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WHEN CAN WE AVOID PARADOXES IN THE CONTACT PROBLEMS OF TWO THERMOELASTIC CYLINDERS

ROMAN D. KULCHYTSKY-ZHYHAILO

Faculty of Mechanical Engineering, Białystok University of Technology

ZBIGNIEW S. OLESIAK

Department of Applied Mathematics and Mechanics, University of Warsaw
e-mail: olesiak@mimuw.edu.pl

We discuss the problem of a contact of two parallel elastic cylinders heated to different temperatures. The purpose of our investigation is to derive the conditions which have to be satisfied so that the solution to the thermoelastic problem is physically meaningfull (i.e. the paradox of "the cooled cylinder"can be avoided). Pertinent formulae have been derived for relations between the contact pressure, geometrical characteristics of the solids and distributions of heat flux over the contacting region. The paper has been illustrated by an example and diagrams.

Key words: thermal stresses, contact of elastic solids, paradox of cooled sphere

1. Introduction

The contact problems of elastic solids in the field of temperature have been atracting many researchers due to the technological importance and theoretical interest. In the paper by George and Sneddon (1963), presumably the first theoretical paper dealing with the contact problems with the heat fluxes taken into account, the authors derived the solutions for axisymmetric shapes of the heated, rigid punches. The boundary value problem satisfied all mathematical requirements of the classical thermoelasticity, and consequently one might think that the method of solution was capable of solving problems for "any" sufficiently smooth surfaces of rigid punches. Many papers have been written in the field of contact problems of thermoelasticity here we mention the monographs: Shlykov and Galin (1963), Shlykov et al. (1977), Nowacki and Olesiak (1991), and original papers: Barber (1973), (1978); Barber and Comninou (1989); Borodachev (1962); Comninou and Barber (1984); Comninou and Dundurs (1979); Generalov et al. (1976); George and Sneddon (1963); Gladwell et al. (1983); Kulchytsky-Zhyhailo et al. (1999). We cite the theoretical papers of arising in the mixed boundary value problems of thermoelasticity. Here we do not discuss the tribological approach, problems of friction and dynamical contacts.

Barber (1978) was the first who pointed out that the classical solution, though mathematically well posed, did not lead to a physically sound solution in the case when the flux of heat was taken in the opposite sense along the normal direction, i.e. when a rigid, in particular case spherical, punch was cooled down. The problem belongs to one of the paradoxes in the mixed boundary value problems of the theory of elasticity (more precisely thermoelasticity). Barber called it "the paradox of the cooled sphere". The original Barber's paper referred to a rigid solid of revolution (a sphere) indented into an elastic semi-space. Since then a number of problems have been discussed taking into account the elastic properties of both the solids in contact. In order to avoid the paradox new, non-classical, models of contact have been devised. To find the limits of the validity of the classical model belongs to important questions considered.

In paper by Kulchytsky et al. (1999) we discussed the case of the axial symmetry while in this paper the case of plane strains. We discuss the stationary problems only. In the case of nonstationary problems it is difficult to find the corresponding relations analytically. Consequently a more general discussions is awkward.

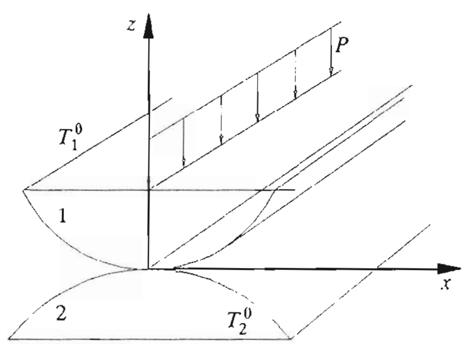

Fig. 1. Two elastic cylinders in contact

The paradox of the cooled sphere, in this paper becomes the paradox of the cooled cylinder, if we we think of an elastic cylinder pressed into an elastic semi-space. Here however we discuss a more complicated case of two parallel elastic cylinders in contact induced by pressure in the presence of heat fluxes. The paradox means that we obtain from the theoretical solution the regions of tensile tractions, and/or the zones of overlapping materials. Both contradicting the physical meaning of the solution. Barber introduced (compare Barber and Comninou, 1989) the condition in the form of "non-tensile inequality" which had to be fulfilled in order to obtain only non-tensile tractions in the contact region. It depends on the direction (more precisely on its sense) of the heat flux flowing into the body of larger distortivity. The conclusion was that in such cases the model of the classical thermoelasticity had to be modified. It could be done in a number of ways. One can assume that the flux of heat depends on the contact pressure, or that the region of the thermal boundary conditions does not coincide with that of the mechanical boundary conditions. Likewise the assumption that the mechanical and/or thermal coefficients depend on the temperature can change the distribution of contact tractions and the theoretically possible regions of tensile surface forces. All these problems are discussed under the assumption that the contact between the solids is frictionless.

