \rangle \ \forall x \in \mathbb{R}^n\}$$

Here $\langle y, x \rangle = y_i x_i$. For more details the reader is referred to Rockafellar (1970). We assume that $K(\sigma_N)$ is convex and closed while the sliding rule has the subdifferential form

$$[\![\dot{\boldsymbol{u}}_T]\!] \in \partial I_{K(\sigma_N)}(\boldsymbol{\sigma}_T) \tag{5.1}$$

where $\dot{\boldsymbol{u}}_T = \partial \boldsymbol{u}/\partial t$ and $I_{K(\sigma_N)}$ is the indicator function of $K(\sigma_N)$, i.e.

$$I_{K(\sigma_N)}(\boldsymbol{\tau}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\tau} \in K(\sigma_N) \\ \infty & \text{otherwise} \end{cases}$$

As usual, $\partial I_{K(\sigma_N)}$ stands for the subdifferential of the function $I_{K(\sigma_N)}$. In the variational formulation given in the next section, the frictional dissipation density will be involved. It is determined by

$$D(\sigma_N, [\![\dot{\boldsymbol{u}}_T]\!]) = \sup\{[\![\dot{\boldsymbol{u}}]\!] \cdot \boldsymbol{\tau} \mid \boldsymbol{\tau} \in K(\sigma_N)\} \tag{5.2}$$

Obviously, $D(\sigma_N, [\![\dot{\boldsymbol{u}}_T]\!]) = [\![\dot{\boldsymbol{u}}_T]\!] \cdot \boldsymbol{\sigma}_T$. Our approach includes anisotropic friction.

In the case of the classical Coulomb friction condition we have

$$D(\sigma_N, [\![\dot{\boldsymbol{u}}_T]\!]) = \mu|\sigma_N| \cdot |[\![\dot{\boldsymbol{u}}_T]\!]|$$

Then (5.1) takes the following form

$$[\![\dot{\boldsymbol{u}}_T]\!] = -\lambda \boldsymbol{\sigma_T} \qquad\qquad \lambda \geqslant 0 \tag{5.3}$$

If the state variables ϑ_p $(p = 1, \dots, n)$ are employed for the description of friction on the fault then the friction condition is assumed in the form

$$f(\sigma_N, \boldsymbol{\sigma}_T, \vartheta_p) \leqslant 0 \tag{5.4}$$

For fixed σ_N and ϑ_p, $p = 1, \dots, n$, the set of admissible tangential stresses is given by

$$K_1(\sigma_N, \vartheta_p) = \{\boldsymbol{\tau} \mid f(\sigma_N, \boldsymbol{\tau}, \vartheta_p) \leqslant 0, \quad \boldsymbol{\tau} \cdot \boldsymbol{N} = 0 \ \text{ on } \ \varGamma_c\} \tag{5.5}$$

In this case the sliding rule may also be assumed in the subdifferential form

$$\boldsymbol{\sigma}_T \in \partial_3 D(\sigma_N, \vartheta_p, [\![\dot{\boldsymbol{u}}_T]\!]) \qquad\qquad p = 1, \dots, n \tag{5.6}$$

to which the evolution equation for ϑ_p should be appended

$$\dot{\vartheta}_p = H_p(t, \sigma_N, \vartheta_m, [\![\dot{\boldsymbol{u}}_T]\!]) \qquad\qquad m, p = 1, \dots, n \tag{5.7}$$

Here $\partial_3 D(\sigma_N, \vartheta_p, [\![\dot{\boldsymbol{u}}_T]\!])$ denotes the subdifferential of the frictional dissipation density with respect to the third variable. Particularly, suppose that the friction condition is given by $f(\sigma_N, \boldsymbol{\sigma}_T, \vartheta_p) = |\boldsymbol{\sigma}_T| - \mu(\vartheta_p)\sigma_N \leqslant 0$. Then we have

$$D(\sigma_N, \vartheta_p, [\![\dot{\boldsymbol{u}}_T]\!]) = |\boldsymbol{\sigma}_T||[\![\dot{\boldsymbol{u}}_T]\!]| = \mu(\vartheta_p)|\sigma_N||[\![\dot{\boldsymbol{u}}_T]\!]| \tag{5.8}$$

We conclude that the friction coefficient may depend on the slip velocity via the state variables.

