JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **38**, 2000

COMPUTATIONAL MODELS OF PERIODIC COMPOSITES. TOLERANCE AVERAGING VERSUS HOMOGENIZATION

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Physical reliability of the known asymptotic homogenization models of periodic composites is discussed. By retaining some terms which are neglected in the homogenization a refined approach to the modelling of composites is proposed. It is shown that this approach results in the non-asymptotic tolerance averaged models which for special problems coincide with the homogenized ones but also are able to describe physical situations which are outside the framework of homogenization theory. The analysis is carried out for the heat conduction problem.

Key words: composites, homogenization, tolerance, modelling

1. Introduction

The exact approach to problems in mechanics of composite materials with a periodic structure is described by partial differential equations with periodic rapidly-oscillating and noncontinuous coefficients. To eliminate the difficulties posed by a direct use of these equations to special problems different approximate mathematical models of periodic composites have been introduced. The simplest and best known models are those based on the concept of homogenization. A general idea of homogenization is that in many special problems the behaviour of a periodic composite body is similar to that of a certain equivalent homogeneous body. It means that the behaviour of composites with a periodic structure, in which periods are much smaller than the minimum characteristic length dimension of a whole composite body, can be described by means of equations with constant coefficients. To detail the concept of homogenization we shall confine ourselves to the heat conduction problem in a composite body. Let this body occupy the region Ω in the reference three-dimensional space and be equipped with the vector basis $\boldsymbol{d}_\alpha$, $\alpha = 1,2,3$ having a periodic structure with periods $l_\alpha = |\boldsymbol{d}_\alpha|$ in direction of vectors $\boldsymbol{d}_\alpha$, respectively. Under the denotation $\Delta = \{\boldsymbol{x} = \eta_\alpha \boldsymbol{d}_\alpha,\ \eta_\alpha \in (-1/2, 1/2),\ \alpha = 1,2,3\}$ this structure will be referred to as Δ-periodic. We shall assume that the maximum length l of the diagonal of Δ is very small compared to the minimum characteristic length dimension L of Ω. Let us denote by $\boldsymbol{x}$ a point of Ω, by t a time coordinate and by $\theta = \theta(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$, a temperature field at an instant t. The heat conduction equation in Ω takes the well known form

$$\nabla \cdot (\mathsf{A} \cdot \nabla\theta) - c\dot{\theta} = f \tag{1.1}$$

where $\mathsf{A} = \mathsf{A}(\boldsymbol{x})$ is the symmetric second order heat conduction tensor, $c = c(\boldsymbol{x})$ is the specific heat and $f = f(\boldsymbol{x}, t)$ is the intensity of heat sources. In the periodic composite under consideration $\mathsf{A}(\cdot)$, $c(\cdot)$ are rapidly oscillating piecewise constant Δ-periodic functions. Generally speaking, by a homogenization we mean here the replacing of Eq (1.1) by what is called the homogenized equation

$$\nabla \cdot (\mathsf{A}^0 \cdot \nabla\theta^0) - c^0\dot{\theta}^0 = f \tag{1.2}$$

where A^0 and c^0 are constant and $\theta^0 = \theta^0(\boldsymbol{x}, t)$ stands for a temperature field at an instant t. Now the question arises how to obtain the values of A^0, c^0, provided that $\mathsf{A}(\cdot)$, $c(\cdot)$ are known and how to determine the temperature field θ in a periodic composite provided that θ^0 has been calculated previously on the basis of Eq (1.2). The question formulated above represents the homogenization problem for the linear heat conduction in a periodic medium. At the same time A^0 will be referred to as the homogenized heat conduction tensor.

The general solution to the homogenization problem can be obtained within the framework of the known homogenization theory, cf e.g. the monographs by Sanchez-Palencia (1980), and Jikov et al. (1994). The extensive list of references on this subject can be found in the second one of the aforementioned monographs. The homogenization theory is now a distinct mathematical discipline which applies the asymptotic analysis to problems of so called micro-heterogeneous media, i.e. periodic media with the periods εl_α, $\alpha = 1,2,3$, where $\alpha \in (0,1]$ ($\varepsilon\Delta$-periodic media). To introduce this concept we define $\mathsf{A}^\varepsilon(\boldsymbol{x}) \equiv \mathsf{A}(\boldsymbol{x}/\varepsilon)$, $c^\varepsilon(\boldsymbol{x}) \equiv c(\boldsymbol{x}/\varepsilon)$ where $\varepsilon \in (0,1]$ and formulate a family of equations

$$\nabla \cdot (\mathsf{A}^\varepsilon \cdot \nabla\theta^\varepsilon) - c^\varepsilon\dot{\theta}^\varepsilon = f \tag{1.3}$$

indexed by $\varepsilon \in (0,1]$ and describing the heat conduction in micro-heterogeneous ($\varepsilon\Delta$-periodic) media. Obviously, Eq (1.3) coincides with Eq (1.1) for