Comninou and Barber (1984) and Comninou and Dundurs (1979) found that if heat flew into the solid body with the smaller distortivity no direct transition from the perfect contact to separation was possible, for the solutions to have a physical meaning, and that an intervening zone of imperfect contact had to exist. In the paper Comninou and Barber (1984) considered the thermoelastic Hertzian problem of two elastic cylinders and parabolic profiles, assuming that a contact resistance was inversely proportional to pressure. The Hertzian contact problem with the heat flow in the case of axial symmetry was considered by Barber and Comninou (1989).

In this paper we have analysed:

- The problem of the loss of contact between the two solids over a central or an outer part of the region in the cases with known contact regions, for a continuous, positively (or negatively) determined functions, characterising the distribution of the heat flux over the contact region.

- The existence problem of physically meaningfull solution for the case with an unknown beforehand contact region. The problem is discussed for continuously differentiable heat flux functions (of constant sign) the absolute values of which increase monotonically. The conditions have been analysed for the cases when the effect of "cooled cylinder" can be avoided.

2. Basic equations

A contact problem is considered between two elastic cylinders kept at different temperatures T_1^0, and T_2^0, respectively. The two cylinders are pressed against each other by a constant force P which is the equipollent to the tractions over the surface of contact. If it is assumed that the cylinderical solids are much longer than the radii of curvatures then the problem can be treated within the framework of the hypotheses of plane strain. It is also assumed that the half width of the contact distance is much smaller than the corresponding radius of the cylinder curvature, and in turn, that it can be modelled by an elastic semi-space.

In the cartesian coordinate system x, z the considered problem can be reduced to the following partial differential equations of thermoelasticity

$$\begin{aligned}
&2(1-\nu_k)u_{k,xx} + (1-2\nu_k)u_{k,zz} + w_{k,xz} = 2(1+\nu_k)\alpha_k T_{k,x} \\
&(1-2\nu_k)w_{k,xx} + 2(1-\nu_k)w_{k,zz} + u_{k,xz} = 2(1+\nu_k)\alpha_k T_{k,z} \\
&T_{k,xx} + T_{k,zz} = 0 \qquad k \equiv 1,2
\end{aligned} \tag{2.1}$$

where

ν_k	–	Poisson ratios
α_k	–	coefficients of the linear thermal expansion of the kth solid
u_k, w_k	–	components of the displacement vectors
T_k	–	temperatures of the solids.

Applying, to Eqs (2.1), the Fourier integral transforms

$$\widetilde{u}_k(\xi,z),\ \widetilde{w}_k(\xi,z),\ \widetilde{T}_k(\xi,z) = \mathcal{F}[u_k(x,z), w_k(x,z), T_k(x,z);\ x \to \xi] \equiv$$

$$\equiv \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} [u_k(x,z), w_k(x,z), T_k(x,z) - T_k^0] \exp(\mathrm{i}\xi x)\, dx$$

we obtain the transforms of the components of displacement vectors, stress tensors, and temperatures in the following form

$$
\begin{aligned}
\widetilde{w}_k(\xi,z) &= \eta_k\Big\{[-A_k(\xi)+(3-4\nu_k-\eta_k|\xi|z)B_k(\xi)]+\\
&\quad +(1+\nu_k)\alpha_k|\xi|^{-1}C_k(\xi)\Big\}\exp(\eta_k|\xi|z)\\
\widetilde{T}_k(\xi,z) &= C_k(\xi)\exp(\eta_k|\xi|z)\\
\widetilde{\sigma}_{zz}^{(k)}(\xi,z) &= 2\mu_k[-A_k(\xi)+(2-2\nu_k-\eta_k|\xi|zB_k(\xi)]|\xi|\exp(\eta_k|\xi|z)\\
\widetilde{\sigma}_{xz}^{(k)}(\xi,z) &= 2\mu_k\eta_k[A_k(\xi)-(1-2\nu_k-\eta_k|\xi|zB_k(\xi)]\xi\mathrm{i}\exp(\eta_k|\xi|z)
\end{aligned}
\tag{2.2}
$$

where $\eta_k \equiv (-1)^k$, $k=1,2$, μ_k – shear moduli.

The unknown functions $A_k(\xi)$, $B_k(\xi)$, $C_k(\xi)$ can be determined from the compatibility equation and the boundary conditions (Olesiak et al., 1995)

$$
\begin{aligned}
&\sigma_{zz}^{(k)}(x,0) = \begin{cases} -p(x) & \text{for } |x| \leqslant a\\ 0 & \text{for } |x| > a\end{cases}\\
&\sigma_{xz}^{(k)}(x,0) = 0 \qquad \text{for } |x| \geqslant 0 \quad k=1,2
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x}(w_2-w_1) = g'(x) && \text{for } |x| \leqslant a \quad z=0\\
&\mathcal{F}_k(T_j,q_j) = 0 && k,j=1,2 \quad \text{for } z=0\\
&\int\limits_{-a}^{a} p(x)\,dx = P
\end{aligned}
\tag{2.4}
$$

here

q_k – heat flux flowing into the kth cylinder through the surface $z=0$

$p(x)$ – contact pressure (on the unit length)

$g_k(x)$ – convex curve describing the form of the kth cylinder.