Remark 5.1. It may happen that the friction condition does not depend on the normal stress σ_N. Specific case is provided by the friction condition used by Cochard and Madariaga (1994). These authors employ the following velocity-dependent condition in the case of antiplane shear, cf Section 6 below

$$f(\sigma_{yz}, [\![\dot{u}]\!]) = |\sigma_{yz}| - \sigma_{yz}^0 \frac{V_0}{V_0 + [\![\dot{u}]\!]} \tag{5.9}$$

where V_0 is a reference velocity that determines the rate of slip velocity weakening and σ_{yz}^0 is the maximum traction drop, reached when the slip velocity is very large. The friction condition can be obtained from the condition depending on an internal variable ϑ. More precisely, let the condition $\widetilde{f}$ depend on σ_{yz} and ϑ, where

$$\frac{d\vartheta}{dt} = H([\![\dot{u}]\!]) \qquad\qquad \vartheta(0) = \vartheta_0 \tag{5.10}$$

Solving the last equation we get

$$\vartheta = h([\![\dot{u}]\!]) \tag{5.11}$$

Substituting (5.10) into $\widetilde{f}(\sigma_{yz}, \vartheta)$ we obtain

$$f(\sigma_{yz}, [\![\dot{u}]\!]) = \widetilde{f}(\sigma_{yz}, h([\![\dot{u}]\!])) \tag{5.12}$$

Remark 5.2. Friction conditions may possess no convexity property. Then the subdifferential ∂ should be replaced by the generalized subdifferential $\overline{\partial}$, cf Panagiotopoulos (1993). Instead of variational inequalities we have to deal then with so-called hemivariational inequalities.

6. Classical and variational formulations of the fault contact problem

Now we pass to the formulation of a quasi-static initial-boundary value problem in the presence of a fault. The fault is treated as a contact surface or interface between two anisotropic, linear-elastic bodies. It can also be modelled as a closed crack in the elastic body. The interface is modelled by the subdifferential sliding rule (5.6). We set $\partial\Omega^a = \overline{\Gamma}{}_0^a \cup \overline{\Gamma}{}_1^a \cup \overline{\Gamma}{}_2^a$, $\Gamma_c = \Gamma_2^1 = \Gamma_2^2$ and formulate the quasi-static contact problem.

Problem (P)

Find $\boldsymbol{u}^a(x,t)$ $(a = 1,2)$ and $\vartheta_p(x,t)$ $(p = 1,\dots,n)$, such that

$$\begin{array}{lll}
\sigma^a_{ij,j}(\boldsymbol{u}^a) + B^a_i = 0 & \text{in} & \Omega^a \times [0,T] \\
\sigma^a_{ij}(\boldsymbol{u}^a) = a^a_{ijkl}\varepsilon_{kl}(\boldsymbol{u}^a) & \text{in} & \Omega^a \times [0,T] \\
\boldsymbol{u}^a(x,t) = 0 & \text{on} & \Gamma^a_0 \times [0,T] \\
\sigma^a_{ij}N^a_j = F^a_i & \text{on} & \Gamma^a_1 \times [0,T] \\
\boldsymbol{\sigma}_T \in \partial_3 D(\sigma_N, \vartheta_m, [\![\dot{\boldsymbol{u}}_T]\!]) & \text{on} & \Gamma_c \times (0,T)
\end{array}$$

$$
\begin{aligned}
&\dot{\vartheta}_p = H_p(t, \sigma_N, \vartheta_m, [\![\dot{\boldsymbol{u}}_T]\!]) && \text{on} \quad \Gamma_c \times (0,T) \quad m = 1,\dots,n \\
&\boldsymbol{u}^a(x,0) = \boldsymbol{u}^a_0(x) && \text{for} \quad x \in \Omega^a \\
&\vartheta_p(x,0) = \vartheta^0_p(x) && \text{for} \quad x \in \Gamma_c
\end{aligned}
$$

where $\varepsilon_{kl}(\boldsymbol{u}) = (u_{k,l} + u_{l,k})/2$, and $\boldsymbol{B}^a$ and $\boldsymbol{F}^a$ are the applied body forces and surface tractions, respectively. The functions H_p, $\boldsymbol{u}^a_0$ and ϑ^0_p are given. The formula $\sigma^a_{ij}(\boldsymbol{u}^a) = a^a_{ijkl}\varepsilon_{kl}(\boldsymbol{u}^a)$ expresses the anisotropic Hooke's law.

To obtain the variational formulation we set

$$
\begin{aligned}
a^a(\boldsymbol{u}^a, \boldsymbol{v}^a) &= \int\limits_{\Omega^a} a^a_{ijkl}\varepsilon_{ij}(\boldsymbol{u}^a)\varepsilon_{kl}(\boldsymbol{v}^a)\, dx \\
a(\boldsymbol{u}, \boldsymbol{v}) &= \sum_{a=1}^{2} a^a(\boldsymbol{u}^a, \boldsymbol{v}^a) \qquad (6.1)\\
L(\boldsymbol{v}) = L(\boldsymbol{v}^1, \boldsymbol{v}^2) &= \sum_{a=1}^{2} \Big(\int\limits_{\Omega^a} B^a_i v^a_i\, dx + \int\limits_{\Gamma^a_1} F^a_i v^a_i\, ds \Big)
\end{aligned}
$$

where $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$, $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$. It can readily be shown that the problem (P) may be transformed to the variational formulation.

