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THIN-WALLED BEAM SUBJECTED TO "WARPING CONSTRAINTS"

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This paper is principally aimed to present an analysis of constraints applied to a thin-walled beam. A particular type of constraints; i.e., the constraints of warping are considered in detail. A theorem on the constraints that make warping impossible has been proved.

Key words: thin-walled beam, constraints, warping

1. Introduction

Unlike a solid bar, a thin-walled one in the static analysis of a frame structure cannot always be represented supplied with by the beam axis only the geometric and sectorial characteristics of the cross-section. It means that the problem cannot be considered as a unidimentional one.

There are only two cases when we are allowed to reduce a thin-walled frame analysis to a one-dimentional task, namely: $\alpha' = 0$ or $\alpha' = \text{const}$ for the extreme cross-sections of all beams that meet at the same node, where $\alpha(x)$ is the angle of rotation about the beam axis.

The aim of this paper is to discuss the first case. If the condition $\alpha' = 0$ is fulfilled, then, in static analysis of a frame, one can use a stiffness matrix similar to that used in the case of frame made of solid bars.

The basic goal of this paper consists in analysing the constraints that are to impose the required condition.

2. Displacements of the middle line of cross-section

According to the basic assumption of the thin-walled bar theory formulated by Vlasov (1961), (1962), a cross-section undergoes the same deformations that are observed in the case of spread diaphragm, fixed against displacements along the y and z axes, but ideally flexible for displacements along the axis x.

On the basis of the above assumption, in the case of fixed cross-section $x = x_0$, the functions of displacements of points belonging to the section middle line defined in the global system x, y, z by the parametric equations $x = x_0$, $y = y(s)$, $z = z(s)$ are determined by the following well-known functions of s (the natural parameter of a middle line) (cf Piechnik, 1999) in the local system x, s, n

$$\begin{aligned} u_x(x,s) &= u(x) - v'(x)y(s) - w'(x)z(s) + \alpha'(x)\omega(s) \\ u_s(x,s) &= v(x)y_{,s}(s) + w(x)z_{,s}(s) - \alpha(x)\rho_n(s) \\ u_n(x,s) &= -v(x)z_{,s}(s) + w(x)y_{,s}(s) + \alpha(x)\rho_s(s) \end{aligned} \tag{2.1}$$

It can be clearly seen that the functions of displacements are determined by seven parameters for the extreme section

$$\begin{array}{lll} a = u(x_0) & b = -v'(x_0) & c = -w'(x_0) \\ d = \alpha'(x_0) & e = v(x_0) & f = w(x_0) \\ g = \alpha'(x_0) & & \end{array} \tag{2.2}$$

The functions $v(x)$, $w(x)$, $u(x)$, $\alpha(x)$, which depend on the applied constraints, must be known to allow one to determine these constants.

It can be proved, on the basis of the equations of *kinematics equivalence* (Piechnik, 1978) of external and internal systems of force that these functions bear the following relations to the functions of cross-sectional forces

$$\begin{array}{ll} u'(x) = \dfrac{F_x(x)}{\widetilde{E}A} & w''(x) = -\dfrac{M_y(x)}{\widetilde{E}I_y} \\[2ex] v''(x) = \dfrac{M_z(x)}{\widetilde{E}I_z} & \widetilde{E}I_\omega\alpha'''(x) - GI_s\alpha'(x) = -M_{Rx}(x) \end{array} \tag{2.3}$$

where $\widetilde{E} = E/(1-\nu^2)$.

General solutions of these equations permit one to determine the following functions with the accuracy defined by the integration constants

$u(x), v(x), w(x)$	–	translation of the points of beam axis along x, translations of bending axis along the global system axes y, z, respectively
$\alpha(x)w'(x), v'(x)$	–	rotation of the cross-section about x, y, z
$\alpha'(x)\omega(s)$	–	warping of the cross-section.

Solutions of Eqs (2.3) can be written as follows

$$
\begin{aligned}
u(x) &= -\int_0^x \frac{N(\zeta)}{\widetilde{E}F}\,d\zeta + C \\
v'(x) &= \int_0^x \frac{M_z(\zeta)}{\widetilde{E}I_z}\,d\zeta + A_1 \\
v(x) &= \int_0^x (x-\zeta)\frac{M_z(\zeta)}{\widetilde{E}I_z}\,d\zeta + A_1 x + A_2 \\
w'(x) &= -\int_0^x \frac{M_y(\zeta)}{\widetilde{E}I_y}\,d\zeta + B_1 \\
w(x) &= -\int_0^x (x-\zeta)\frac{M_y(\zeta)}{\widetilde{E}I_y}\,d\zeta + B_1 x + B_2 \\
\alpha(x) &= \alpha_s(x) + D_1 + D_2\sinh(\gamma x) + D_3\cosh(\gamma x)
\end{aligned}
\tag{2.4}
$$

where

$\alpha_s(x)$ – particular integral of non-homogeneous equation

γ – decay factor, and

$$
\gamma^2 \stackrel{def}{=} \frac{GI_s}{\widetilde{E}I_\omega}
$$

3. Constraints of warping

Generally the analysis of the imposed constraints on translational displacement $u(x)$, $v(x)$, $w(x)$, and rotations about the axes y and z of the cross-section does not present any problem. On the other hand, the analysis of the warping-restraining constraints whose equations depend on the derivative of the angle of torsion may cause a problem, due to excessively numerous ambiguities, evident errors which can be encountered in the literature, or simple avoidance of this topic in the literature on thin-walled beams. This fact deserves a special attention.