Without the loss of generality we can assume that $g_1(x)+g_2(x)=g(x)$. The convexity of the curve at the point $x=0$ means that $g'(0)=0$, and $g''(x) \geqslant 0$ in its vicinity. Eqs $(2.4)_2$ afford the conditions of the thermal contact between the solids and environment. From the condition of the balance of energy we obtain

$$
q_1+q_2=0 \qquad \text{for } |x| \leqslant b \quad z=0 \quad b \geqslant a
\tag{2.5}
$$

It is assumed that the region of the thermal contact can be larger than that of the mechanical one. After some algebra we obtain from the solution to the boundary value problem (compare the Appendix)

$$p(x) = \begin{cases} \dfrac{\phi_0}{\sqrt{a^2-x^2}} + \gamma^{-1}p_g(x) - \beta^* p_q(x) & \text{for} \quad x \geqslant 0 \\ p(-x) & \text{for} \quad x < 0 \end{cases} \tag{2.6}$$

where

$$\begin{aligned} p_g(x) &= \frac{2}{\pi}\int\limits_x^a \frac{y\,dy}{\sqrt{y^2-x^2}} \int\limits_0^y \frac{g''(t)\,dt}{\sqrt{y^2-t^2}} \\ p_q(x) &= \frac{2}{\pi}\int\limits_x^a \frac{y\,dy}{\sqrt{y^2-x^2}} \int\limits_0^y \frac{q(t)\,dt}{\sqrt{y^2-t^2}} \\ \beta^* &= \frac{1}{\gamma}\Big[\frac{(1+\nu_1)\alpha_1}{\lambda_1} - \frac{(1+\nu_2)\alpha_2}{\lambda_2}\Big] \\ \gamma &= \frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2} \qquad\qquad q(x) \equiv q_2(x) = -q_1(x) \end{aligned} \tag{2.7}$$

The unknown constants ϕ_0 or a can be determined from the equilibrium condition $(2.4)_3$ which together with conditions (2.6) and (2.7) can be reduced to the following form

$$P = \pi\phi_0 + \gamma^{-1}P_g - \beta^* P_q \tag{2.8}$$

where

$$P_g = 2\int\limits_0^a g''(x)\sqrt{a^2-x^2}\,dx \qquad\qquad P_q = 2\int\limits_0^a q(x)\sqrt{a^2-x^2}\,dx \tag{2.9}$$

It is evident from Eqs (2.6)÷(2.9) that the distribution of the contact tractions is determined by the compressive force, geometry of the contacting surfaces, and heat flux through the contact region. Let us note that $q(x) > 0$ for $T_1^0 > T_2^0$, and $q(x) < 0$ for $T_1^0 < T_2^0$. Since the character of thermal effects depends on $\beta^* q(x)$ where the parameter $\beta^* \in (-\infty, +\infty)$ it is sufficient to discuss one of the cases, say, when $T_1^0 > T_2^0$.

Taking into account the definition of heat fluxes $q_i = \eta_i\lambda_i\partial_z T_i$ we obtain that the heat flux has to be directly proportional to $\Delta T(\lambda_1^{-1}+\lambda_2^{-1})^{-1}b^{-1}$ and can be represented in the form

$$q(x) = \Delta T\lambda b^{-1}q_0\Big(\frac{x}{b}\Big) \qquad\qquad |x| < b \tag{2.10}$$

where $\lambda = (\lambda_1^{-1} + \lambda_2^{-1})^{-1}$, while $q_0(x/b)$ denotes a certain dimensionless function responsible for the character of the heat flux distribution between the solids and outer environment $(2.4)_2$. By substitution of (2.10) into (2.6) and (2.8), respectively, we obtain

$$p(x) = \frac{\phi_0}{\sqrt{a^2 - x^2}} + \gamma^{-1} p_g(x) - \beta^* \Delta T \lambda b^{-1} p_{q_0}(x) \tag{2.11}$$

$$P = \pi\phi_0 + \gamma^{-1} P_g - \beta^* \Delta T \lambda b^{-1} P_{q_0} \tag{2.12}$$

3. Case of a fixed contact region

First we shall discuss the case of a fixed region of contact. Then the regions of the mechanical and thermal contact coincide.