Problem (P_v)

Find $\boldsymbol{u}^a = \boldsymbol{u}^a(x,t)$, $x \in \Omega^a$ $(a = 1,2)$, $t \in [0,T]$ and ϑ_p $(p = 1,\dots,n)$, such that $\boldsymbol{u}^a(x,0) = \boldsymbol{u}^a_0$ $(x \in \Omega^a)$, $\vartheta_p(x,0) = \vartheta^0_p(x)$ $(x \in \Gamma_c)$ and

$$
\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + \int\limits_{\Gamma_c} D(\sigma_N, \vartheta_p, [\![\boldsymbol{v}_T]\!])\, d\Gamma - \int\limits_{\Gamma_c} D(\sigma_N, \vartheta_p, [\![\dot{\boldsymbol{u}}_T]\!])\, d\Gamma \geqslant L(\boldsymbol{v} - \dot{\boldsymbol{u}}) \\
&\int\limits_{\Gamma_c} \Big[\dot{\vartheta}_p - H_p(\sigma_N, \vartheta_m, [\![\dot{\boldsymbol{u}}_T]\!]\Big]\eta_p(x)\, d\Gamma = 0
\end{aligned}
\qquad (6.2)
$$

for all test functions $\boldsymbol{v} = \boldsymbol{v}(x)$, $\eta_p = \eta_p(x)$. The inequality $(6.2)_1$ provides an example of an implicit variational inequality; more precisely, it is a variational inequality of the second kind. From the physical point of view, it represents the principle of virtual velocities in the presence of friction.

The variational formulation proposed can be used for the derivation of numerical procedures, cf Johansson (1992) and Section 7.

7. Green's function and contact stresses

Starting from the problem (P_v) we can derive the so-called dual formulation for the determination of stresses σ_N and $\boldsymbol{\sigma}_T$ on the interface, i.e., on the fault. Applying the procedure developed by the second author, see Telega (1991), we arrive at the dual problem.

Problem (P_G)

Find $\boldsymbol{\Sigma} = (\sigma_{ij}n_j) = (\sigma_N(x,t), \boldsymbol{\sigma}_T(x,t))$, $\boldsymbol{\sigma}_T \in K_1(\sigma_N, \vartheta_p)$, and $\vartheta_p(x,t)$, $x \in \Gamma_c$, such that for all $t \in (0,T)$ such that

$$\begin{gathered}
\int\limits_{\Gamma_c} \Big\langle \boldsymbol{s}_T(x) - \boldsymbol{\sigma}_T(x,t), \frac{d}{dt}[[\![\widehat{\boldsymbol{u}}(x,t)]\!]_T + (G^1+G^2)\boldsymbol{\Sigma}(x,t)]_T \Big\rangle \, d\Gamma(x) \geqslant 0 \\
\int\limits_{\Gamma_c} \Big([\![\widehat{\boldsymbol{u}}(x,t)]\!]_N + [(G^1+G^2)\boldsymbol{\Sigma}(x,t)]_N \Big) \varphi(x)\, d\Gamma(x) = 0 \\
\int\limits_{\Gamma_c} \Big(\dot{\vartheta}_p(x,t) - H_p(\sigma_N, \vartheta_m, [\![\dot{\boldsymbol{u}}_T]\!]) \Big) \eta_p(x)\, d\Gamma(x) = 0 \\
\vartheta_p(x,0) = \vartheta_p^0(x) \qquad \boldsymbol{\Sigma}(x,0) = \boldsymbol{\Sigma}_0(x) \qquad x \in \Gamma_c
\end{gathered} \tag{7.1}$$

for all $\boldsymbol{S} = (s_N, \boldsymbol{s}_T)$, $\boldsymbol{s}_T \in K_1(\sigma_N, \vartheta_p)$ and for all sufficiently regular φ. Similarly to Section 4, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. Obviously G^a $(a = 1,2)$ denotes the Green function for the domain Ω^a; moreover

$$\hat{u}_k^a(x,t) = \int\limits_{\Omega^a} B_i^a(y,t) G_{ik}^a(x,y) d\Omega^a(y) + \int\limits_{\Gamma_1^a} F_i^a(y,t) G_{ik}^a(x,y)\, d\Gamma(y). \tag{7.2}$$

In Eqs $(7.1)_{1,2}$ the following notation is used

$$[(G^1+G^2)\boldsymbol{\Sigma}]_i(x,t) = \int\limits_{\Gamma_c} [G_{ij}^1(x,\xi) + G_{ij}^2(x,\xi)] \Sigma_j(\xi,t)\, d\Gamma(\xi)$$

We observe that the dual problem (P_G) enables us to find the *normal* and *tangential* stresses on the fault surface Γ_c. The inequality $(7.1)_1$ is a quasi-variational inequality since the set of constraints K_1 depends on the solution.

