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RIGID SHEET-LIKE INTERFACE INCLUSION IN AN INFINITE BIMATERIAL PERIODICALLY LAYERED COMPOSITE

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This paper presents a potential function method for solving three-dimensional interface inclusion problem of a two-layered periodic space, treated within the framework of linear elasticity with microlocal parameters. By constructing the appropriate harmonic functions, the resulting boundary-value problems involving a thin rigid interface inclusion are reduced to classical mixed problems of potential theory. Further, an integral equation formulation for an arbitrary shaped inclusion is given to be used in numerical techniques.

Key words: periodic two-layered space, rigid interface thin-sheet inclusion, integral equations

1. Introduction

In the present contribution a continuation of earlier studies (see Yevtushenko et al., 1995; Kaczyński and Matysiak, 1997) into interface crack and thin rigid inclusion in periodic two-layered elastic composites is given. Effective results have been obtained using the method of microlocal homogenization presented by Woźniak (1987) and Matysiak and Woźniak (1988).

We aim at solving and carrying out the analysis of three-dimensional problems of a rigid lamellate interface inclusion embedded in a periodic two-layered unbounded composite.

In Section 2 the governing equations of homogenized model of the linear elasticity with microlocal parameters are briefly reviewed. The general representations of the displacement and stress distribution are also given. In Section 3, the problem of a sheet-like rigid inclusion lying on one of the interfaces of two-layered laminated space is posed and solved by using a method of potential functions similar in the form to that used in formulating the analogous crack problem. Next, the problem in terms of integral equations for an arbitrary shaped inclusion is given in Section 4.

The literature on the subject related to the problems under study but concerned with the inclusions in homogeneous solids is extensive (see, for example, Collins, 1962; Keer, 1965; Kassir and Sih, 1968; Selvadurai, 1980; Mura, 1982; Silovanyuk, 1984; Panasyuk et al., 1986, which are pertinent to the present study, and references therein).

2. Governing equations

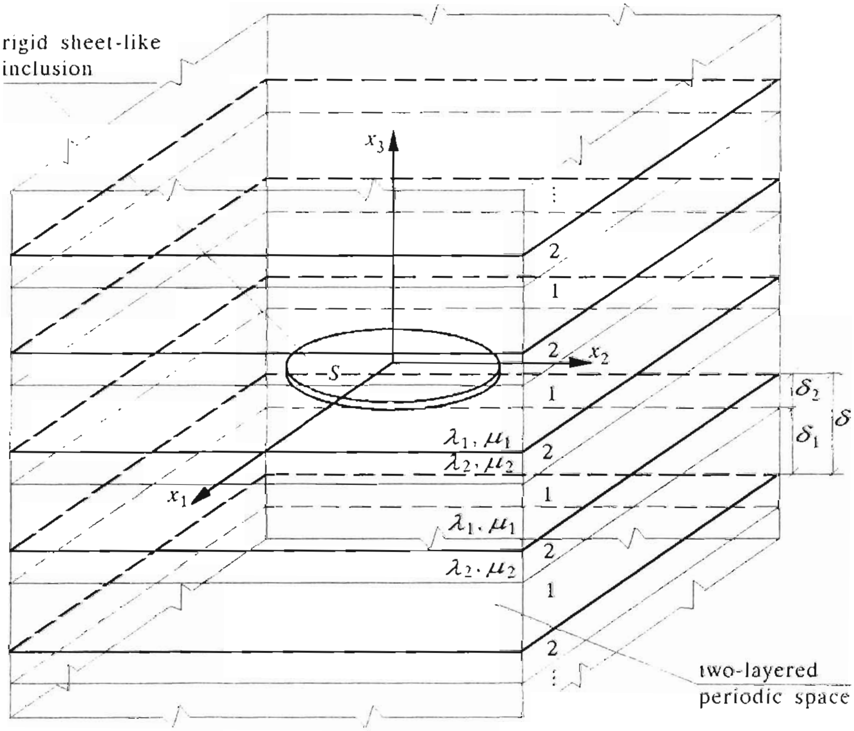

Fig. 1. Two-layered periodic space with an interface rigid plane inclusion

Consider a composite consisting of an infinite number of periodically repeating isotropic elastic layers of two different types, with thicknesses δ_l, $l = 1, 2$ ($\delta = \delta_1 + \delta_1$) and characterized by the Lamé constants λ_l, μ_l as shown in Fig.1;