It results from Eq (2.12) that the case of a fixed contact region makes sense when the obvious condition

$$\gamma^{-1} P_g - \beta^* \Delta T \lambda a^{-1} P_{q_0} < P \tag{3.1}$$

holds true. In such a case the distribution of contact tractions has a one over square root singularity at the boundary of the contact region. If condition (3.1) does not hold one has to assume that the value of constant ϕ_0, appearing in Eqs (2.6) and (2.11) vanishes, and has to solve the problem of contact with an unknown beforehand region of contact.

In the case of convex surfaces and any positive function q_0 we obtain that the quantities P_g and P_{q_0} are likewise positive. Consequently, we obtain that for the negative β^* there exists such a critical value

$$\Delta T_{cr} = \frac{(P_g - \gamma P)a}{\beta^* \lambda \gamma P_{q_0}}$$

for which (for $\Delta T > \Delta T_{cr}$) inequality (3.1) is no longer true. This means that in the neighbourhood of the contact region boundary the contact between solids is lost. The maximum value of ΔT_{cr} is obtained in the case when the bottom of punch is flat.

On the other hand, for $\beta^* > 0$ the distribution of contact stresses has a one over square root singularity at the end of the contact region provided:

$$\gamma^{-1} P_g < P \tag{3.2}$$

Kulchytsky-Zhyhailo et al. (1999) discussed the possibility of the loss of contact over the central region of contact. Now, after finding from Eq (2.12) the value of constant ϕ_0 and substituting for it into Eq (2.11), taking into account Eqs (2.7) and (2.9) we find that

$$p(0) = \frac{P}{\pi a} + \frac{2}{\pi\gamma}\int\limits_0^a g''(t)G(t)\;dt - \frac{2}{\pi a}\beta^* \Delta T\lambda \int\limits_0^a q_0(t)G(t)\;dt \tag{3.3}$$

where

$$G(t) = \ln\frac{a+\sqrt{a^2-t^2}}{t} - \frac{\sqrt{a^2-t^2}}{a}$$

It can be shown that $G(t) \geqslant 0$ for $t \in [0,a]$, consequently for any positive function $q(x)$ the integral on the right hand side of Eq (3.3) is also positive, moreover there exists the critical value

$$\Delta T_{cr} = \frac{P + 2a\gamma^{-1}\int\limits_0^a g''(t)G(t)\;dt}{2\beta^*\lambda\int\limits_0^a q_0(t)G(t)\;dt}$$

such that for $\Delta T > \Delta T_{cr}$ we obtain a region of negative tractions in a neighbourhood of $x = 0$, i.e. there exists a region of the lack of contact in the central part. The smallest value of ΔT_{cr} is obtained for the flat bottom punch.

4. Case of solids with smooth contacting surfaces

In the case when the region of contact is not known beforehand we have to assume that ϕ_0 entering Eqs (2.11) and (2.12) vanishes, and we have to find the width of the contact region from Eq (2.12) while the distribution of the contact tractions from Eq (2.11). It results from Eqs (2.9) and (2.12) that for $\beta^* < 0$ the contact region width is smaller as compared with that for the corresponding isothermal problem, on the other hand, for $\beta^* > 0$ it is bigger. From the papers, devoted to the problems of thermal contact, it is known that for $\beta^* > 0$ we can obtain the diagram of contacting tractions with varying sign i.e. the problem does not have a physical meaning. We shall call it the paradox of the "cooled sphere" (after Barber, 1978), here rather the paradox of the "cooled cylinder". We shall analyse when such a problem appeares, the effect of the heat flux and the distribution of contact pressure for $x \to a$. Taking into account the theorem on the mean value of the integral in formulae Eqs (2.7) we find that

$$p_g = \widehat{p}_g(\tau_1)\sqrt{a^2 - x^2} \qquad\qquad p_{q0} = \widehat{p}_{q0}(\tau_2)\sqrt{a^2 - x^2} \tag{4.1}$$

where

$$\widehat{p}_g(\tau_1) = \frac{2}{\pi}\int\limits_0^{\tau_1} \frac{g''(t)dt}{\sqrt{\tau_1^2 - t^2}} \qquad\qquad \widehat{p}_{q0}(\tau_2) = \frac{2}{\pi}\int\limits_0^{\tau_2} \frac{q_0\left(\frac{t}{b}\right)dt}{\sqrt{\tau_2^2 - t^2}} \tag{4.2}$$

where τ_1, τ_2 denote certain unknown numbers from the interval $[x, a]$. Tending to the limit $x \to a$ we obtain

$$\lim_{x\to a} \frac{p_g(x)}{\sqrt{a^2 - x^2}} = \widehat{p}_g(a) \qquad\qquad \lim_{x\to a} \frac{p_{q0}(x)}{\sqrt{a^2 - x^2}} = \widehat{p}_{q0}(a) \tag{4.3}$$

From Eqs (2.11) and (4.3) it is evident that for $x \to a$ there exists a positive pressure provided

$$\gamma^{-1}\widehat{p}_g(a) > \beta^* \Delta T \lambda b^{-1}\widehat{p}_{q0}(a) \tag{4.4}$$

Otherwise, when condition (4.4) is not satisfied, we obtain, in the vicinity of the contact contour, a region of tensile tractions, i.e. the effect of "cooled cylinder".