Remark 7.1. Dual formulation for the static problem with friction was examined in Bielski and Telega (1985).

8. Antiplane deformation

In order to illustrate the approach developed in the previous section, we consider the antiplane crack problem in an isotropic infinite space. Now $\Omega = \mathbb{R}^3$ and the contact surface, i.e. the crack Γ_c, is defined by

$$\Gamma_c = \Big\{(x,y,z) \in \mathbb{R}^3 \mid -\ell \leqslant x \leqslant \ell, \ \ -\infty < y < \infty, \ \ z = 0\Big\} \tag{8.1}$$

where $\ell > 0$. The displacement vector field is assumed to be continuous on $\Omega \backslash \Gamma_c$. In the case of the antiplane deformation we have, see Cochard and Madariaga (1994)

$$\boldsymbol{u}(x,z) = [0, u(x,z), 0] \tag{8.2}$$

The strain tensor has the following form

$$\mathbf{e} = \frac{1}{2}\left[\begin{array}{ccc} 0 & u_{,x} & 0 \\ u_{,x} & 0 & u_{,z} \\ 0 & u_{,z} & 0 \end{array}\right] \tag{8.3}$$

The stress tensor $\boldsymbol{\sigma} = 2\mu\mathbf{e} + \lambda \operatorname{tr}\mathbf{e}$ reduces to

$$\boldsymbol{\sigma} = \mu\left[\begin{array}{ccc} 0 & u_{,x} & 0 \\ u_{,x} & 0 & u_{,z} \\ 0 & u_{,z} & 0 \end{array}\right] \tag{8.4}$$

The equilibrium equation is expressed by

$$\mu \Delta u(x,z,t) = 0 \qquad \text{in} \quad \mathbb{R}^3 \backslash \Gamma_c \tag{8.5}$$

Now u depends also on time t since the problem under consideration is quasistatic. Here Δ denotes the Laplacian with respect to the variables x, z. The fundamental solution or the Green function for the last equation is, see Vladimirov (1984)

$$G(x,z;\xi) = \frac{1}{2\pi\mu}\ln r \tag{8.6}$$

where $r^2 = (x-\xi)^2 + z^2$.

From Betti's formula we find the displacement inside the elastic body

$$u(x,z,t) = \mu \int\limits_{-\ell}^{\ell} [\![u(\xi,t)]\!] \frac{\partial G(x,z;\xi)}{\partial z}\, d\xi \tag{8.7}$$

The problem considered is two-dimensional, since it does not depend on y.

Now we have $u_N = 0$ and the tangent displacement $u_T = u$ is in the direction of the axis y. We set

$$[\![u(x,t)]\!] = u(x,0^+,t) - u(x,0^-,t) \tag{8.8}$$

Then

$$[\![u(x,t)]\!] = 0 \quad \text{for} \quad |x| > \ell$$
$$[\![u(x,t)]\!] \neq 0 \quad \text{for} \quad |x| \leqslant \ell$$

Eq. (8.5) is completed with
(i) the sliding rule

$$\sigma = \sigma_T \in \partial_2 D(\vartheta_p, [\![\dot{u}]\!]) \qquad \text{for} \qquad |x| < \ell \qquad p = 1,\dots,n \tag{8.9}$$

(ii) the evolution equation for ϑ_p

$$\dot{\vartheta}_p = H_p(\vartheta_m, [\![\dot{u}]\!]) \qquad m,p = 1,\dots,n \qquad |x| < \ell \tag{8.10}$$

(iii) the initial conditions

$$\vartheta_p(x,0) = \vartheta_p^0(x) \qquad \sigma(x,0) = \sigma^0 \tag{8.11}$$

We observe that now the friction condition does not depend on the normal stress, cf Remark 8.1.

In the case of the antiplane deformation, the dual problem (P_G) reduces to:

Find $\sigma(x,t)$, $x \in \mathit{\Gamma}_c$, $t \in [0,T]$ such that (8.9) is satisfied and

$$\int\limits_{-\ell}^{\ell}\int\limits_{-\ell}^{\ell} [s(x) - \sigma(x,t)] G(x,z;\xi)\dot{\sigma}(\xi,t)\; d\xi dx \geqslant 0 \qquad \forall s(x) \in K_1 \tag{8.12}$$

The indicator function of the set $K_1(\vartheta_p)$ is a dual of $D(\vartheta_p,\cdot)$. Once the density of frictional dissipation is known, one can also find the set $K_1(\vartheta_p)$. Eq. (8.2) is now trivially satisfied. We observe that in the case of the friction condition used in Cochard and Madariaga (1994), the problem (P_G) for the antiplane shear does not involve Eq. (8.9).

Remark 8.1.

(i) The antiplane problem significantly simplifies the fault deformation. Here it has been used to show how the method of duality may be employed to study the friction problem in the neighborhood of the fault.

