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PARAMETRIC STRUCTURAL SHAPE OPTIMIZATION USING THE GLOBAL TREFFTZ APPROACH

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The global Trefftz approach is applied to optimization of certain types of structures. Numerical examples show advantages of the method proposed. The algorithm can be extended to cover optimization using the Trefftz finite element solutions.

Key words: Trefftz method, shape optimization

1. Introduction

In any algorithm of structural optimization an approximate solution of a particular boundary value problem is calculated many times inside the optimizing loops. Therefore, one of the basic directions of its improvement is minimizing the computer time in each single step of the procedure without loss of the solution accuracy. The proposed Trefftz method, applied to structural elements like plates (Jirousek, 1987; Jin and Cheung, 1999), helical springs (Karaś and Zieliński, 1998a), 2D and 3D elastic objects (Jirousek and Venkatesh, 1992; Peters, 1994) etc. has proved to be very efficient in accurate numerical approximations of numerous boundary problem solutions. The basic idea of the method consists in application of analytical trial functions (called T-complete systems) identically fulfilling the governing differential equations of the problem. The unknown coefficients of the functions are calculated from the given boundary and sometimes also connectivity conditions.

The global Trefftz approach, the basic formulation of the method, yields very accurate results. However, it rather cannot be applied to structures of a complex form due to possible problems with conditioning of the solution matrices (Zieliński and Herrera, 1987). In such cases it is necessary to divide the investigated object into subregions applying one of the formulations of Trefftz finite elements (T-elements) (Jirousek and Zieliński, 1997; Jirousek and Wróblewski, 1996). Hence, the present investigations of the global version of the method take into consideration structures of relatively simple original shapes. On the other hand, this is a necessary stage of invesigation of the optimization algorithm before the application of the T-element solution mentioned above.

2. T-complete systems

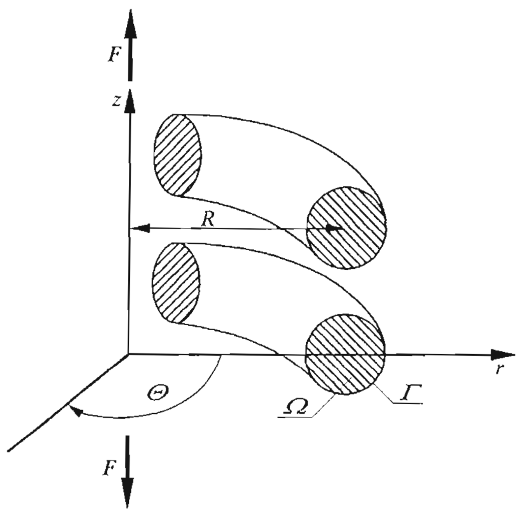

Fig. 1. Scheme of helical spring

The main limitation of the Trefftz approach is existance of the Trefftz-type functions fulfilling particular differential equations. There are different possibilities of derivation of such functions (Zieliński, 1995). A good example of such a procedure is given by Karaś and Zieliński (1998a) for the equation of the stress function in a helical spring (Fig.1)

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{3}{r}\frac{\partial \Phi}{\partial r} + 2c = 0 \tag{2.1}$$

where c is a constant dependent on the load F while

$$\tau_{r\theta} = \frac{GR^2}{r^2}\frac{\partial \Phi}{\partial z} \qquad\qquad \tau_{\theta z} = -\frac{GR^2}{r^2}\frac{\partial \Phi}{\partial r} \tag{2.2}$$

where
G – Kirchhoff modulus
$\tau_{r\theta}, \tau_{\theta z}$ – tangent stresses in the cross-section of the spring.