herein, all the quantities (material constants, stresses, etc.) pertaining to the layers denoted by 1 and 2 will be associated with the index l or (l) taking the values 1 and 2, respectively.

Referring to the Cartesian coordinate system (x_1, x_2, x_3) with the x_3-axis normal to the layering, denote at the point $\boldsymbol{x} = (x_1, x_2, x_3)$ the displacement vector by $\boldsymbol{u} = [u_1, u_2, u_3]$ and the stresses by σ_{11}, σ_{12}, σ_{22}, σ_{13}, σ_{23}, σ_{33},.

We take into consideration the specific homogenization procedure called microlocal modelling (cf Woźniak, 1987; Matysiak and Woźniak, 1988) leading to certain macro-homogeneous model of the treated body with the following approximations[1]

$$\begin{aligned}
&u_i \cong w_i \qquad\qquad u_{i,\alpha}^{(l)} \cong w_{i,\alpha} \qquad\qquad u_{i,3}^{(l)} \cong w_{i,3} + h^{(l)} d_i\\
&\sigma_{\alpha\beta}^{(l)} \cong \mu_l(w_{\alpha,\beta} + w_{\beta,\alpha}) + \delta_{\alpha\beta}\lambda_l(w_{i,i} + h^{(l)} d_3)\\
&\sigma_{\alpha 3}^{(l)} \cong \mu_l(w_{\alpha,3} + w_{3,\alpha} + h^{(l)} d_\alpha)\\
&\sigma_{33}^{(l)} \cong (\lambda_l + 2\mu_l)(w_{3,3} + h^{(l)} d_3) + \lambda_l w_{\gamma,\gamma}
\end{aligned} \tag{2.1}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta and $h^{(l)}$ is the derivative of the assumed δ-periodic, sectionally linear shape function, defined as

$$\begin{aligned}
&h^{(l)} = \begin{cases} 1 & \text{if} \quad l = 1 \quad (\boldsymbol{x} \text{ belongs to 1st layer})\\ -\eta/(1-\eta) & \text{if} \quad l = 2 \quad (\boldsymbol{x} \text{ belongs to 2nd layer}) \end{cases}\\
&\eta = \frac{\delta_1}{\delta}
\end{aligned} \tag{2.2}$$

Moreover, w_i and d_i are unknown functions interpreted as the macrodisplacements and microlocal parameters, respectively.

The asymptotic approach to the modelling of the periodic laminated space under consideration leads to the following governing equations and constitutive relations of the homogenized model, given (after eliminating microlocal parameters and in the absence of body forces) in terms of the macrodisplacements w_i as follows (see Kaczyński, 1993)

$$\begin{aligned}
&\frac{1}{2}(c_{11} + c_{12}) w_{\gamma,\gamma\alpha} + \frac{1}{2}(c_{11} - c_{12}) w_{\alpha,\gamma\gamma} + c_{44} w_{\alpha,33} + (c_{13} + c_{44}) w_{3,3\alpha} = 0\\
&(c_{13} + c_{44}) w_{\gamma,\gamma 3} + c_{44} w_{3,\gamma\gamma} + c_{33} w_{3,33} = 0
\end{aligned} \tag{2.3}$$

[1]The indices i, j run over $1,2,3$ and are related to the Cartesian coordinates while the indices α, β, γ run over $1,2$. Summation convention holds unless orthewise stated.