(ii) Okubo (1989) defined the fault as the plane $x_3 = 0$ in an infinite, homogeneous, elastic whole space. In such case $\mathit{\Gamma}_c = \mathbb{R}^2$ and our duality method can also be applied.

9. Time and space discretizations

Now we pass to time discretization of problem (P_G) in the case of the anti-plane shear. First, we observe that then Green's function G of the considered problem does not depend on time t, cf Eq. (8.6). Thus we have $dG\boldsymbol{\sigma}/dt = G\dot{\boldsymbol{\sigma}}$. Therefore we can introduce the following approximation of the stress derivative with respect to time t. Let the time interval $[0,T]$ be divided into L intervals (t_{l-1}, t_l) for $l = 1,\ldots,L$ and $0 = t_0 < t_1 < \ldots < t_L = T$. The time derivative is approximated by the backward finite difference in the following way

$$\dot{\sigma}(x,t_l) \approx \frac{\sigma(x,t_l) - \sigma(x,t_{l-1})}{t_l - t_{l-1}} \tag{9.1}$$

The evolution equation of the internal variables takes the form

$$\dot{\vartheta}_p(x,t) = H_p(\vartheta_m, [\![\dot{u}]\!])$$

After discretization in time we write

$$\dot{\vartheta}_p(x,t_l) \approx \frac{\vartheta_p(x,t_l) - \vartheta(x,t_{l-1})}{t_l - t_{l-1}} = H_p(\vartheta_m(x,t_{l-1}), [\![\dot{u}(x,t_{l-1})]\!]) \tag{9.2}$$

Substituting (9.1) into (P_G) we get the following problem for the interval (t_{l-1}, t_l). After time discretization, the quasi-variational inequality (8.12) is written as the sequence of quasi-variational inequalities.

Problem (P_G^l)

Find $\sigma(x,t_l)$ and $\vartheta_p(x,t_l)$, $l = 1,\ldots,L$; $0 = t_0 < t_1 < \ldots < t_L = T$, $x \in (-\ell,\ell)$ such that for all admissible stresses $s = s(x) \in K_1(\vartheta_p(x,t_l))$,

$$\begin{aligned}
&\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} G(x,\xi)\sigma(\xi,t_l)[s(x) - \sigma(x,t_l)]\,dxd\xi \geqslant \\
&\geqslant \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} G(x,\xi)\sigma(\xi,t_{l-1})[s(x) - \sigma(x,t_l)]\,dxd\xi \\
&\frac{\vartheta_p(x,t_l) - \vartheta_p(x,t_{l-1})}{t_l - t_{l-1}} = H_p(\vartheta_m(x,t_{l-1}), [\![\dot{u}(x,t_{l-1})]\!]) \\
&\sigma(x,0) = \sigma_0(x) \qquad \vartheta(x,0) = \vartheta_0(x) \qquad x \in [-\ell,\ell]
\end{aligned} \tag{9.3}$$

where G is given by (8.6)

Remark 9.1. Cochard and Madariaga (1994) have considered dynamical problem of fault friction for slip velocity-dependent model of friction in the case of antiplane deformation of the whole space. In this paper, we consider a quasi-static deformation. We take into account the friction condition and the sliding rule. We observe that our approach can be generalized to the dynamic case. Then Green's tensor will depend on time.

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References

1. Beeler N.M., Tullis T.E., Blanpied M.L., Weeks J.D., 1996, Frictional Behavior of Large Displacement Experimantal Faults, *J. Geophys. Res.*, **101**, 8697-8715
2. Ben-Zion Y., Rice J.R., 1995, Slip Patterns and Earthquake Populations Along Different Classes of Faults in Elastic Solids, *J. Geophys. Res.*, **100**, 12959-12983
3. Ben-Zion Y., Rice J.R., 1997, Dynamic Simulations of Slip on a Smooth Fault in an Elastic Solid, *J. Geophys. Res.*, **102**, 17771-17784
4. Beroza G., Mikumo T., 1996, Short Slip Duration in Dynamic Rupture in the Presence of Heterogeneous Fault Properties, *J. Geophys. Res.*, **101**, 22449-22460
5. Bielski W.R., Telega J.J., 1985, A Contribution to Contact Problem for a Class of Solids and Structures, *Arch. Mech.*, **37**, 303-320
6. Bielski W.R., Telega J.J., 1994, Friction on Geological Faults: Modelling and Variational Formulation of Initial-Boundary Value Problems, *IFTR Reports*, **31**, in Polish
7. Bielski W.R., Telega J.J., 2000, A Contribution to Modelling of Friction on Tectonic Faults, Bull Pol. Acad, Sci, Ser. Earth Sci., **48**, 33-43
8. Blanpied M.L., Tullis T.E., Weeks J.D., 1998, Effects of Slip, Slip Rate, and Shear Heating on the Friction of Granite, *J. Geophys. Res.*, **103**, 489-511
9. Boatwright J., Cocco J., 1996, Frictional Constraints on Crustal Faulting, *J. Geophys. Res.*, **101**, 13895-13909