After some transformations we obtain

$$\Phi_0 = 1 \qquad \Phi_2 = z^2 + \frac{1}{2}r^2$$
$$\Phi_1 = z \qquad \Phi_3 = z^3 + \frac{3}{2}r^2 z \tag{2.3}$$
$$\Phi_{j+4} = \sum_{k=0}^{\left[\frac{j}{2}\right]} a_{jk} r^{4+2k} z^{j-2k} \qquad j = 0, 1, 2, ...$$

where $[x]$ is the integer part of the real number x and

$$a_{j0} = 1 \qquad a_{j(k+1)} = -a_{jk}\frac{(j-2k)(j-2k-1)}{[4+2(k+1)]2(k+1)} \tag{2.4}$$

The inductive process of derivation ensures the completness of the system. To present it we consider the action of the operator

$$L = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{3}{r}\frac{\partial}{\partial r} \tag{2.5}$$

resulting from Eq (2.1), on the monomials $ar^\alpha z^\beta$, $br^{\alpha+2}z^{\beta-2}$, $cr^{\alpha+4}z^{\beta-4}$

$$\begin{aligned}
L(ar^\alpha z^\beta) &= a\alpha(\alpha-4)r^{\alpha-2}z^\beta + a\beta(\beta-1)r^\alpha z^{\beta-2}\\
L(br^{\alpha+2}z^{\beta-2}) &= b(\alpha+2)(\alpha-2)r^\alpha z^{\beta-2} + b(\beta-2)(\beta-3)r^{\alpha+2}z^{\beta-4}\\
L(cr^{\alpha+4}z^{\beta-4}) &= c\alpha(\alpha+4)r^{\alpha+2}z^{\beta-4} + c(\beta-4)(\beta-5)r^{\alpha+4}z^{\beta-6}
\end{aligned} \tag{2.6}$$

Now we choose the coefficients a, b, c so that

$$\begin{aligned}
a\beta(\beta-1) + b(\alpha+2)(\alpha-2) &= 0\\
b(\beta-2)(\beta-3) + c\alpha(\alpha+4) &= 0
\end{aligned} \tag{2.7}$$

Then, summing Eqs (2.6) we obtain

$$\begin{aligned}
&L(ar^\alpha z^\beta) + L(br^{\alpha+2}z^{\beta-2}) + L(cr^{\alpha+4}z^{\beta-4}) =\\
&= a\alpha(\alpha-4)r^{\alpha-2}z^\beta + c(\beta-4)(\beta-5)r^{\alpha+4}z^{\beta-6}
\end{aligned} \tag{2.8}$$

Now, if

$$\alpha(\alpha-4)=0 \qquad \text{and} \qquad (\beta-4)(\beta-5)=0 \tag{2.9}$$

the sum (2.8) vanishes. Following this procedure we finally obtain the complete system of polynomials (2.3) and (2.4). More details of the derivation can be found in Karaś and Zieliński (1998a).

3. Preliminary numerical investigations

The problem of the helical spring cross-section representing a simply-connected region was chosen for the first numerical example. Prior to the application of optimization algorithm it was necessery to carry out certain preliminary numerical investigations of convergence of the solution proposed. Because of the symmetry of the problem with respect to the r-axis (see Fig.1; compare also Eq (4.2)) only the even functions Φ_j, $(j=0,2,4,...)$ were taken into account. The first calculations were done for a circular cross-section. The T-complete set was supplemented by the solution

$$\Phi^p(r,z)=\frac{c}{4}(r^2-2z^2) \tag{3.1}$$

The boundary equation of the problem

$$\Phi_\Gamma=0 \tag{3.2}$$

was approximately fulfilled in a system of equidistant collocation points. To reduce the boundary oscillations, the number of collocations points exceeded the number of unknown Trefftz coefficients and the least square procedure was applied.