$\varepsilon = 1$ and hence $\theta^1 = \theta$. The solution to the aforementioned homogenization problem can be obtained using the method of asymptotic expansions. For the sake of simplicity for the time being we restrict ourselves to the steady-state problems neglecting the terms $c^\varepsilon \dot{\theta}^\varepsilon$ in Eq (1.3) and we look for the first approximation of solution to Eq (1.3) in the form $\theta_1^\varepsilon(\boldsymbol{x}) = \theta^0(\boldsymbol{x}) + \varepsilon\theta_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$, where $\theta_1(\boldsymbol{x}, \cdot)$ is a Δ-periodic function and $\theta^0 \in C^2(\overline{\Omega})$. Define by $\langle f \rangle$ the mean value of Δ-periodic function f on the cell Δ. Then for every $f \in H^{-1}(\Omega)$ and $\varepsilon \to 0$ we can prove that θ_1^ε tends weakly to θ^0 in $H^1(\Omega)$ and $\mathbf{A}^\varepsilon \cdot \nabla\theta_1^\varepsilon$ tends weakly to $\mathbf{A}^0 \cdot \nabla\theta^0$ in $L^2(\Omega)$. Moreover, the homogenized heat conduction tensor is given by $\mathbf{A}^0 = \langle \mathbf{A} + \mathbf{A} \cdot \nabla\boldsymbol{N} \rangle$, where $\boldsymbol{N}$ is a Δ-periodic vector field which is a solution to the auxiliary periodic problem given by $\nabla \cdot (\mathbf{A} \cdot \nabla\boldsymbol{N}) = -\nabla \cdot \mathbf{A}$, $\boldsymbol{N} \in (H^1(\Delta); R^3)$ with $(H^1(\Delta); R^3)$ as the Sobolev space of Δ-periodic vector functions. If $\boldsymbol{N}$ is a solution to this problem then the condition $\langle \nabla\vartheta^* \cdot \mathbf{A} \cdot \nabla\boldsymbol{N} \rangle = -\langle \nabla\vartheta^* \cdot \mathbf{A} \rangle$ has to hold for every $\vartheta^* \in H^1(\Delta)$. Apart from one-dimensional periodic structures and some special forms of $\mathbf{A}$ (cf Jikov et al., 1994) the Δ-periodic solution $\boldsymbol{N}$ to the above problem can be obtained exclusively in an approximate form, e.g. by a discretization of Δ and by assuming $\boldsymbol{N} = g^A(\boldsymbol{y})\boldsymbol{Q}^A$, $A = 1, ..., N$, summation convention holds, where $g^A(\cdot)$ are postulated *a priori* Δ-periodic shape functions such that $\langle g^A \rangle = 0$ and $\boldsymbol{Q}^A$ are unknown vectors. Using the orthogonalization method from the above variational condition we obtain the system of linear algebraic equations for $\boldsymbol{Q}^A$ given by $\langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle \boldsymbol{Q}^B = -\langle \nabla g^A \cdot \mathbf{A} \rangle$. Hence the final (approximate) formula for the homogenized matrix is $\mathbf{A}^0 = \langle \mathbf{A} \rangle + \langle \mathbf{A} \cdot \nabla g^A \rangle \boldsymbol{Q}^A$. At the same time, the temperature field will be approximated by $\theta = \theta^0 + g^A W^A$ with $W^A = \boldsymbol{Q}^A \cdot \nabla\theta^0$; it follows that $\theta_1 = g^A W^A = g^A \boldsymbol{Q}^A \cdot \nabla\theta^0$. It will be tacitly assumed that the approximation formula $\boldsymbol{N}(\boldsymbol{y}) = g^A(\boldsymbol{y})\boldsymbol{Q}^A$ represents $\boldsymbol{N}(\cdot)$ with a sufficient accuracy. At last, we can prove that $\theta_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) = \boldsymbol{N}(\boldsymbol{x}/\varepsilon) \cdot \nabla\theta^0(\boldsymbol{x})$ and $\|\theta^\varepsilon - \theta_1^\varepsilon\| < K\sqrt{\varepsilon}$, where $\|\cdot\|$ is the norm in $H^1(\Omega)$ and K depends on θ^0 being independent of ε. The proof of the above statements can be found in Jikov et al. (1994). The obtained formula for the homogenized matrix $\mathbf{A}^0$ holds true also for non-stationary problems where we obtain $c^0 = \langle c \rangle$. At the same time, for an arbitrary heat source intensity f and for a sufficiently small ε the solutions θ^ε to the boundary-value problems related to Eq (1.3) can be approximated by

$$\theta_1(\boldsymbol{x}, t) = \theta^0(\boldsymbol{x}, t) + \varepsilon\boldsymbol{N}(\boldsymbol{x}/\varepsilon) \cdot \nabla\theta^0(\boldsymbol{x}, t) \tag{1.4}$$

where θ^0 is the solution to a pertinent problem for the homogenized equation (1.2) with $\mathbf{A}^0$ and c^0 defined before.

