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COMPUTATION OF PLANE TURBULENT FLOW USING PROBABILITY DENSITY FUNCTION METHOD[1]

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Application of the PDF method to turbulence modelling is surveyed. Important parts of the algorithm and of variance reduction techniques used therein are described. The computation results for a free-shear flow (mixing layer) and a wall-bounded flow in simple geometry (channel flow) are reported and perspectives for further development of the method are suggested.

Key words: turbulence, stochastic processes, PDF method

1. Introduction

The Probability Density Function (PDF) method represents a statistical tool for description and computation of turbulent flows (Pope, 1985). Contrary to the classical Eulerian approach, where averaged equations representing conservation laws are considered, in the PDF method turbulent flow is described with the use of the one-point probability density for instantaneous velocities and locations of fluid elements, as well as other variables (passive or reactive scalars, such as chemical composition, temperature, etc.).

The PDF method represents the Lagrangian approach, due to the fact that equations for instantaneous variables are modelled directly (Pozorski, 1997). Stochastic particles (representing fluid elements) are introduced, with a set of variables (location, velocity, possibly scalars) attached to them, and their time evolution is followed in the solution domain. Models used to describe the evolution of the variables are usually stochastic in nature. Such a description

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gives a closed PDF transport equation. Equivalent to the equation is a system of stochastic differential equations (SDE) that is numerically solved. Statistical averaging (cf Monin and Yaglom, 1971) is applied to the realisations of the stochastic process, and data on the flow are obtained (as moments of the distribution): mean velocity field, turbulent stress tensor, skewness and flatness coefficients of velocity components, intermittency factor, etc.

2. Governing equations

The crux of the PDF method is the closure of the evolution equation for the probability distribution of instantaneous velocity and other variables (depending on the level of description taken) that characterise the flow; in the presented work, knowledge of the spatial flow structure is input in the form of the dissipation rate ϵ of the turbulent kinetic energy k.

Consider high Reynolds number turbulent incompressible flow of density ρ and kinematic viscosity ν. After the Reynolds decomposition into the mean and fluctuation

$$\Phi = \langle \Phi \rangle + \phi' \tag{2.1}$$

where Φ stands for any variable describing the flow (velocity component U_i, pressure P, etc.), the Navier-Stokes equation in the Lagrangian approach writes

$$dU_i = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial x_i}\,dt + \nu\nabla^2\langle U_i \rangle\,dt - \frac{1}{\rho}\frac{\partial p'}{\partial x_i}\,dt + \nu\nabla^2 u_i'\,dt \tag{2.2}$$

Now, fluid elements are modelled as the stochastic particles with a set of attached (Lagrangian) variables. Their positions evolve with

$$dx_i = U_i\,dt \tag{2.3}$$

and the Navier-Stokes equation is modelled as (Pope and Tchen, 1990; Pope, 1991)

$$dU_i = -\frac{1}{\rho}\frac{\partial \langle P \rangle}{\partial x_i}\,dt + D_i\,dt + B_{ij}\,dW_j \tag{2.4}$$

Here, $d\boldsymbol{W}$ stands for an increment of the Wiener process (white noise). It is readily seen that the mean viscous term has been neglected and the sum of the fluctuating terms (pressure gradient and diffusion of momentum) has been replaced by a stochastic process of the diffusion type (Sobczyk, 1991). Arguments can be put forward in favour of such a closure model (Pope, 1994; Minier and Pozorski, 1997b).

On the other hand, the evolution equation for the dissipation rate ϵ has been modelled altogether through a stochastic process of the diffusion type, formulated in terms of the characteristic turbulent frequency $\omega = \epsilon/k$, with the account taken of the standard Eulerian modelling of the ϵ-equation (Pope, 1991)

$$d\omega = D_\omega \, dt + B_\omega \, dW_\omega \tag{2.5}$$

The components of the drift vector $\boldsymbol{D}$ and the diffusion matrix B are, in general, rather complicated functions of location $\boldsymbol{x}$, time t, flow variables $(\boldsymbol{U}, \omega)$ and their statistics.