In the Trefftz approach the extreme errors occur on the boundary Γ of the investigated area Ω. This suggested introduction of the following boundary error, related to $\widetilde{\Phi}_8(R,0)$ which was very near to $\Phi_{\max}$

$$\varepsilon=\frac{\max|\widetilde{\Phi}|_\Gamma}{\widetilde{\Phi}_8(R,0)}\cdot 100\% \tag{3.3}$$

where $\widetilde{\Phi}$ stand for the approximated, numerically calculated value of Φ and $\widetilde{\Phi}_8$ is the result obtained for eight Trefftz functions. The relative error (3.3) illustrates the quality of the whole approximate solution. Its accuracy is visible in Fig.2 in which the distribution of the boundary error is presented. We can

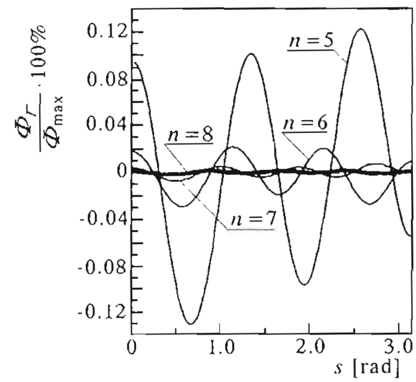

Fig. 2. Distributions of boundary solution error for different numbers of Trefftz functions ($N = 2n$)

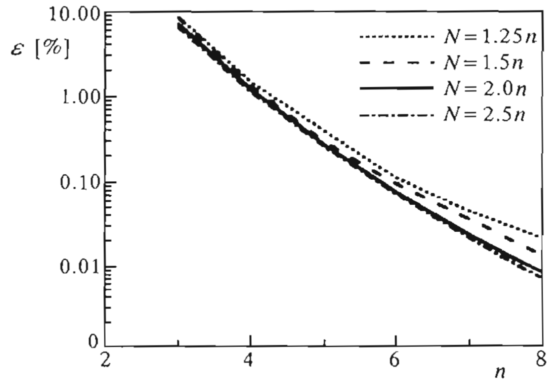

Fig. 3. Error measure ε versus the number of trial functions

observe the quick decrease in the error with the increase in the number n of trial functions.

As it has been already mentioned, the number N of the collocation points should be larger than n, but not too much. In Fig.3 we can observe that an excessive number of the collocation points is not profitable, in view of an obvious increase in the computing time.

The double-connected region was here investigated on an example of a rectangular plate with a circular hole of different diameters. The plate was subjected to uniaxial tension (Fig.4). Different variants of collocation on the external boundary were compared (equidistant, at the Gaussian points, at the Lobbato points). For a square plate the results were very similar, while for a

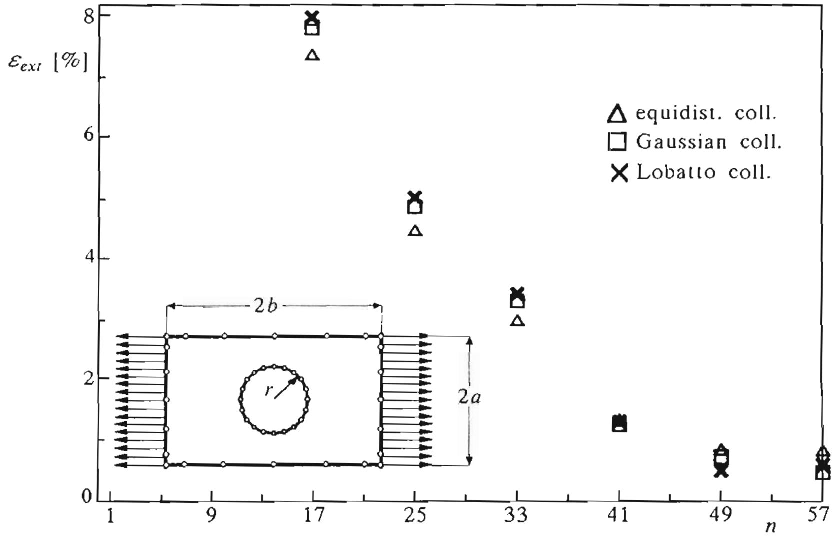

Fig. 4. Investigated plate; different variants of boundary collocation $(b/a = 2,$ $r/a = 0.25, \overline{N}_{ext} = \overline{N}_{int} = 72)$

rectangular plate and a larger number of functions the orthogonal collocation appeared to be optimal (Fig.4; see also Zieliński and Herrera, 1987). Here we considered the error measure:

$$\varepsilon_{ext} = \frac{\max|t - \overline{t}|_{\Gamma_{ext}}}{p} \cdot 100\% \tag{3.4}$$

where
- $t, \overline{t}$ – calculated and given tractions, respectively
- p – defined in Fig.8.