The above results describe the heat conduction in micro-heterogeneous media on condition that $\varepsilon \to 0$. However, heat conduction problems for the composite body under consideration are described not by a family of equations (1.3) but by the single equation (1.1) which coincides with Eqs (1.3) only for $\varepsilon = 1$. Thus, using exclusively the homogenization theory we are not able to answer the next important question whether the homogenized equation (1.2) together with Eq (1.4) for $\varepsilon = 1$ are physically reliable for the given *a priori* boundary and initial conditions and the known heat source intensity f. Generally speaking, the homogenized procedures are not equipped with any *a posteriori* estimates for solutions to special problems. In order to answer this new question we shall pass in Section 2 from Eq (1.1) to Eq (1.2) not using the asymptotic approach. We are to show that this procedure makes it possible to formulate certain heuristic homogenization assumptions under which the homogenized equation (1.2) together with Eq (1.4) for $\varepsilon = 1$ have a physical sense. Moreover, in Section 3 we shall reject some of the heuristic homogenization assumptions in order to remove certain restrictions imposed on the homogenization theory and formulate conditions on which its results are physically reliable. In this way we shall formulate what will be called the Tolerance Averaging Approach (TAA) which from the point of view of applications of the theory can be treated as an extension of the homogenization approach to cover the heat transfer problems in periodic composites. In this paper the TAA will be not presented in the form of a mathematical theory like the homogenization but the main attention is to be given to the physical reliability of TAA. It will be shown that, contrary to the homogenization theory, the TAA makes it possible to estimate *a posteriori* an approximation of the obtained solutions to special problems. To this end we shall introduce to the TAA the concept of a tolerance relation. Denoting by S a certain linear normed (function) space, for every $s', s'' \in S$ we shall write $s' \approx s''$ iff $\|s' - s''\| < e_S$ where e_S describes the admissible error related to computations of elements of S. In the general case $\approx$ will stand for different tolerance relations describing the required accuracy of measurements and/or computations in the problem under consideration, cf also Zeeman (1965), Woźniak (1983), for the concept of tolerance space.

2. Reliability of the homogenization approach

We begin with some auxiliary concepts. To this end, define $\Delta(\boldsymbol{x}) := \boldsymbol{x} + \Delta$ as a cell with a center at $\boldsymbol{x}$ and $\Omega^0 := \{\boldsymbol{x} \in \Omega : \Delta(\boldsymbol{x}) \subset \Omega\}$. Because of $l \ll L$ set $\Omega \backslash \Omega^0$ constitutes a thin near boundary layer in Ω. To an arbitrary integrable function φ defined in Ω we shall assign its averaging $\langle\varphi\rangle$ defined in Ω^0 setting

$$\langle\varphi\rangle(\boldsymbol{x}) = \frac{1}{\mathrm{mes}\Delta} \int\limits_{\Delta(\boldsymbol{x})} \varphi(y)\, dy \qquad \boldsymbol{x} \in \Omega^0$$

If φ is Δ-periodic then $\langle\varphi\rangle = \mathrm{const}$ and for φ depending also on t we shall write $\langle\varphi\rangle(\boldsymbol{x}, t)$.

For an arbitrary sufficiently regular function $F(\cdot)$ defined in Ω and belonging to a certain normed linear function space, we shall write $F \in SV(\Omega)$ provided that the approximation condition $\langle F\rangle(\boldsymbol{x}) \approx F(\boldsymbol{x})$ holds in Ω^0. In this case $F(\cdot)$ will be referred to as a *Δ-slowly varying function* (i.e. slowly varying with respect to the periods l_α, $\alpha = 1, 2, 3$). It has to be remembered that if F_1, F_2 are differentiable functions belonging to a certain normed linear space then the tolerance relation $F_1 \approx F_2$ implies a similar relation for derivatives of F_1, F_2; to emphasizes this fact we shall also write $\nabla F_1 \approx \nabla F_2$.