In the case when for $x \to a$ the distribution of the heat flux $p_{q0}(x)$ tends to infinity as $1/\sqrt{a^2 - x^2}$ and $\widehat{p}_{q0}(x \to a)$ tends also to infinity, however this time logarithmically. Then the paradox takes place for arbitrary mechanical and thermal parameters entering the problem. Consequently, we can avoid the paradox of "cooled cylinder" if the prescribed thermal contact $(2.4)_2$ is such that the function $q_0(x)\sqrt{a^2 - x^2}$ in the region of the mechanical contact is regular and $\lim\limits_{x\to a} q_0(x)\sqrt{a^2 - x^2} = 0$.

More precise analysis how to avoid the problem of the "cooled cylinder" will be discussed for a particular case of the paraboloidal cylinder $g(x) = x^2/2R$. Then we obtain

$$\begin{aligned} P &= \frac{\pi a^2}{2R\gamma} - 2\beta^* \Delta T \lambda a \epsilon I_1(q_0(\epsilon y)) \\ \frac{p(\zeta \to 1)}{\sqrt{1 - \zeta^2}} &= \frac{a}{R\gamma} - \frac{2}{\pi}\beta^* \Delta T \lambda \epsilon I_2(q_0(\zeta y)) \end{aligned} \tag{4.5}$$

where

$$I_1(q_0(\epsilon y)) = \int\limits_0^1 q_0(\epsilon y)\sqrt{1-y^2}\,dy$$

$$I_2(q_0(\epsilon y)) = \int\limits_0^1 \frac{q_0(\epsilon y)\,dy}{\sqrt{1-y^2}} \qquad \epsilon \equiv \frac{a}{b} \qquad \zeta \equiv \frac{x}{a}$$

5. Conclusions

For any monotonic increasing function $q_0(\epsilon y)$, $y \in [0,1]$ it can be shown that:

(a) For coinciding regions of the mechanical and thermal contacts $(\epsilon = 1)$ there exists P_{cr} such that for $P < P_{cr}$ tensile tractions $p(\zeta)$ appear in a certain neighbourhood of $\zeta = 1$.

(b) Parameter ϵ determined from the condition $p(\zeta \to 1)/\sqrt{1-\zeta^2} = 0$ satisfies inequality $p(\zeta) > 0$ for any $\zeta \in [-1,1]$.

(c) In the case when parameter ζ is smaller than parameter ϵ determined from the condition at point (b) then $p(\zeta) > 0$ for any $|\zeta| < 1$.

It results from points (b) and (c) that there exists ϵ_{max} (correspondingly b_{min}) for which the problem has a physical meaning. To determine ϵ_{max} one has to solve the following nonlinear equation

$$\epsilon_{max} I_2(q_0(\epsilon_{max} y)) = \frac{\pi a}{2\beta^* \Delta T \lambda R \gamma} \tag{5.1}$$

It results from point (a) that at least for small values of compressive force $\epsilon_{max} < 1$. It means that at least for small values of P we have to assume that $b > a$ to obtain the physically meaningfull solution to the contact problem. If ϵ_{max}, determined from Eq (5.1) is greater than 1, it is natural to assume that $\epsilon_{max} = 1$, i.e. the mechanical and thermal regions of contact are equal.

Substituting Eq (5.1) into $(4.5)_1$ we obtain the dependance between the compressive force and the width of mechanical contact region for small values of P (ϵ has been determined from Eq (5.1) and is less than 1)

$$P = \frac{\pi a^2}{2R\gamma}\Big[1 - \frac{2I_1(q_0(\epsilon y))}{I_2(q_0(\epsilon y))}\Big] \tag{5.2}$$

The above formula (5.2) provides the relationship between the compressive force and the width of the contact region for small values of P (in the limit we obtain for $P \to 0$ that $a \to 0$). We should note that from Eq $(4.5)_1$ we obtain in the limit $P \to 0$ a finite width of the mechanical contact region. The knowledge of the width of contact let us find, from Eq (5.1) the relation between ϵ_{max}, distribution of the heat flux and characteristics of the problem. We obtain

$$\epsilon_{max} I_2(q_0(\epsilon_{max} y)) \sqrt{1 - \frac{2I_1(q_0(\epsilon_{max} y))}{I_2(q_0(\epsilon_{max} y))}} = \frac{\pi a_H}{2\beta^* \Delta T \lambda R \gamma} = \chi \tag{5.3}$$

where $a_H^2 = 2\pi^{-1} PR\gamma$ denotes the square of the width of the contact region in the corresponding isothermal Hertz's problem.