10. Brace W.F., Byerlee J.D., 1966, Stick Slip as a Mechanism for Earthquakes, *Science*, **153**, 990-992

11. Brown S.R., 1998, Frictional Healing on Faults: Stable Sliding Versus Stick Slip, *J. Geophys. Res.*, **103**, 7413-7420

12. Brogliato B., 1999, *Nonsmooth Mechanics*, Springer, London

13. Byerlee J.D., 1970, The Mechanics of Stick-Slip, *Tectonophysics*, **9**, 475-486

14. Byerlee J.D., 1978, Friction of Rocks, *Pageoph.*, **116**, 615-626

15. Campillo M., Ionescu I.R., 1997, Initiation of Antiplane Shear Instability Under Slip Dependent Friction, *J. Geophys. Res.*, **102**, 20363-20371

16. Campillo M., Ionescu I.R., Paumier J.C., Renard Y., 1996, On the Dynamic Sliding with Friction of a Rigid Block and of an Infinite Elastic Slab, *Phys. Earth and Planetary Interiors*, **96**, 15-23

17. Cao T., Aki K., 1986, Seismicity Simulation with a Rate- and State-Dependent Friction Law, *Pageoph*, **124**, 487-513

18. Chester F.M., 1994, Effects of Temperature on Friction: Constitutive Equations and Experiments with Quartz Gouge, *J. Geophys. Res.*, **99**, 7247-7261

19. Cochard A., Madariaga R., 1994, Dynamic Faulting Under Rate-Dependent Friction, *Pageoph*, **142**, 419-445

20. Cochard A., Rice J.R., 2000, Fault Rupture Between Dissimilar Materials: Ill-posedness, Regularization, and Slip-Pulsed Response, *J. Geophysical Res.*, preprint

21. Cotton F., Campillo M., 1995, Frequency Domain Inversion of Strong Motions: Application to the 1992 Landers Earthquake, *J. Geophys. Res.*, **100**, 3961-3975

22. Dascalu C., Ionescu I.R., Campillo M., 2000, Fault Finiteness and Initiation of Dynamic Shear Instability, *Earth and Planetary Science Letters*, **177**, 163-176

23. Dieterich J.H., 1972, Time Dependent Friction in Rocks, *J. Geophys. Res.*, **77**, 3690-3797

24. Dieterich J.H., 1978, Time-Dependent Friction and the Mechanics of Stick-slip, *Pure. Appl. Geophys.*, **116**, 790-806

25. Dieterich J.H., 1979, Modeling of Rock Friction 1. Experimental Results and Constitutive Equations, *J. Geophys. Res.*, **84**, 2161-2168

26. Dieterich J.H., 1992, Earthquake Nucleation on Faults with Rate- and State-Dependent Strength, *Tectonophysics,* **211**, 115-134

27. Dieterich J.H., Conrad G. , 1984, Effect of Humidity on Time- and Velocity-Dependent Frictions in Rocks, *J. Geophys. Res.*, **89**, 4196-4202

28. Dieterich J.H., Kilgore B.D., 1994, Direct Observations of Frictional Contact: New Insights for State-Dependent Properties, *Pageoph*, **143**, 283-302

29. Favreau P., Campillo M., Ionescu I.R., 1999, Initiation of In-Plane Shear Instability under Slipp-Dependent Friction, *Bull., Seism. Soc. Amer.*, **89**, 1280-1295

30. Fukuyama E., Madariaga R., 1998, Rupture Dynamics of a Planar Fault in a 3D Elastic Medium: Rate- and Slip-Weakening Friction, *Bull. Seism. Soc. Amer.*, **88**, 1-17

31. Ide S., Takeo M., 1997, Determination of Constitutive Relations of Fault Slip Based on Seismic Wave Analysis, *J.Geophys. Res.*, **102**, 27379-27391

32. Iio Y., 1997, Frictional Coefficient on Faults in a Seismogenic Region Inferred from Earthquake Mechanism Solutions, *J. Geophys. Res.*, **102**, 5403-5412

33. Ionescu I.R., Campillo M., 1996, On the Contact Problem with Slip Displacement Dependent Friction in Elastostatics, *Int. J. Engng. Sci.*, **34**, 471-491

34. Ionescu I.R., Campillo M., 1999, Influence of the Shape of the Friction Law and Fault Finitness on the Duration if Initiation, *J. Geophys. Res.*, **104**, 3013-3024

35. Ionescu I.R., Paumier J.-C., 1997, Instabilities in Slip Dependent Friction, *ESAIM: Proc.*, **2**, 99-111, URL: http://www.emath.fr/proc/Vol2/