Analogically, ε_{int} was defined on the internal (hole) boundary.

The most characteristic phenomenon observed while investigating the double-connected plate was instability of the solution for a too small number of the collocation points either on the internal or external boundary. According to our investigations, the final relation ensuring the stability could be written as

$$\overline{N}_{ext} > n_f \qquad \text{and} \qquad \overline{N}_{int} > n_f \tag{3.5}$$

where
- n_f – number of T-functions
- $\overline{N}_{ext}, \overline{N}_{int}$ – numbers of equations resulting from the collocation points on the external and internal boundary, respectively.

The form of Eq (3.5) can be explained by a specific structure of the T-complete system (Jirousek and Venkatesh, 1992). Because of existence of both positive and negative powers, the influence of the collocation on the internal and external boundaries differs substantially for particular terms. For $k > 0$ the external boundary results in great values and for $k < 0$ small values of the approximating functions. On the internal boundary the situation is opposite. This results in "numerical separation" of the terms with positive and negative powers and the characteristic behaviour illustrated in Fig.5.

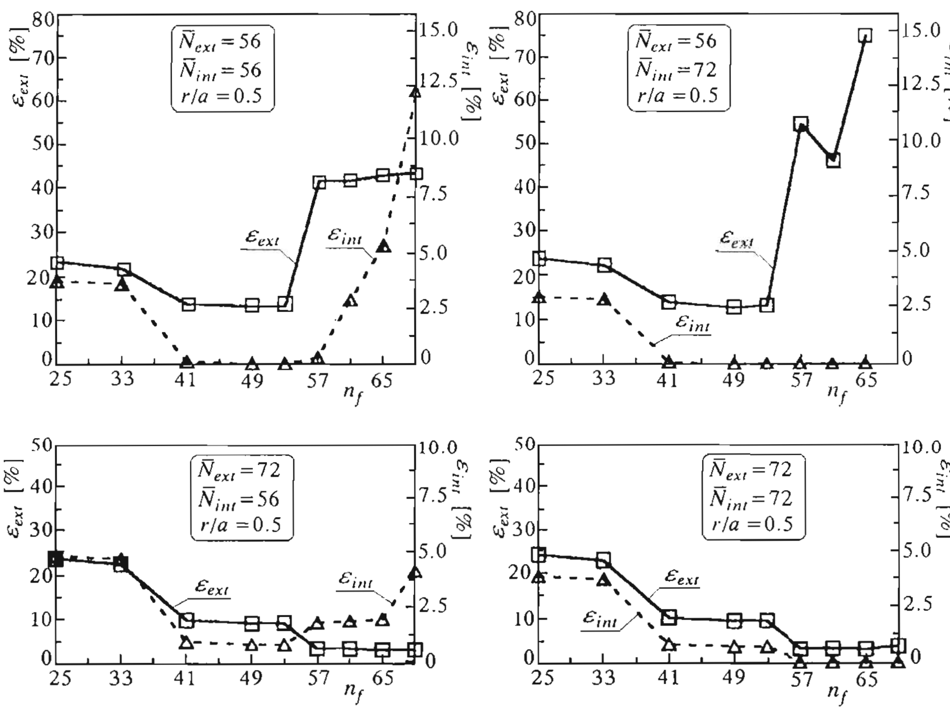

Fig. 5. Square plate $(a = b)$ with a hole; investigation of the solution stability

Generally, the external error ε_{ext} ocurred to be bigger then internal error ε_{int} and the extreme error values were situated near the corners of the investigated plate (which was predictable for the Trefftz-type solution). However, the stress concentration near the internal hole determined the maximal effort of the structure and this region was more important in the optimization procedure.