From the regularity of distribution of the heat flux we obtain that for $\epsilon \in [0,1]$ the left hand side of Eq (5.3) is finite, therefore there exists a critical value χ_{cr} such that for $\chi > \chi_{cr}$ the roots of Eq (5.3) do not belong to the segment $\epsilon_{max} \in [0,1]$. This means that in the interval $[\chi_{cr}, \infty)$ the solution in the case of a regular monotonic increasing function $q_0(\epsilon y)$ has a physical meaning already for $a = b$. Thus, the regions of the mechanical and thermal contacts coincide.

In this way, for $\chi \in [0, \chi_{cr}]$ we find from Eq (5.3) the parameter ϵ, from Eq (5.2) the region of the mechanical contact, and from the relation $b\epsilon = a$ the width of the thermal contact region. For $\chi \in [\chi_{cr}, +\infty]$ the physical solution takes place for $a = b$.

We introduce a dimensionless quantity of the mechanical contact region $a_0 = a/a_H$. For $\chi \in [0, \chi_{cr}]$ we obtain from Eq (5.2) the following relation

$$a_0 = \sqrt{1 - \frac{2I_1(q_0(\epsilon_{max} y))}{I_2(q_0(\epsilon_{max} y))}} \tag{5.4}$$

where ϵ_{max} has been determined from Eq (5.3).

For $\chi \in [\chi_{cr}, \infty)$ Eq $(4.5)_1$ reduces to the form

$$a_0^2 - 2\chi^{-1} a_0 I_1(q_0(y)) = 1 \tag{5.5}$$

Thus the parametr $\chi = \pi a_H/(2\beta^* \Delta T \lambda R \gamma)$ plays an essential role as the characteristics of the effect of temperature difference on the contact region.

6. Example

The effects appearing in the thermal contact between two convex solids and

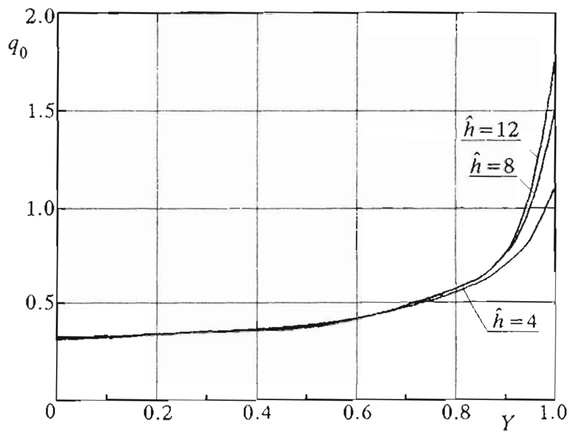

Fig. 2. Dimensionless heat flux over the contact region

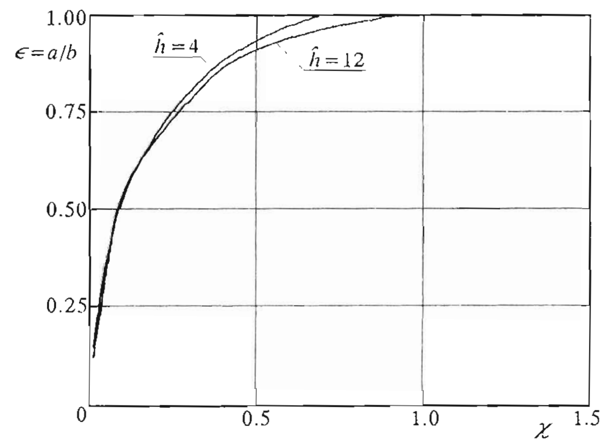

Fig. 3. The ratio of the width of the region to that of the thermal one for varying parameter χ

the formulae derived above will be discussed on an example of two cylindrical rollers.

We assume that the thermal contact is non-ideal in a thin strip, namely:

$$\begin{aligned} q_1 + q_2 &= 0 \qquad & |x| < b \quad z = 0 \\ q_1 - q_2 &= h(T_2 - T_1) \qquad & |x| < b \quad z = 0 \end{aligned} \tag{6.1}$$

Outside the contact the condition of the heat conduction between the solids and the environment obeys Newton's law

$$q_1 = \widehat{\alpha}_1(T_e^{(1)} - T_1) \qquad q_2 = \widehat{\alpha}_2(T_e^{(2)} - T_2) \qquad |x| > b \quad z = 0 \tag{6.2}$$

where $T_e^{(1)}$, $T_e^{(2)}$ denote the temperatures of the environment, $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ the coefficients of the heat exchange. We assume that the temperature of the environment varies exponentially $T_e^{(1)} = T_1^0 - \Delta T \exp[-\kappa(|x| - b)]$, for $|x| \to \infty$, $T_e^{(1)} = T_1^0$ while for $T_e^{(1)} = T_2^0$, $|x| = b$, $T_e^{(2)} \equiv T_2^0$.