Let $\psi_{\boldsymbol{x}}(\boldsymbol{y})$ be for every $\boldsymbol{x} \in \Omega^0$ a Δ-periodic function of $\boldsymbol{y}$ (hence defined almost everywhere in R^3) and for every $\boldsymbol{y}$ be a Δ-slowly varying function of $\boldsymbol{x}$. Then setting $\psi(\boldsymbol{x}) = \psi_{\boldsymbol{x}}(\boldsymbol{x})$ in Ω^0 and assuming that $\psi(\cdot)$ belongs to a certain linear normed function space, we shall write $\psi \in PL(\Omega)$ and refer $\psi(\cdot)$ to as a *Δ-periodic like function*. Roughly speaking, every Δ-periodic like function $\psi(\cdot)$ after restricting its domain to an arbitrary cell $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega^0$, can be approximated in this cell (within to a certain tolerance) by a Δ-periodic function $\psi_{\boldsymbol{x}}(\cdot)$. It means that $\langle\psi\rangle(\boldsymbol{x}) \approx \langle\psi_{\boldsymbol{x}}\rangle(\boldsymbol{x})$ and $\psi_{\boldsymbol{x}}$ will be called a Δ-periodic approximation of $\psi(\cdot)$ in $\Delta(\boldsymbol{x})$. It follows that if $\psi \in PL(\Omega)$ then $\langle\psi\rangle \in SV(\Omega)$. If $\psi \in PL(\Omega)$ and either $\langle\psi\rangle = 0$ or $\langle c\psi\rangle = 0$ then ψ will be referred to as an *oscillating Δ-periodic like function*, $\psi \in OPL(\Delta)$; here $\langle\psi\rangle = 0$ is called the normalizing condition.

By a tolerance averaging, which is the main mathematical tool of the modelling proposed in this contribution, we shall mean the tolerance relations

$$\langle\varphi F\rangle(\boldsymbol{x}) \approx \langle\varphi\rangle(\boldsymbol{x})F(\boldsymbol{x}) \qquad \langle\varphi\psi\rangle(\boldsymbol{x}) \approx \langle\varphi\psi_{\boldsymbol{x}}\rangle(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega^0 \tag{2.1}$$

which have to hold for every $F \in SL(\Delta)$, $\psi \in OPL(\Delta)$ with φ as an arbitrary integrable function defined in Ω, and which make it possible to replace the left-hand sides of Eqs (2.1) by their right-hand sides. At the same time formulae similar to those in Eq (2.1) have to hold also for all derivatives of $F(\cdot)$ and $\varphi(\cdot)$. All the aforementioned concepts require the specification of tolerance relations $\approx$ in calculations of averages (2.1) which depends on the accuracy of computations for the problem under consideration.

The heuristic passage from the problems governed by Eq (1.1) to their approximate solutions described by Eq (1.4) for $\varepsilon = 1$ together with the homogenized equation (1.2) will be based on three assumptions. The first of them restricts the class of temperature fields θ under consideration to that which can be represented by the decomposition $\theta = \theta^0 + \vartheta$, with $\theta^0(\cdot, t) \in SV(\Delta)$, $\vartheta(\cdot, t) \in OPL(\Delta)$ for every t; here the normalizing condition is assumed in the form $\langle \vartheta \rangle = 0$ but the alternative form $\langle c\vartheta \rangle = 0$ can be also taken into account where $c(\cdot)$ is the periodic specific heat function, cf Eq (1.1). The aforementioned decomposition of θ is well motivated from a physical point of view but can be verified only *a posteriori* because $\theta^0(\cdot, t)$ and $\vartheta(\cdot, t)$ are not known *a priori*. Using this decomposition, averaging Eq (1.1) and applying the tolerance averaging formula (2.1) we obtain

$$\nabla \cdot \langle \mathsf{A} \cdot (\nabla \theta^0 + \nabla \vartheta_{\boldsymbol{x}}) \rangle(\boldsymbol{x}, t) - \langle c \rangle \dot{\theta}^0(\boldsymbol{x}, t) - \langle c \dot{\vartheta}_{\boldsymbol{x}} \rangle(\boldsymbol{x}, t) = \langle f \rangle(\boldsymbol{x}, t) \tag{2.2}$$

for every $\boldsymbol{x} \in \Omega^0$. At the same time multiplying Eq (1.1) by an arbitrary Δ-periodic test function $\vartheta^* \in H^1_{per}(\Delta)$ such that $\langle \vartheta^* \rangle = 0$, integrating this equation over $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega^0$, and using again the tolerance averaging formula (2.1) we arrive at the variational conditions

$$\begin{aligned} &\langle \nabla \vartheta^* \cdot \mathsf{A} \cdot \nabla \vartheta_{\boldsymbol{x}} \rangle(\boldsymbol{x}, t) + \langle \vartheta^* c \dot{\vartheta}_{\boldsymbol{x}} \rangle(\boldsymbol{x}, t) = \\ &= \langle \vartheta^* f \rangle(\boldsymbol{x}, t) - \langle \vartheta^* c \rangle \dot{\theta}^0(\boldsymbol{x}, t) - \langle \nabla \vartheta^* \cdot \mathsf{A} \rangle \cdot \nabla \theta^0(\boldsymbol{x}, t) \end{aligned} \tag{2.3}$$

which hold for every test function ϑ^* and for an arbitrary but fixed $\boldsymbol{x} \in \Omega^0$.