4. Optimization of structural elements using the global Trefftz approach – assumptions

Optimization of the cross-section of a helical spring was interesting not only as a sample test of the procedure for a simply-connected region but also as a solution of a specific engineering problem. Higly loaded springs in small spaces (e.g. in car making industry) require optimal shapes. This was investigated in a series of Japanese papers (Nagaya, 1985), also with the help of the Boundary Element Method (Kamiya and Kita, 1990a,b). The Trefftz approach ocurred to be very convenient in this case.

The circular cross-section boundary described as

$$r = R_0 + \rho_0 \cos\varphi \qquad\qquad z = \rho_0 \sin\varphi \tag{4.1}$$

varied to

$$r = R + \rho_\Gamma(\varphi)\cos\varphi \qquad\qquad z = \rho_\Gamma \sin\varphi \tag{4.2}$$

where

$$\rho_\Gamma(\varphi) = \rho_0 \sum_{k=0}^{K} C_k \cos(k\varphi) \tag{4.3}$$

with $C_0, C_1, ..., C_K, R$ as optimization (shape) variables. Introducing the condition of the constant volume (constant cross-section area and average radius of the spring) we obtained the two constraint equations for the constants C_0 and R (Karaś and Zieliński, 1998b)

$$\begin{aligned} &C_0^2 + \frac{1}{2}\sum_{k=0}^{K} C_k^2 = 1 \\ &R = R_0 - \frac{\rho_0}{3\pi} \sum_{k,l,m=0}^{K} C_k C_l C_m A_{klm} \end{aligned} \tag{4.4}$$

where

$$A_{klm} = \int\limits_0^{2\pi} \cos(k\varphi)\cos(l\varphi)\cos(m\varphi)\cos\varphi \, d\varphi$$

A different parametrization was proposed in the investigation of the plate with a hole (Fig.4). We assumed the possibility of the additional ellipse-type modification

$$x = a\rho(\varphi)\cos\varphi \qquad\qquad y = b\rho(\varphi)\sin\varphi \tag{4.5}$$

with

$$\rho(\varphi) = \rho_0 \sum_{k=0}^{K} C_k \cos(k\varphi) \tag{4.6}$$

The constant volume was here ensured by

$$ab = 1 \qquad \text{and} \qquad C_0^2 + \frac{1}{2}\sum_{k=0}^{K} C_k^2 = 1 \tag{4.7}$$

which eliminated b and C_0 from the set of the independent optimization variables.

In both above examples the main objective function was defined as the maximal equivalent stress σ_0 of the structure ($\sigma_{0\,\max}$), which was minimized for varying shapes (contures). In the case of spring the uniform contour effort ($\Delta\sigma_0 = \sigma_{0\,\max} - \sigma_{0\,\min} \to \min$) was also taken into account. The simple gradient procedure appeared to be sufficient in evaluation of the minimum of the objective functions.

5. Numerical examples of the optimization algorithms

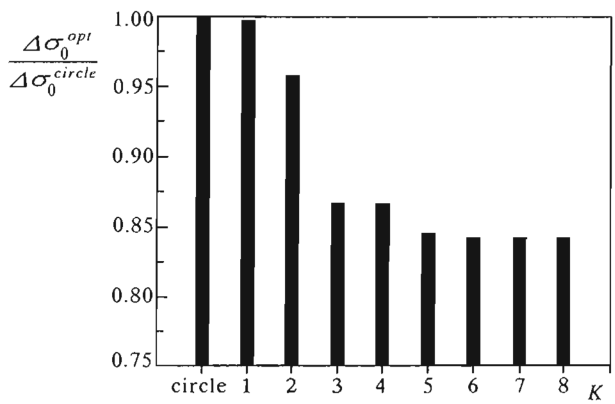

Fig. 6. Calculated minimum value of the objective function related to $\Delta\sigma_0^{circle}$ (for circular cross-section); example of spring. The influence of number of optimization parameteres K