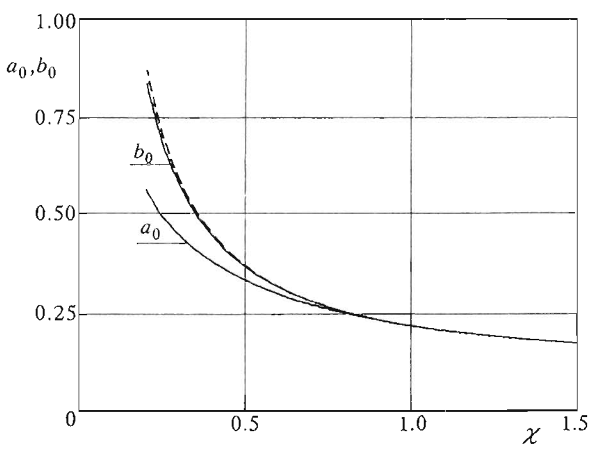

Fig. 4. The ratio of the width of the thermal and mechanical contact to that from the solution to Hertz's problem

Now we apply the exponential Fourier transforms to the problem governed by Eqs $(2.1)_3$, (6.1) and (6.2). Taking into account the symmetry of the contact problem with respect to $x = 0$ we obtain

$$\frac{T_1 - T_1^0}{\Delta T} = -t_1(f_1) - t_1^0 \qquad \frac{T_2 - T_2^0}{\Delta T} = t_2(f_2) \tag{6.3}$$

where

$$t_k(f_k) = \frac{2}{\pi}\int\limits_0^\infty \frac{\cos sY\, ds}{s+\mathrm{Bi}^{(k)}}\int\limits_0^1 f_k(y)\cos sy\, dy \qquad k=1,2$$

$$t_1^0 = \frac{2}{\pi}\mathrm{Bi}^{(1)}\int\limits_0^\infty \frac{\kappa_0\cos s - s\sin s}{k_0^2+s^2}\,\frac{\cos sY}{s+\mathrm{Bi}^{(1)}}\, ds$$

$$f_1(Y) = \Delta T^{-1}\Big[\frac{\partial T_1}{\partial\eta} - \mathrm{Bi}^{(1)}(T_1 - T_1^0)\Big]$$

$$f_2(Y) = \Delta T^{-1}\Big[\frac{\partial T_2}{\partial\eta} - \mathrm{Bi}^{(2)}(T_2 - T_2^0)\Big] \qquad |Y|<1$$

and $Y = x/b$, $\eta = z/b$, Biot's number: $\mathrm{Bi}^{(k)} = \widehat{\alpha}_k b/\lambda_k$, $k=1,2$, $\kappa_0 = \kappa b$.

The problem can be reduced to the following system of integral equations for the unknown functions $f_1(Y)$, $f_2(Y)$

$$\begin{aligned} &2\psi f_1(Y) + (\widehat{h} - 2\psi\mathrm{Bi}^{(1)})t_1(f_1) + \widehat{h}t_2(f_2) = \widehat{h} - (\widehat{h} - 2\mathrm{Bi}^{(1)}\psi)t_1^0 \\ &2(1-\psi)f_2(Y) + [\widehat{h} - 2(1-\psi)\mathrm{Bi}^{(2)}]t_2(f_2) + \widehat{h}t_1(f_1) = \widehat{h} - \widehat{h}t_1^0 \end{aligned} \tag{6.4}$$

where $\widehat{h} = hb/(\lambda_1+\lambda_2)$, $\psi = \lambda_1/(\lambda_1+\lambda_2)$. The solution of the system of integral equations $(6.4)_2$ can be obtained by using polynomial expansions of the required functions, reducing it to the system of infinite series, and after proving the convergence of the process, to truncating the system of algebraic equations. We assume that

$$f_i(Y) = \sum_{j=0}^{\infty} a_j^{(i)} Y^{2j} \tag{6.5}$$

Similarly the integrals entering the formulae for $t_i(f_i)$, $i=1,2$ and t_1^0 have been approximated by the series of polynomials with the use of the method of least squares

$$t_i(f_i) = \sum_{j=0}^{m} a_j^{(i)} \sum_{k=0}^{m} t_{kj}^{(i)} Y^{2k} \qquad\qquad t_1^0 = \sum_{j=0}^{m} t_j^0 Y^{2j}$$

From the solution of system of algebraic equations we obtain the values of the coefficients $a_j^{(i)}$, $j=0,...,m$; $i=1,2$, and eventually the required functions. In order to analyse the stress-strain state it is sufficient to compute the function

$$q_0 = \frac{bq_2(Y)}{\Delta T\lambda} = \psi^{-1}[f_2(Y) - \mathrm{Bi}^{(2)}t_2(f_2)] = \sum_{j=0}^{m} q_0^{(j)} Y^{2j}$$

The numerical analysis can be simplified if we assume that $\widehat{\alpha}_1 = \widehat{\alpha}_2 = \widehat{\alpha}$; $\mathrm{Bi}^{(1)} = \mathrm{Bi}/\psi$; $\mathrm{Bi}^{(2)} = \mathrm{Bi}/(1-\psi)$; $\mathrm{Bi} = \widehat{\alpha}b/(\lambda_1+\lambda_2) = 0.1$; $\kappa_0 = 1$.