The second heuristic assumption leading to the homogenized model restricts the class of heat sources under consideration to that satisfying the condition $f \in SV(\Delta)$. In this case the first term on the right-hand side of Eq (2.3) can be replaced by $\langle \vartheta^* \rangle(\boldsymbol{x}) f(\boldsymbol{x}, t)$ and by means of $\langle \vartheta^* \rangle = 0$ drops out from Eq (2.3); at the same time the right-hand side of Eq (2.2) can be replaced by $f(\boldsymbol{x})$.

The third assumption required for obtaining the homogenized model of periodic medium requires that the terms in Eqs (2.2), (2.3) involving averages $\langle \vartheta^* f \rangle$, $\langle c\dot{\vartheta}_{\boldsymbol{x}} \rangle$, $\langle \vartheta^* c \dot{\vartheta}_{\boldsymbol{x}} \rangle$ and $\langle \vartheta^* c \rangle$ be sufficiently small compared to the remaining terms and can be therefore neglected. In this case Eq (2.3) represents the periodic problem for $\vartheta_{\boldsymbol{x}}(\boldsymbol{y}, t)$, $\boldsymbol{y} \in \overline{\Delta}(\boldsymbol{x})$, which for $\vartheta_{\boldsymbol{x}}, \vartheta^* \in H^1_{per}(\Omega)$, $\langle \vartheta_{\boldsymbol{x}} \rangle = 0$, $\langle \vartheta^* \rangle = 0$, has a unique solution and yields $\vartheta_{\boldsymbol{x}} = \boldsymbol{N}(\boldsymbol{y}) \cdot \nabla \theta^0(\boldsymbol{x}, t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, where $\boldsymbol{N}(\cdot)$ was defined in Section 1.

Summing up, it can be seen that the aforementioned three assumptions lead to the homogenized equation (1.2) and to the formula (1.4) for $\varepsilon = 1$ after using the denotations $\theta = \theta^1_1$, $\vartheta = \boldsymbol{N} \cdot \nabla \theta^0$. The physical reliability condition of the homogenization approach, which is implied by the first assumption and can be verified only *a posteriori* has the form: $\theta^0 \in SV(\Delta)$; it holds only if $f \in SV(\Delta)$, i.e. it implies the second heuristic assumption leading to the homogenized model of periodic medium.

3. Tolerance averaging approach

Let us observe that contrary to the first and second assumptions it is rather difficult to verify whether the third of the aforementioned assumptions is physically reliable i.e. if it can be applied to the heat conduction problem we are to solve. Moreover, this assumption does not have sufficiently reasonable physical interpretation. Thus, the *crucial idea of the refined averaging mathematical modelling* of heat transfer problems in periodic composites, *is to reject the third one of the above assumptions.* Moreover, it can be shown that the second assumption is implied by the first one and will be formulated now in a less restrictive form; namely *we have to confine ourselves to the class of heat sources which are periodic like,* $f \in PL(\Delta)$, *and hence can be represented by a sum* $f = f^0 + \varphi$ *where* $f^0 \in SV(\Delta)$ *and* $\varphi \in OPL(\Delta)$. At the same time we leave the first heuristic assumption unchanged. It means that the basis of the approach detailed below is given by Eqs (2.2), (2.3) where now the right-hand side of Eq (2.2) is equal to $f^0 \in SV(\Delta)$ and where the term $\langle\vartheta^* f\rangle(\boldsymbol{x}, t)$ in Eq (2.3) has to be replaced by $\langle\vartheta^*\varphi\rangle(\boldsymbol{x}, t)$. It has to be emphasized that from now on all approximations of Eq (1.1) are due exclusively to the tolerance averaging (2.1) and a similar averaging for all derivatives of F and ψ. That is why the proposed approach will be referred to as the *tolerance averaging approach.* In the sequel, stationary and nonstationary heat conduction problems in periodic composites will be analyzed separately within the framework of the proposed tolerance averaging approach.