Denote by $(\boldsymbol{V}, \theta)$ the sample space associated with the flow variables $(\boldsymbol{U}, \omega)$. The evolution equations for the stochastic particle locations (2.3), velocity (2.4) and turbulent frequency (2.5) give the closed transport equation for the probability density $f(\boldsymbol{V}, \theta; \boldsymbol{x}, t)$ in the phase space of flow variables. It is a PDE of the parabolic type (the Fokker-Planck equation, see e.g. Sobczyk, 1991)

$$\begin{gathered}\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} + \frac{\partial}{\partial V_i}(D_i f) + \frac{\partial}{\partial \theta}(D_\omega f) = \\ = \frac{1}{2} \frac{\partial^2}{\partial V_i \partial V_j}(B_{ik} B_{jk} f) + \frac{1}{2} \frac{\partial^2}{\partial \theta^2}(B_\omega^2 f)\end{gathered} \tag{2.6}$$

From the above closed formula, all moment equations can be derived (Reynolds equation, transport equations for the turbulent stress tensor components, etc.).

3. Numerical algorithm

Because of high dimensionality of the phase space, Eq (2.6) is not solved with the help of classical difference method. Rather, the use is made of equivalence of the closed PDF Eq (2.6) and the system of SDE for flow variables (2.3) and (2.4), and the latter is integrated using the Monte Carlo technique.

Evolution equations (2.3) and (2.4) are solved in time; because of their features, different from those of ODE, higher order numerical schemes become complicated. In the case of wall-bounded flows, the appropriate boundary conditions are added to simulate the presence of the solid walls (Pozorski and Minier, 1998a).

Contrary to the Eulerian approach, in the PDF method for incompressible flow the continuity equation is not solved explicitly. It is shown (Pope, 1985) that satisfaction of this equation is equivalent to the set of two conditions: constant spatial concentration of stochastic particles, zero-divergence of the mean velocity field. To satisfy both conditions, at every time step the correction equations (of the elliptic type) for instantaneous particle locations and velocities are solved (Pozorski and Minier, 1998b). Actually, the latter correction (that of particle velocities) is akin to the pressure correction algorithm (correction of mean velocity field) well known in the Eulerian approach.

A characteristic feature of the PDF method in the present implementation is a twofold description of flow: basic one, with stochastic particles, and so-called secondary, with mean variables computed within cells of a computational mesh superimposed on the solution domain. The numerical algorithm must take account of both descriptions respectively and of the (otherwise evident) relationship between them, known as the "particle-mesh coupling". The relationship is accounted for by procedures of computing statistical moments (means, variances, etc.) on the one hand, and interpolation of the statistics to the particle locations – on the other hand (Hockney and Eastwood, 1981).

Because of the fact that the Monte Carlo technique is used in the PDF method, the obtained solution is always flawed due to the statistical error. The ability to reduce this error for a given amount of computational work and within an assumed accuracy (spatial and temporal resolution) is thus of crucial importance in view of the practical application of the method. The variance reduction techniques (VRT), or, in other words, techniques of improving the solution quality, have been used in various ways in Monte Carlo algorithms (Kalos and Whitlock, 1986); in the PDF method for turbulent flows new techniques are developed. The quality improvement applied to the present algorithm has several aspects (Pozorski and Minier, 1998b). First, the computation of statistical moments takes an important place. In the simplest way, they are found using the NGP (nearest-grid-point) method where the identical weights are ascribed to all particles with regard to their cell center where the average values are computed. Alternatively, in the more precise CIC (cloud-in-cell) method, the particle weights reflect particle location with respect to the centers of all neighbouring meshes. Another possibility to improve the solution quality is to consider the evolution equations for some statistical moments $(\langle U_i\rangle, \langle u_i' u_j'\rangle, \langle\omega\rangle)$, and integrate them at the time step; next, instantaneous values are corrected according to the computed moments. Yet, another possibility is to integrate instantaneous evolution equations using a set of properly prepared vectors of random variables and to take their average as the final value (so-called tetrahedron method).