36. Jaeger J.C., Cook N.G.W., 1976, *Fundamentals of Rock Mechanics*, 2nd ed., London, Chapman and Hall

37. Johansson L., 1992, Elastic and Thermoelastic Contact Problems with Friction and Wear, Linköping Studies in Science and Technology, Dissertation No. 266, Linköping

38. Kilgore B.D., Blanpied M.L., Dieterich J.H., 1993, Velocity and Normall Stress Depend Friction of Granite, *Geophys. Res. Lett.*, **20**, 903-906

39. King G.C.P., Cocco M., 2001, Fault Interaction by Elastic Stress Changes: New Clues from Earthquake Sequences, in: *Advances in Geophysics*, edit. by Dmowska R., and Saltzman B., **44**, 1-38, Acadamic Press, San Diego

40. Lapusta N., Rice J.R., Ben-Zion Y., Zheng G., 2000, Elastodynamic Analysis for Slow Tectonic Loading with Spontaneous Rupture Episodes on Faults with Rate- and State-Dependent Friction, *J. Geophys. Res.*, **105**, 23765-23789

41. Lockner D. A., 1998, A Generalized Law for Brittle Deformation of Westerly Granite, *J. Geophys. Res.*, **103**, 5107-5123

42. Madariaga R., Olsen K., Archuleta R., 1998, Modeling Dynamic Rupture in a 3D Earthquake Fault Model, *Bull. Seism. Soc. Am.*, **88**, 1182-1197

43. Mair K., Marone C., 1999, Friction of Simulated Fault Gouge for a Wide Range of Velocities and Normal Stresses, *J. Geophys. Res.*, **104**, 28899-28898

44. Morrow C.A., Moore D.E., Lockner D.A., 2000, The Effect of Mineral Bond Strength and Absorbed Water on Fault Gouge Frictional Strength, *Geophys. Res. Lett.*, **27**, 815-818

45. Nielsen S.B., Carlson J.M., Olsen K.B., 2000, Influence of Friction and Fault Geometry on Earthquake Rupture, *J. Geophys. Res.*, **105**, 6069-6088

46. Okubo P.G., 1989, Dynamic Rupture Modeling with Laboratory-Derived Constitutive Relations, *J. Geophys. Res.*, **94**, 12321-12335

47. Oglesby D.D., Archuleta R.J., Nielsen S.B., 2000, The Dynamics of Dip-Slip Faulting: Exploration in Two Dimensions, *J. Geophys. Res.*, **105**, 13643-13653

48. Olsen M.P., Scholz C.H., Leger A., 1998, Healing and Scaling of a Simulated Fault Gouge Under Hydrothermal Conditions: Implications for Fault Healing, *J. Geophys. Res.*, **103**, 7421-7430

49. Place D., Mora P., 1999 The Lattice Solid Model to Simulate the Physics of Rocks and Earthquakes: Incorporation of Friction, *J. Comput. Phys.*, **150**, 332-372

50. Panagiotopoulos P.D., 1993, *Hemivariational Inequalities: Applications to Mechanics and Engineering*, Springer-Verlag, Berlin

51. Perrin G.J., Rice J.R., Zheng G., 1995, Self-Healing Slip Pulse on a Frictional Surface, *J. Mech. Phys. Solids*, **43**, 1461-1495

52. Press F, Siever R., 1986, *Earth*, Freeman and Company, New York

53. Rabinowicz E., 1965, *Friction and Wear of Materials*, John Wiley, New York

54. Ranjith K., Rice J.R., 2001, Slip Dynamics at an Interface Between Dissimilar Materials, *J. Mech. Phys. Solids*, **49**, 2, 341-361

55. Rice J.R., 1983, Constitutive Relations for Fault Slip and Earthquake Instabilities, *Pageoph*, **121**, 443-475

56. Rice J.R., 1993, Spatio-Temporal Complexity of Slip on a Fault, *J. Geophys. Res.*, **98**, 9885-9907

57. Richardson E., Marone C., 1999, Effects of Normal Stress Vibration on Frictional Healing, *J. Geophys. Res.*, **104**, 28859-28878

58. Rockafellar R.T., 1970, *Convex Analysis*, Princeton, New Jersey, Princeton University Press

59. Rowshandel B., Nemat-Nasser S., 1986, A Mechanical Model for Deformation and Earthquakes on Stroke-slip Faults, *Pageoph*, **124**, 531-566

60. Roy M., Marone C., 1996, Earthquake Nucleation on Model Faults with Rate- and State-Dependent Friction: Effects of Inertia, *J. Geophys. Res.*, **101**, 13919-13932

61. Rudnicki J., Wu M., 1995, Mechanics of Dip-Slip Faulting in an Elastic Half-Space, *J. Geophys. Res.*, **100**, 22173-22186