$$\begin{aligned}
&\sigma^{(l)}_{\alpha 3} = c_{44}(w_{\alpha,3} + w_{3,\alpha}) && \sigma^{(l)}_{33} = c_{13}w_{\gamma,\gamma} + c_{33}w_{3,3} \\
&\sigma^{(l)}_{12} = \mu_l(w_{1,2} + w_{2,1}) && \sigma^{(l)}_{11} = d^{(l)}_{11}w_{1,1} + d^{(l)}_{12}w_{2,2} + d^{(l)}_{13}w_{3,3} \\
&\sigma^{(l)}_{22} = d^{(l)}_{12}w_{1,1} + d^{(l)}_{11}w_{2,2} + d^{(l)}_{13}w_{3,3}
\end{aligned} \tag{2.4}$$

Positive coefficients appearing in the above equations, describing the material and geometric properties of the composite constituents, are given in the Appendix. It should be emphasized that the condition of perfect bonding between the layers is satisfied (the components $\sigma_{3\alpha}$, σ_{33} do not depend on l implying the continuity of the stress vector at the interfaces). Finally observe, that setting $\mu_1 = \mu_2 \equiv \mu$, $\lambda_1 = \lambda_2 \equiv \lambda$ we obtain $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, passing directly to the well-known equations of elasticity for a homogeneous isotropic body (cf Kassir and Sih, 1975).

According to the results obtained by Kaczyński (1993), the general solution of the governing equations (2.3) may be expressed in terms of three harmonic potentials. However, the form of the representation depends on the material constants of the sublayers and will be given below in two cases[2].

Case 1: $\mu_1 \neq \mu_2$

Here, the displacement field is expressed by the potentials denoted by $\widehat{\varphi}_i(x_1, x_2, z_i)$, $z_i = t_i x_3$, such that

$$\nabla^2 \widehat{\varphi}_i = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z_i^2}\Big)\widehat{\varphi}_i = 0 \qquad \forall i \in \{1,2,3\}$$

as follows

$$\begin{aligned}
&w_1 = (\widehat{\varphi}_1 + \widehat{\varphi}_2)_{,1} - \widehat{\varphi}_{3,2} \qquad\qquad w_2 = (\widehat{\varphi}_1 + \widehat{\varphi}_2)_{,2} + \widehat{\varphi}_{3,1} \\
&w_3 = m_1 t_1 \frac{\partial \widehat{\varphi}_1}{\partial z_1} + m_2 t_2 \frac{\partial \widehat{\varphi}_2}{\partial z_2}
\end{aligned} \tag{2.5}$$

From the stress-displacement relations (2.4), the stresses σ_{3i} are expressed as

$$\begin{aligned}
\sigma_{31} &= c_{44}\Big[(1+m_1)t_1\frac{\partial \widehat{\varphi}_1}{\partial z_1} + (1+m_2)t_2\frac{\partial \widehat{\varphi}_2}{\partial z_2}\Big]_{,1} - t_3\frac{\partial^2 \widehat{\varphi}_3}{\partial z_3 \partial x_2} \\
\sigma_{32} &= c_{44}\Big[(1+m_1)t_1\frac{\partial \widehat{\varphi}_1}{\partial z_1} + (1+m_2)t_2\frac{\partial \widehat{\varphi}_2}{\partial z_2}\Big]_{,2} + t_3\frac{\partial^2 \widehat{\varphi}_3}{\partial z_3 \partial x_1} \\
\sigma_{33} &= c_{44}\Big[(1+m_1)\frac{\partial^2 \widehat{\varphi}_1}{\partial z_1^2} + (1+m_2)\frac{\partial^2 \widehat{\varphi}_2}{\partial z_2^2}\Big]
\end{aligned} \tag{2.6}$$

[2]The constants t_i, m_α appearing in the representations are defined in the Appendix.

For the purpose of further discussion the remaining stresses $\sigma_{\alpha\beta}^{(l)}$ are not of immediate interest.