In the numerical example of optimization of the spring cross-section 10 even Trefftz functions and 20 collocation points were applied. The modification of the original circular form included up to 8 trigonometric terms. Starting from C_3 we already obtained considerable improvement of the spring effort (see Fig.6). This stands in contrast with the number of 52 shape optimization parameters used in the BEM algorithm (Kamiya and Kita, 1990b). The optimized shape and the distribution of stresses along the boundary (for $\sigma_{0\,\max} \to \min$) are presented in Fig.7. The minimization of $(\Delta\sigma_0$ led to different shape and stress distribution in this case (Karaś and Zieliński, 1998b).

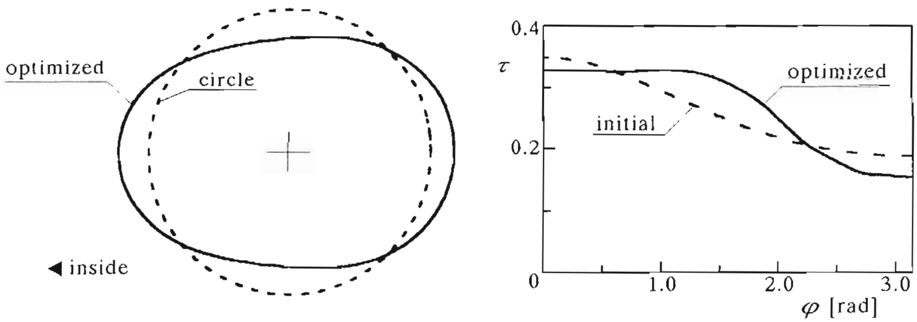

Fig. 7. Optimal cross-section of helical spring and distribution of boundary stresses (for minimization of $\sigma_{0\,\max}$)

Fig.8 presents the results for a square plate with a hole. Also in this case the three optimization variables $a, C_1 C_2$ were sufficient to describe the modifications of the opening. The results proved the possibilities of efficient application of the special Trefftz-type finite element with an hole (Zieliński, 1997) to the structural optimization.

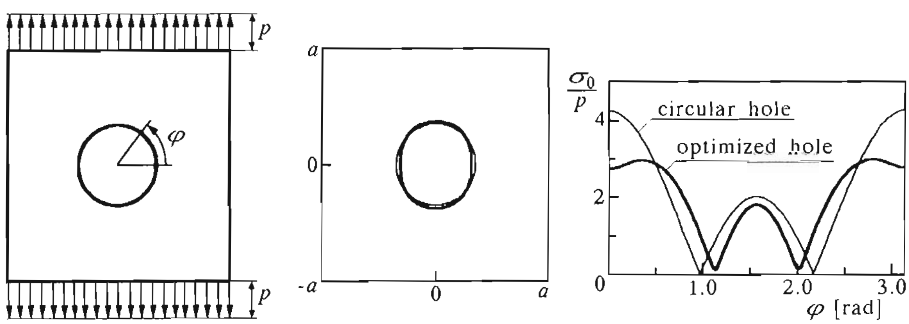

Fig. 8. Optimal shape of a hole and improved stresses along its boundary ($a/r = 0.35$)

6. Final remarks

The Trefftz method appeared to be a very convenient tool for the structural optimization. It can be used both in the shape modification and in the optimization of different structural parameters (e.g. positions of holes in Zieliński and Sanecki (1998)). However, certain rules of its application should be observed, which was investigated in the present paper. The global formulation can be applied to relatively simple structures, however, its investigation is very useful before using the finite T-element approach.

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Parametryczna optymalizacja kształtu konstrukcji z użyciem globalnej metody Trefftza

Streszczenie

Globalną metodę Trefftza zastosowano do optymalizacji pewnych typów elementów konstrukcji. Przykłady numeryczne potwierdzają zalety zastosowanej metody. Proponowany algorytm może być rozszerzony do optymalizacji z użyciem elementów skończonych typu Trefftza.

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