The linearity in every straight section of the middle line is the characteristic feature of warping of an arbitrary cross-section, which follows from the definition. A possible warping diagram is presented in Fig.1 as an example.

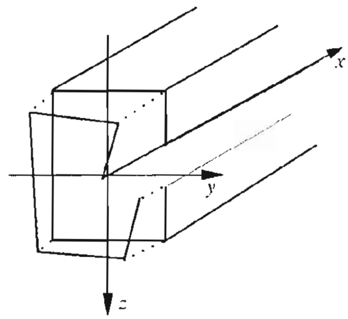

Fig. 1.

Calculation of the warping value at particular points of a cross-section does not seem to present any serious problem. Two of the constants that appear in Eq $(2.4)_6$ are calculated from the known values of bimoments on extreme cross-section areas. Once the relation between the bimoment and the second derivative of the angle of torsion (Piechnik, 1999) known

$$\alpha''(x) = \frac{B(x)}{\widetilde{E}I_\omega}$$

the values of these derivatives on the beam-ends can be easily found if we know all the forces applied to these cross-sections that yield the components collinear with the axis x

$$\alpha''(0) = \frac{B(0)}{\widetilde{E}I_\omega} \qquad\qquad \alpha''(l) = \frac{B(l)}{\widetilde{E}I_\omega}$$

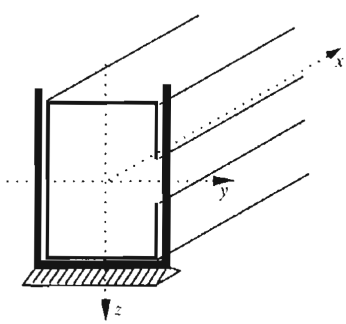

Fig. 2.

There may arise problems with formulation of the third constraint equation. The problem disappears only in the case when the constraints block the

x-rotation of one of the extreme cross-sections. An example of those constraints is presented in Fig.2.

Let us consider an arbitrary system of constraints.

In the case of statically determinate beam, the number of constraints normal to the cross-section ends cannot exceed three for obvious reasons (all constraints block out all the six degrees of freedom). In that case the values of reactions are calculated from the equations of equilibrium.

If constraint-limiting displacements in the direction of the beam axis are additionally applied, for instance, at the point D to the cross wall (see Fig.3), then the problem becomes statically indeterminate. The reaction magnitude of this constraint can be determined e.g. by the force method.

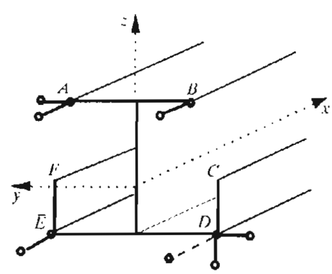

Fig. 3.

There-after we will consider the indeterminate cases due to the application of more than three constraints perpendicular to the extreme cross-section.

One can distinguish two cases of the above mentioned joints, namely, when the rigid constraints join the extreme cross-section of the thin-walled beam with:

- the rigid body or with a deformable body that contains a "rigid diaphragm" element within it that corresponds to the shape of the thin-walled beam cross-section,

- the deformable body.

The first case will be discussed further on. Among a number of elements of the set of possible to be applied constraints, we may distinguish the subset of constraints that make warping impossible. The elements of that subset will be called *warping constraints.*

Let us prove the following theorem, which is also a consequence of the Vlasov hypothesis (Vlasos, 1961).

> If at least 4 points of the cross-section at which the constraints have been applied are situated in the same plane after deformation of the bar, then, the entire cross section does not warp at all (it means that the cross-section moves like a rigid plate). The selected points, however, have to satisfy the following conditions: no three of them can lie on one line and the linear combination of their sectorial co-ordinates is not identical with the linear combination of Cartesian co-ordinates.