4. Computation results

To illustrate possible application areas of the PDF method, the results of computations for two flow configurations will now be reported and compared to available experimental data.

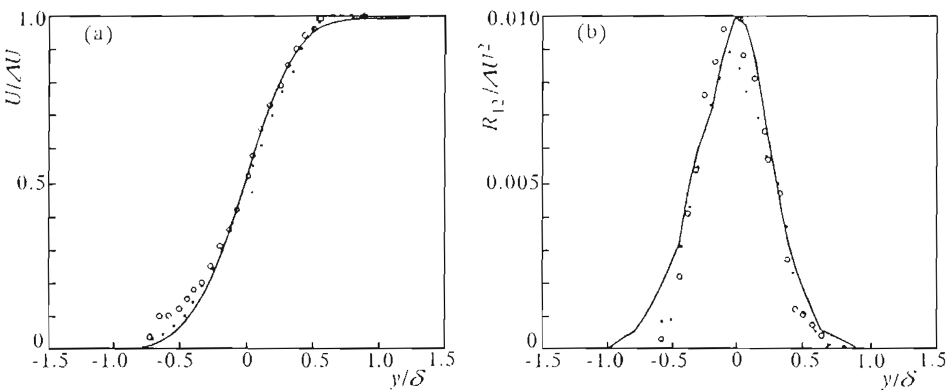

Fig. 1. Turbulent mixing layer: (a) mean axial velocity, (b) turbulent shear stress; (—) computation results; (∘) experiment of Patel (1973); (×) Wygnanski and Fiedler (1970)

First case of study was the turbulent mixing layer (Minier and Pozorski, 1995). It is characterized by the external intermittency; thus, the entrainment of the laminar flow into the turbulent core has to be accounted for in a proper way, through the input of stochastic particles at the edges of the mixing layer. Fig.1 presents some of the computed statistics: mean axial velocity and turbulent shear stress (obviously, all components of the Reynolds tensor are available) versus the cross-stream coordinate y normalized by the local flow width δ (thickness of the layer).

Second case being computed was the developed (i.e. statistically stationary) turbulent flow in a plane channel (Minier and Pozorski, 1997a). Fig.2 presents the characteristic fluctuating velocity in the cross-stream direction and the turbulent shear stress. Both variables are made non-dimensional with the use of the friction velocity u_τ, and the cross-stream coordinate y is normalised with the channel half-width h.

The following scatter plot (Fig.3) presents other type of results available from the PDF computation. Instantaneous streamwise velocities of stochastic particles are plotted versus their locations. From these data, local profiles of the probability density for the velocity component can be obtained (this holds also for any other variable attached to particles).

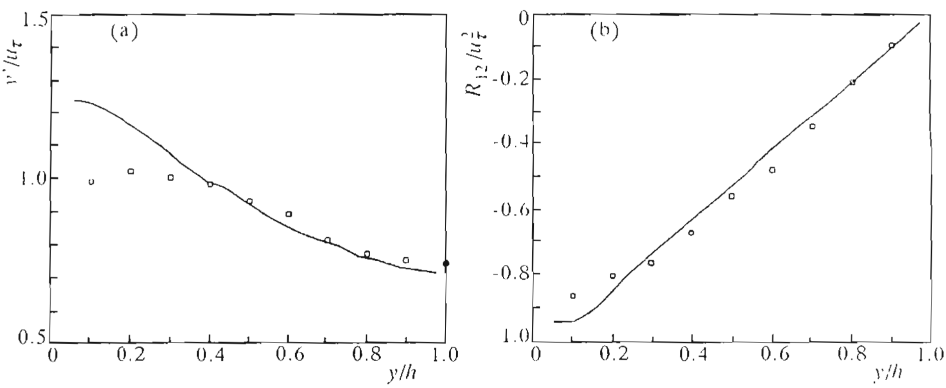

Fig. 2. Turbulent channel flow: (a) characteristic turbulent cross-stream velocity, (b) turbulent shear stress. Computation results (solid line) and experimental data of Comte-Bellot (1965)