A. Appendix

From the boundary conditions (2.3) we find

$$A_i(\xi) = B_i(\xi)(1-2\nu_i) \qquad\qquad B(\xi) = 2\mu_1 B_1(\xi) = 2\mu_2 B_2(\xi) \tag{A.1}$$

From the last equation of Eqs $(2.3)_1$ and condition $(2.4)_1$ we obtain the system of dual integral equations for parameter $B(\xi)$

$$\begin{aligned} &\gamma \int_0^\infty B(\xi)\xi \sin x\xi \; d\xi + \sum_{i=1}^{2} \alpha_i(1+\nu_i) \int_0^\infty C_i(\xi) \sin x\xi \; d\xi = \\ &= -\sqrt{\frac{\pi}{2}} g'(x) \qquad\qquad x \in [0,a) \\ &\int_0^\infty B(\xi)\xi \cos x\xi \; d\xi = 0 \qquad\qquad x \in (a,\infty) \end{aligned} \tag{A.2}$$

where $\gamma = (1-\nu_1)/\mu_1 + (1-\nu_2)/\mu_2$. Unknown functions $C_i(\xi)$ can be found from conditions $(2.4)_2$. Solving dual integral equations (A.2) we took into account the evenness of the functions $B(\xi)$ and $C_i(\xi)$. The solution is presented in the form

$$\xi B(\xi) = \sqrt{\frac{\pi}{2}} \int_0^a \phi(t) J_0(\xi t) \; dt - \sqrt{\frac{\pi}{2}} \phi_0 J_0(\xi a) \tag{A.3}$$

We note that the second term in Eq (A.3) constitutes the solution of the corresponding homogeneous system of the dual integral equations. The first term satisfies identically the second of Eqs (A.2), while from the first of Eq (A.2) we obtain

$$\gamma \int_0^x \frac{\phi(t)\; dt}{\sqrt{x^2-t^2}} = -g'(x) - \sqrt{\frac{2}{\pi}} \sum_{i=1}^{2} \alpha_i(1+\nu_i) \int_0^\infty C_i(\xi) \sin x\xi \; d\xi \tag{A.4}$$

The distribution of the contact pressure takes the following form

$$p(x) = -\int_x^a \frac{\phi(t)\,dt}{\sqrt{t^2-x^2}} + \frac{\phi_0}{\sqrt{a^2-x^2}} \qquad x \geqslant 0 \quad p(-x) = p(x) \tag{A.5}$$

By applying the inverse Abel transform of the first kind to Eq (A.4) and taking into account the cosine transforms

$$q_i(x) = \lambda_i \int_0^\infty C_i(\xi)\xi \cos x\xi \; d\xi$$

we find

$$\phi(y) = -\frac{2y}{\pi\gamma} \int_0^y \frac{g''(x)\,dx}{\sqrt{y^2-x^2}} + \frac{2y\beta^*}{\pi} \int_0^y \frac{q(x)\,dx}{\sqrt{y^2-x^2}} \tag{A.6}$$

where

$$\beta^* = \frac{1}{\gamma}\Big[\frac{(1+\nu_1)\alpha_1}{\lambda_1} - \frac{(1+\nu_2)\alpha_2}{\lambda_2}\Big]$$

$$q(x) \equiv q_2(x) = -q_1(x) \qquad |x| < a$$

The substitution of Eq (A.6) into Eq (A.5) provides us with formulae (2.6) and (2.7).

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Kiedy można uniknąć paradoksów w zagadnieniu termosprężystego kontaktu dwóch walców

Streszczenie

Rozpatrujemy zagadnienie kontaktu ściskanych sprężystych walców o równoległych tworzących w polu temperatury. Głównym celem pracy jest wyprowadzenie odpowiednich warunków gwarantujących fizyczny sens rozwiązań odpowiednich zagadnień termosprężystości (tzn. uniknięcie paradoksu "chłodnego walca"). W tym celu wyprowadzone zostały wzory na związki łączące ciśnienie na powierzchni kontaktu, geometryczne charakterystyki cylindrów i rozkład strumienia ciepła poprzez powierzchnię kontaktu. Pracę zilustrowano przykładem liczbowym i wynikającymi stąd wykresami.

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