3.1. Tolerance averaging of stationary processes

In this case if $\varphi_{\boldsymbol{x}} \in L^2_{per}(\Delta)$ then Eq (2.3) for every $\boldsymbol{x} \in \Omega^0$ leads to the separate periodic problem: find $\vartheta_{\boldsymbol{x}} \in H^1_{per}(\Delta)$ such that

$$\langle\nabla\vartheta^* \cdot \mathbf{A} \cdot \nabla\vartheta_{\boldsymbol{x}}\rangle(\boldsymbol{x}) = -\langle\nabla\vartheta^* \cdot \mathbf{A}\rangle(\boldsymbol{x}) \cdot \nabla\theta^0(\boldsymbol{x}) - \langle\vartheta^*\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x}) \tag{3.1}$$

holds for every $\vartheta^* \in H^1_{per}(\Omega)$. Because of $\langle\varphi_{\boldsymbol{x}}\rangle = 0$ the solution $\vartheta_{\boldsymbol{x}}$ to the above problems is unique to within an arbitrary constant, cf Jikov et al. (1994). To obtain this solution we shall discretize Δ introducing Δ-periodic shape functions $g^A(\cdot)$, $A = 1, ..., N$, $\langle g^A\rangle = 0$ as it was done in the case of homogenization approach. We shall also look for the approximate solution to

Eq (3.1) in the form $\vartheta_{\boldsymbol{x}} = g^A(\boldsymbol{y})W^A(\boldsymbol{x})$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, with $W^A(\boldsymbol{x})$ as new unknowns which will be determined by the orthogonalization method. Hence, Eq (2.2) for the stationary processes together with the orthogonalization conditions implied by (3.1) yield the following equations with constant coefficients for θ^0, W^A, $A = 1, ..., N$

$$
\begin{aligned}
&\nabla \cdot \Big(\langle \mathbf{A} \rangle \cdot \nabla \theta^0 \Big) + \langle \mathbf{A} \cdot \nabla g^A \rangle W^A = f^0 \\
&\langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle W^B(\boldsymbol{x}) = -\langle \nabla g^A \cdot \mathbf{A} \rangle \cdot \nabla \theta^0 - \langle g^A \varphi_{\boldsymbol{x}} \rangle(\boldsymbol{x})
\end{aligned}
\tag{3.2}
$$

and a temperature field is given by $\theta = \theta^0 + g^A W^A$. Solutions to Eq (3.2) are physically reliable if $\theta^0 \in SV(\Delta)$ and $W^A \in SV(\Delta)$; in this case Eqs (3.2) represent *the tolerance averaging model of stationary heat conduction processes in a periodic composite.*

If $\varphi \in PL(\Delta)$ and $\varphi \in H^1_{per}(\Delta)$ then $\varphi = \varphi_{\boldsymbol{x}}$ on every $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega^0$, and hence Eq (3.1) represents for every $\boldsymbol{x} \in \Omega^0$ the same periodic problem. In this case we can assume $W^A = \boldsymbol{Q}^A \cdot \nabla\theta^0 + R^A$ where $\boldsymbol{Q}^A$, R^A are constant being solutions to

$$
\langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle \boldsymbol{Q}^B = -\langle \nabla g^A \cdot \mathbf{A} \rangle
$$

$$
\langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B \rangle R^B = \langle g^A \varphi \rangle
$$

respectively. For θ^0 we shall obtain the homogenized equation

$$
\nabla \cdot (\mathbf{A}^0 \cdot \nabla \theta^0) = f^0 \qquad\qquad \mathbf{A}^0 = \langle \mathbf{A} \rangle + \langle \mathbf{A} \cdot \nabla g^A \rangle \cdot \boldsymbol{Q}^A
$$

and the temperature will be given by $\theta = \theta^0 + g^A \boldsymbol{Q}^A \cdot \nabla\theta^0 + g^A R^A$. This is a special case of Eqs (3.2).

Let us observe that if $\varphi = 0$ (i.e., if $f = f^0 \in SV(\Delta)$) then for stationary processes the tolerance averaging model coincides with the homogenized one. Hence the final conclusion is that in stationary problems the homogenized model is incapable of describing the effect of heat source oscillations φ on the heat transfer in a periodic medium.

3.2. Tolerance averaging of nonstationary processes

In nonstationary processes $\vartheta_{\boldsymbol{x}}$ for every $\boldsymbol{x} \in \Omega^0$ is described by the variational problem (2.3) with $\langle \vartheta^* f \rangle = \langle \vartheta^* \varphi \rangle$. This problem is rather complicated and will be solved using the refined version of orthogonalization method. To this end we shall assign to Eq (1.1) the periodic eigenvalue problem for Δ-periodic function h satisfying in Δ the equation

$$\nabla \cdot (\mathbf{A} \cdot \nabla h) + \lambda c h = 0 \tag{3.3}$$

Leading to the condition $\langle ch \rangle = 0$, where λ is an eigenvalue. The function h has to satisfy in Δ the regularity conditions similar to those imposed on the temperature field θ. Let us denote by h_A, $A = 1, 2, \ldots$ the sequence of pertinent eigenfunctions related to the aforementioned problem and define $h^A = h_A - \langle h_A \rangle$ for $A = 1, \ldots, N$. For an arbitrary $\boldsymbol{x} \in \Omega^0$ we shall seek an approximate solution to Eq (2.3) in the form