Case 2: $\mu_1 = \mu_2 \equiv \mu$, $\lambda_1 \neq \lambda_2$

The governing equations (2.3) take the simple form

$$(B+\mu)w_{i,ij} + \mu w_{j,ij} = 0 \tag{2.7}$$

where

$$B = \frac{\lambda_1\lambda_2 + 2\mu[\eta\lambda_1 + (1-\eta)\lambda_2]}{(1-\eta)\lambda_1 + \eta\lambda_2 + 2\mu}$$

and the displacement representation is given now in terms of the three space harmonic functions $\varphi_i(x_1, x_2, x_3)$ as

$$\begin{aligned} &w_1 = (\varphi_1 + x_3\varphi_2)_{,1} - \varphi_{3,2} \qquad\qquad w_2 = (\varphi_1 + x_3\varphi_2)_{,2} + \varphi_{3,1} \\ &w_1 = \varphi_{1,3} + x_3\varphi_{2,3} - \frac{B+3\mu}{B+\mu}\varphi_2 \end{aligned} \tag{2.8}$$

The formulae for the corresponding stresses are

$$\begin{aligned} \sigma_{31} &= 2\mu\Big[\varphi_{1,3} - \frac{\mu}{B+\mu}\varphi_2 + x_3\varphi_{2,3}\Big]_{,1} - \mu\varphi_{3,23} \\ \sigma_{32} &= 2\mu\Big[\varphi_{1,3} - \frac{\mu}{B+\mu}\varphi_2 + x_3\varphi_{2,3}\Big]_{,2} + \mu\varphi_{3,13} \\ \sigma_{33} &= 2\mu\Big[\varphi_{1,33} - \frac{B+2\mu}{B+\mu}\varphi_{2,3} + x_3\varphi_{2,33}\Big] \end{aligned} \tag{2.9}$$

Note that putting in this case $\lambda_1 = \lambda_2 \equiv \lambda$ (then $B = \lambda$) we pass to the case of homogeneous isotropic body with the Lamé constants λ, μ.

3. Formulation of rigid plane inclusion problem and the method of solution

Consider a two-layered periodic space with a rigid sheet-like inclusion occupying the region S (of an arbitrary shape with a smooth boundary) in the x_1x_2-plane being one of the interfaces of the materials.

Within the framework of the homogenized model presented in Section 2 and applying the principle of superposition to satisfy the global mechanical boundary condition ensuring that the faces of inclusion are free from displacements, the problem is separated into two parts: the first one representing the multilayered space in the absence of the inclusion with the applied external loads and the second, corrective part in which the negative of the displacements (denoted by $w_{0i}(x_1, x_2)$) generated at the prospective inclusion faces in the first part are prescribed to the sites of the inclusion. Next, the attention is focused at the non-trivial perturbed problem, solution to which tends to zero at infinity and satisfies the necessary boundary condition

$$w_i = -w_{0i} \qquad \forall i \in \{1,2,3\} \quad \forall (x_1, x_2) \in S \tag{3.1}$$

assuming that the functions $w_{0i}(x_1, x_2)$ (including rigid body motion) are known from the solution to the first problem (i.e. without the inclusion).

Proceeding as in the homogeneous case considered by Silovanyuk (1984), we shall seek a solution in a potential form in the half-space $x_3 \geqslant 0$ taking into account the symmetry of stress state by summing solutions to two problems (denoted by (A) and (B)) with the following boundary conditions

$$\text{(A)} \quad \begin{cases} w_1 = w_2 = 0 & \forall (x_1, x_2) \in Z \\ w_3 = -w_{03} & \forall (x_1, x_2) \in S \\ \sigma_{33} = 0 & \forall (x_1, x_2) \in Z - S \end{cases} \tag{3.2}$$

$$\text{(B)} \quad \begin{cases} w_3 = 0 & \forall (x_1, x_2) \in Z \\ w_1 = -w_{01} \quad w_2 = -w_{02} & \forall (x_1, x_2) \in S \\ \sigma_{31} = \sigma_{32} = 0 & \forall (x_1, x_2) \in Z - S \end{cases} \tag{3.3}$$

where Z denotes the entire x_1x_2-plane.