Adopting the following denotations we will carry out the proof of the above-presented theorem:

— At an arbitrary point of the cross-section: $A_i(x_0, y_i, z_i)$ where $y_i = y(s_i)$, $z_i = z(s_i)$, the relation $(2.1)_1$ can be formulated as

$$u_x(x_0, y_i, z_i) = a + by_i + cz_i + d\omega_i \tag{3.1}$$

— After deformation, the point A_i will be localised at the point A'_i, whose co-ordinates are

$$A'_i[x_0 + u_x(x_0, y_i, z_i), y_i + u_s(x_0, y_i, z_i), z_i + (x_0, y_i, z_i)]$$

— If the denotation $x_i = x_0 + u_x(x_0, s_i)$ is adopted, Eq (3.1) will take the following form

$$-(x_i - x_0) + by_i + cz_i + d\omega_i + a = 0 \tag{3.2}$$

— Assuming the definition of the following vectors

$$\boldsymbol{n} = [-1, b, c] \qquad\qquad \boldsymbol{r}_i = [x_i - x_0, y_i, z_i] \tag{3.3}$$

one can reformulate Eq (3.2) as

$$\boldsymbol{r}\boldsymbol{n} = -d\omega_i - a \tag{3.4}$$

Assumptions

Let us consider four points A'_i of radius-vectors $\boldsymbol{r}_i$ and four values ω_i $(i = 1, 2, 3, 4)$, for which Eq (3.4) is satisfied, and let us adopt the following assumptions:

1. All points A_i' lie on one plane $(\boldsymbol{r}-\boldsymbol{r}_0)\boldsymbol{n}=0$ and no three vectors $\boldsymbol{r}_i$ lie in one plane (it follows that no three points A_i lie on one straight line)
2. Linear combination of radius-vectors is not identical with the linear combination of sectorial co-ordinate values

$$\boldsymbol{r}_k=\sum_{i=1}^{3}\eta_i\boldsymbol{r}_i \Rightarrow \begin{cases} x_k-x_0=\sum\limits_{i=1}^{3}\eta_i(x_i-x_0) \Rightarrow x_k-x_0=\sum\limits_{i=1}^{3}\eta_i x_i - x_0\sum\limits_{i=1}^{3}\eta_i \\ y_k=\sum\limits_{i=1}^{3}\eta_i y_i \\ z_k=\sum\limits_{i=1}^{3}\eta_i z_i \end{cases}$$

$$\omega_k\neq\sum_{i=1}^{3}\eta_i\omega_i \tag{3.5}$$

<u>*Thesis*</u>

$\alpha'(x_0)=d=0.$

<u>*Proof*</u>

It follows from assumption 2. that the sum of linear combination coefficients is equal to unity

$$\eta_1+\eta_2+\eta_3=1 \tag{3.6}$$

Developing Eq (3.4) for the kth vector, one obtains the following equation

$$\boldsymbol{r}_k\boldsymbol{n}=-d\omega_k-a \tag{3.7}$$

The left-hand side of the above relation will be rewritten, considering $(3.5)_1$ and (3.6)

$$\boldsymbol{r}_k\boldsymbol{n}=\sum_{i=1}^{3}\eta_i\boldsymbol{r}_i\boldsymbol{n}=\sum_{i=1}^{3}\eta_i(-d\omega_i-a)=-\sum_{i=1}^{3}\eta_i d\omega_i-a\sum_{i=1}^{3}\eta_i=-\sum_{i=1}^{3}\eta_i d\omega_i-a \tag{3.8}$$

Substituting the result into Eq (3.7) yields the following equation

$$d\sum_{i=1}^{3}\eta_i\omega_i=d\omega_k \ \Rightarrow\ d\Big(\sum_{i=1}^{3}\eta_i\omega_i-\omega_k\Big)=0 \tag{3.9}$$

As to assumption $(3.5)_2$, it is a must that $d=0$.

This proves the theorem on movement of the cross-section as a rigid disc.

4. Conclusions

- It follows from the above theorem that the derivative of the angle of section torsion to which the constraints are applied equals zero, irrespective of the point of application, provided the basic assumptions of the theorem are satisfied.
- Applications of any kind of warping constraints influence the boundary problem governing the solution of a thin-walled beam only by values of bimoments. The boundary condition $\alpha'(l) = 0$ is always the same.
- The second case of linking mentioned above – i.e. when the rigid constraints join the cross-section of the thin-walled beam with a freely deformable body – should be solved according to the known algorithm of the method of forces while the exact points of application to the cross-section are recognised.

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The importance of this paper as a contribution to the theory of thin-walled beams is out of proportion to prof. Michał Życzkowski contribution to author's scientific career at its every step. The professor whose jubilee is here celebrated was the supervisor of the author's doctoral thesis and a kind reviewer at the moment of receiving every other scientific title.

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"Więzy deplanacji" przyłożone do belki cienkościennej

Streszczenie

Celem pracy jest analiza więzów zewnętrznych, którym poddana jest belka o profilu cienkościennym. W szczególności omówione zostały więzy uniemożliwiające spaczenie ścianki poprzecznej, do której zostały one przyłożone. Sformułowano i udowodniono stosowne twierdzenie.

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