62. Ruina A.L., 1980, Friction Laws and Instabilities: A Quasistatic Analysis of Some Dry Frictional Behaviour, Ph.D. Division of Engineering, Brown Unuversity

63. Ruina A.L., 1983, Slip Instability and State Variable Friction Laws, *J. Geophys. Res.*, **88**, 10359-10370

64. Rybicki K.R., 1992, Strike-Slip Faulting in the Presence of Low-Rigidity Inhomogeneities, *Bull. Seism. Soc. Am.*, **82**, 2170-2190

65. Savage J.C., Lockner D.A., Byerlee J.D., 1996, Failure in Laboratory Fault Models in Triaxial Tests, *J. Geophys. Res.*, **101**, 22215-22224

66. Scholz C.H., 1994, *The Mechanics of Earthquakes and Faulting*, Cambridge University Press

67. Scholz C.H., 1998, Earthquakes and Friction Laws, *Nature*, **391**, 37-42

68. Scholz C.H., 1996, Faults without Friction?, *Nature*, **381**, 556-557

69. Segall P., Rice J.R., 1995, Dilatency, Compaction, and Slip Instability of a Fluid-Infiltrated Fault, *J. Geophys. Res.*, **100**, 22155-22171

70. Senatorski P., 2000a, Model of Earthquake Cycles with Slip-Dependent Constitutive Law. Part I-Theory, *Acta Geophys. Pol.*, **48**, 299-316

71. Senatorski P., 2000b, Model of Earthquake Cycles with Slip-Dependent Constitutive Law. Part II-Simulation, *Acta Geophys. Pol.*, **48**, in press *Acta Geophys. Pol.*

72. Shillor M., Sofonea M., Telega J.J., 2002, Advances in Variational Solution of Quasistatic Contact Problems, *Appl. Mech. Rev.*, in preparation

73. Sleep N.H., 1995, Frictional Heating and the Stability of Rate and State Dependent Frictional Sliding, *Geophys. Res. Letters*, **22**, 2785-2788

74. Sleep N.H., 1997, Application of a Unified Rate and State Dependent Friction Theory to the Mechanism of Fault Zones with Strain Localization, *J. Geophys. Res.*, **102**, 2875-2895

75. Sleep N.H., 1998, Rate Dependent Rate and State Friction, *J. Geophys. Res.*, **103**, 7111-7119

76. Sleep N.H., 1999, Effects of the Extrusion of Fault Gouge on Frictional Sliding, *J. Geophys. Res.*, **104**, 23023-23032

77. Teisseyre R., 1985, *Continuum Theories in Solid Earth Physics*, PWN and Elsevier, Warszawa-Amsterdam

78. Teisseyre R., (edit.), 1995, *Theory of Earthquake Premonitory and Fracture Processes*, Polish Scientific Publishers PWN, Warszawa

79. Teisseyre R., Majewski E., (edit.), 2001, *Earthquake Thermodynamics and Phase Transitions in the Earth's Interior*, Academic Press, San Diego

80. Telega J.J., 1988, Topics on Unilateral Contact Problems of Elasticity and Inelasticity, in: *Nonsmooth Mechanics and Applications*, J.J. Moreau, P. D. Panagiotopoulos (edit.), Springer-Verlag Wien-New York, 341-462

81. Telega J.J., 1991, Quasi-Static Signorini's Contact Problems with Friction and Duality, in: International Series of Numerical Mathematics, **101**, Birkhäuser, 199-214

82. Turcotte D.L., Schubert G., 1982, *Geodynamics: Applications of Continuum Physics to Geological Problems*, John Wiley, New York

83. Vladimirov V.S., 1984, *Equations of Mathematical Physics*, Mir Publishers Moscow

84. Weeks J., Tullis T., 1985, Frictional Sliding of Dolomite: A Variation in Constitutive Behavior, *J. Geophys. Res.*, **90**, 7821-7826

85. Zheng G., Rice J.R., 1998, Conditions Under which Velocity-Weakening Friction Allows a Self-Healing Versus a Cracklike Mode of Rupture, *Bull. Seism. Soc. Am.*, **88**, 1466-1483

Modelowanie zagadnień kontaktowych z tarciem w mechanice uskoków geologicznych

Streszczenie

Cel pracy jest dwojaki. Po pierwsze, omówiono modele tarcia stosowane w geofizyce. Modele te obejmują tarcie zależne od strun, prędkości i poślizgu.

Po drugie, zaproponowano nowy opis tarcia w języku nowoczesnej mechaniki kontaktu, wprowadzając prawa poślizgu wiążące naprężenie poślizgu z prędkością poślizgu. Prawa poślizgu sformułowano w postaci subróżniczkowej. Zagadnienia początkowo-brzegowe sformułowano w postaci silnej i wariacyjnej. Stosując funkcję Greena zaproponowano sformułowanie wariacyjne pozwalające wyznaczyć normalne i styczne naprężenia kontaktowe.

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