$$\vartheta_{\boldsymbol{x}}(\boldsymbol{y}, t) = h^A(\boldsymbol{y}) V^A(\boldsymbol{x}, t) \qquad \boldsymbol{y} \in \Delta(\boldsymbol{x}) \tag{3.4}$$

where the summation convention over $A = 1, \ldots, N$ holds, $h^A(\cdot)$, $A = 1, \ldots, N$ are called the mode shape functions and $V^A(\boldsymbol{x}, t)$ are unknowns. Here the positive integer N is arbitrary but fixed and hence we can look for solutions to Eq (2.3) on different levels of accuracy. Unknowns $V^A(\boldsymbol{x}, t)$ for every $\boldsymbol{x} \in \Omega^0$ have to satisfy the orthogonality conditions which can be obtained from Eq (2.3) by setting $\vartheta^* = h^A$, $A = 1, 2, \ldots, N$. Let us observe that $\vartheta \in PL(\Delta)$ implies $V^A(\cdot, t) \in SV(\Delta)$. Taking into account the last condition, from Eq (2.2) and the aforementioned orthogonality condition we obtain for θ^0, V^A, $A = 1, 2, \ldots, N$ the following system of equations with constant coefficients

$$\begin{aligned} &\nabla \cdot \Big(\langle \mathbf{A} \rangle \cdot \nabla \theta^0 + \langle \mathbf{A} \cdot \nabla h^A \rangle V^A \Big) - \langle c \rangle \dot{\theta}^0 - \langle c h^A \rangle \dot{V}^A = f^0 \\ &\langle c h^A h^B \rangle \dot{V}^B + \langle \nabla h^A \cdot \mathbf{A} \cdot \nabla h^B \rangle V^B + \langle c h^A \rangle \dot{\theta}^0 + \langle \nabla h^A \cdot \mathbf{A} \rangle \cdot \nabla \theta^0 = -\langle h^A \varphi \rangle \end{aligned} \tag{3.5}$$

where the summation convention over $B = 1, \ldots, N$ holds and $\langle c h^A h^B \rangle = 0$ for $A \neq B$. Hence the temperature field in the composite under consideration with the required accuracy can be approximated by $\theta = \theta^0 + h^A V^A$, $A = 1, 2, \ldots, N$, where θ^0, V^A is the solution to a certain initial-boundary value problem related to Eqs (3.5). The obtained solution is physically reliable if $\theta^0, V^A \in SV(\Delta)$ and Eqs (3.5) together with the above formula for θ represent *the tolerance averaging model of nonstationary heat conduction problems in a periodic composite.*

The characteristic feature of Eq (3.5) is that for unknowns V^A, $A = 1, 2, \ldots, N$, we have obtained the system of ordinary differential equations involving only the time derivatives of V^A. Hence, there are no boundary conditions for V^A and we deal here with a situation similar to that discussed by Woźniak (1997). That is why the functions V^A can be called internal variables and the obtained model of heat conduction in a periodic composite will be referred to as the internal variable model. Application of Eqs (3.5) to the analysis of stationary problems is possible from a formal viewpoint but the eigenvalue problem related to Eq (3.3) and also the choice of mode shape functions h^A have no physical motivation. In this case we can assume h^A as shape functions derived from the discretization. In this way for stationary problems we obtain from Eqs (3.5) after neglecting time derivatives the system of linear algebraic equations for V^A. Hence the formula $\theta(\boldsymbol{x}) = \theta^0(\boldsymbol{x}) + h^A(\boldsymbol{x})V^A(\boldsymbol{x})$ will represent a certain approximate solution to the stationary periodic problem related to Eq (1.1).

At the end of this Section let us also observe that for a homogeneous body not subjected to the oscillations of heat sources and initial temperature ($\varphi = 0$ and $V^A(\boldsymbol{x}, t_0) = 0$) Eqs (3.5) for V^A yield the trivial solution $V^A = 0$ and the first equation of Eqs (3.5) reduces to the form $\nabla \cdot (\mathsf{A} \cdot \nabla\theta) - c\dot{\theta} = f$ with A and c constant.

4. Conclusions

The first main result of this paper is that the tolerance averaging models of heat conduction in periodic composites, given by Eqs (3.2) and (3.5) together with the pertinent formulae for θ, are based on the only physical assumption that in periodic media a temperature field has to satisfy the condition $\theta \in PL(\Delta)$. In general, this condition may be not satisfied in the near boundary layer of the medium; for the sake of simplicity we assume that the boundary conditions for temperature are imposed exclusively on θ^0; the discussion of boundary conditions in periodic media can be found in Bensoussan et al. (1978).