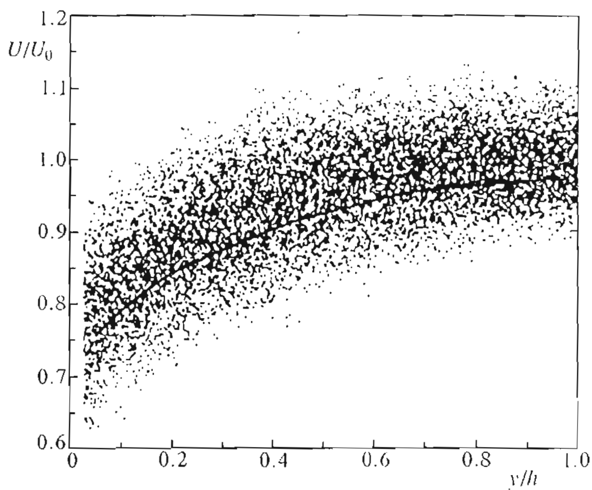

Fig. 3. Turbulent channel flow – streamwise velocity: mean flow velocity (solid line) and instantaneous velocities of stochastic particles (points)

5. Perspectives

The PDF method differs in some respects from more traditional proposals based on the closure of equations for moments of the distribution ($k - \epsilon, R_{ij} - \epsilon$). It is formulated in a different way, available set of data is much more extensive, numerical solution of the flow problem is different. Another important feature of the method is that it gives an accurate description (without modelling) of convective terms in the transport equations (gradient hypotheses, often erroneous, are thus avoided). Similarly, source terms for passive and reactive scalars (chemical composition, temperature) are also exact. For this reason, the method is promising, in particular for physically compound problems, such as combustion (Anand et al., 1997) or two-phase flows with the dispersed phase.

The results presented above have been obtained for statistically two-dimensional flows of a simple geometry. However, generalisation of the method to the 3D case or/and more complex geometrical configuration does not present a major difficulty. Parts of the algorithm dealing with stochastic particles (evolution equations, boundary conditions, computation of statistical averages) do not need to be modified, basically. On the other hand, the routines involving the difference mesh (description of geometry, Poisson solver) have to be generalised – in a similar way as in any numerical method based on the Eulerian approach.

Work performed to date confirms the potential of the PDF method. A general algorithm for computation of non-homogeneous turbulent flows has been written and tested. The boundary conditions for free-shear flows (with laminar flow entrainment) and bounded flows (in the presence of the wall) have been formulated. Nevertheless, as in any Monte Carlo method, slow statistical convergence remains a limitation. Actually, these methods used "as such", without quality improvement, tend to be prohibitively expensive with regard to the computational cost. To make the PDF method a viable tool for practical (engineering) computations, its constituent elements have to be further developed. They include: higher order numerical schemes for stochastic differential equations, variance reduction techniques, dynamical particle management (cloning, fusion) during simulation, and possibly also a coupling of the particle method with an Eulerian solver.

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Obliczenie płaskiego przepływu turbulentnego metodą funkcji gęstości prawdopodobieństwa

Streszczenie

Omówiono zastosowanie metody funkcji gęstości prawdopodobieństwa (PDF) do modelowania turbulencji, wraz z przykładami własnych obliczeń dla swobodnego przepływu ścinającego (strefa mieszania) i przepływu w prostej geometrii w obecności ścianek (kanał płaski). Wyszczególniono elementy algorytmu metody oraz sposoby redukcji szumu statystycznego uzyskiwanego rozwiązania. Omówiono perspektywy dalszego rozwoju metody PDF.

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