The second main result is that for solutions to the problems formulated within the framework of TAA we have obtained *a posteriori* estimates implied by the conditions θ^0, $V^A \in SV(\Delta)$. The accuracy of the solution obtained depends also on the form of expansions (3.4). It has to be emphasized that finding solutions to the eigenvalue problem (3.3) is rather a difficult task and in most cases the eigenfunctions h^A have to be obtained using approximate methods. However, this difficulty is also typical for the asymptotic homogenization where the solution $\boldsymbol{N}(\cdot) = g^A(\cdot)\boldsymbol{Q}^A$ to the auxiliary periodic problem (apart from one-dimensional problems) was also obtained using approximate methods, see Section 1.

In the paper it was shown that the proposed models have been derived by rejecting some from heuristic assumptions included *implicite* in the reliability conditions of the known homogenization approach. Hence the proposed tolerance averaging approach is able to describe physical phenomena which cannot be investigated within the framework of homogenized model, described by Eq (1.2). For stationary heat conduction processes it is the effect of heat source cell oscillations (i.e., the oscillations within every periodicity cell $\Delta(\boldsymbol{x})$) on the temperature distribution. This effect is described by the term depending on φ in Eqs (3.2). For nonstationary processes the aforementioned effect is represented by the terms $\langle h^A\varphi\rangle$ on the right-hand sides of Eqs (3.5). Moreover, contrary to the homogenized model, the tolerance averaging approach describes the effect of periodicity cell size on the heat conduction in periodic composites. This effect is due to the coefficients $\langle ch^Ah^B\rangle$ in Eqs (3.5) which by means of Eq (3.3) depend on the cell length dimensions. Let us observe that the terms $\langle g^A\varphi_{\boldsymbol{x}}\rangle$ in Eqs (3.2) as well as the term $\langle h^A\varphi\rangle$ in Eqs (3.5) also depend on the cell size and hence the effect of heat source cell oscillations is coupled with that of the cell size. Let us also observe that there is no physical motivation for the formal introducing to the stationary problems the divergence terms depending on the length scales as it was done by Lewiński and Kucharski (1992) because the truncation relation (6.8) in the aforementioned paper hes to imply the first heuristic assumption which eliminates these terms. Moreover, the tolerance averaging approach makes it possible to investigate the effect of temperature oscillation at the initial instant $t = t_0$ provided that those oscillations with a sufficient accuracy can be determined by Eq (3.4) for $t = t_0$. This effect is described by the initial conditions for internal variables $V^A(\boldsymbol{x},\cdot)$, $\boldsymbol{x} \in \Omega^0$.

At the beginning of Section 2 it was stated that it is difficult to verify whether the homogenized model of heat transfer in periodic composites has a physical sense in the problem under consideration since the terms in Eqs (2.2), (3.3) involving averages $\langle c\dot{\vartheta}_{\boldsymbol{x}}\rangle$, $\langle\vartheta^* c\dot{\vartheta}_{\boldsymbol{x}}\rangle$ and $\langle\vartheta^* c\rangle$ which are neglected in the asymptotic homogenization approach, can influence a solution to the problem. Within the framework of the proposed tolerance averaging method the aforementioned terms are not rejected and the physical reliability of solutions have to be verified *a posteriori* by means of the condition $\theta^0(\cdot) \in SV(\Delta)$ in stationary processes and the conditions $\theta^0(\cdot,t)$, $V^A(\cdot,t) \in SV(\Delta)$ for every t in nonstationary processes.

Some applications of Eqs (3.5) can be found in Woźniak et al. (1996) and the discussion of a special case of these equations was carried out by Ignaczak (1998), and in the subsequent papers of this Author. For the sake of simplicity all considerations carried out throughout this contribution were restricted to the second-order parabolic equations. However, the proposed refined averaging approach can be also applied to different problems in thermomechanics of composites with a periodic structure; for the references see Woźniak (1997).

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Modele kompozytów o budowie periodycznej. Uśrednianie tolerancyjne a homogenizacja

Streszczenie

Celem pracy jest krytyczne spojrzenie na fizyczną poprawność rezultatów znanej w literaturze metody homogenizacji asymptotycznej, por. np. V.V. Jikov i inni

(1994), Sanchez-Palencia (1980), zwrócenie uwagi na jej usterki i ograniczenia a następnie propozycja usunięcia tych niedomagań poprzez osłabienie fizycznych założeń homogenizacji. Wykazano, że proponowane podejście prowadzi do pewnych tolerancyjnie uśrednionych modeli nieasymptotycznych, które w szczególnych przypadkach są zgodne z rezultatami uzyskanymi metodą homogenizacji, lecz umożliwiają także badanie zjawisk, których nie opisuje teoria homogenizacji. Rozważania przeprowadzono na przykładzie równania przewodnictwa cieplnego.

Manuscript received October 18, 1999; accepted for print December 10, 1999