We now proceed to reduction of the above-mentioned problems to the mixed boundary-value problems of potential theory related to a half-space. It will be done by constructing the potential functions well suited to the boundary conditions defined by Eqs (3.2) and (3.3).

Case 1, Problem (A)

The potentials representing the displacements (see Eqs (2.5)) are expressed by in terms of harmonic function $\widehat{f}(x_1, x_2, x_3)$ as follows

$$\begin{aligned} \widehat{\varphi}_1(x_1, x_2, z_1) &= -\widehat{f}(x_1, x_2, z_1) \\ \widehat{\varphi}_2(x_1, x_2, z_2) &= \widehat{f}(x_1, x_2, z_2) \\ \widehat{\varphi}_3 &\equiv 0 \end{aligned} \tag{3.4}$$

Note that on the boundary $x_3 = 0$ (then $z_1 = z_2 = 0$) this suitable representation automatically satisfies the condition $w_1 = w_2 = 0$ appearing in Eqs

(3.2). In addition, in view of Eqs (2.6), the quantities of interest are

$$
\begin{aligned}
w_3(x_1, x_2, 0^+) &= (m_2 t_2 - m_1 t_1)\Big[\widehat{f}_{,3}(x_1, x_2, x_3)\Big]_{x_3=0^+} \\
\sigma_{33}(x_1, x_2, 0^+) &= c_{44}(m_2 - m_1)\Big[\widehat{f}_{,33}(x_1, x_2, x_3)\Big]_{x_3=0^+}
\end{aligned}
\tag{3.5}
$$

Thus, the inclusion problem described by Eqs (3.2) is reduced to the classical mixed problem in the potential theory (cf Sneddon, 1966) of finding the harmonic function $\widehat{f}$ in the half-space $x_3 \geqslant 0$ with the boundary conditions

$$
\begin{aligned}
\Big[\widehat{f}_{,3}(x_1, x_2, x_3)\Big]_{x_3=0^+} &= -\frac{1}{m_2 t_2 - m_1 t_1} w_{03}(x_1, x_2) \qquad && \forall (x_1, x_2) \in S \\
\Big[\widehat{f}_{,33}(x_1, x_2, x_3)\Big]_{x_3=0^+} &= 0 && \forall (x_1, x_2) \in Z - S
\end{aligned}
\tag{3.6}
$$

Case 2, Problem (A)

An appropriate displacement representation in terms of a single harmonic function $f(x_1, x_2, x_3)$, which frees the plane $x_3 = 0$ of the displacement w_1, w_2 is obtained by taking in the general solution (2.8)

$$
\varphi_1 = \varphi_3 \equiv 0 \qquad \qquad \varphi_2 = f_{,3}
\tag{3.7}
$$

On the plane $x_3 = 0$ the corresponding displacement and stress components become

$$
w_3 = -\frac{B + 3\mu}{B + \mu} f_{,3} \qquad \qquad \sigma_{33} = -2\mu \frac{B + 2\mu}{B + \mu} f_{,33}
\tag{3.8}
$$

Application of the conditions (3.2) yields a similar problem to that appearing in Eqs (3.6)

$$
\begin{aligned}
\Big[f_{,3}(x_1, x_2, x_3)\Big]_{x_3=0^+} &= \frac{B + \mu}{B + 3\mu} w_{03}(x_1, x_2) \qquad && \forall (x_1, x_2) \in S \\
\Big[f_{,33}(x_1, x_2, x_3)\Big]_{x_3=0^+} &= 0 && \forall (x_1, x_2) \in Z - S
\end{aligned}
\tag{3.9}
$$

Case 1, Problem (B)

The procedure for satisfying the conditions (3.3) is more sophisticated than that used in the problem (A) and follows similarly to that used in the skew-symmetrical crack problem (cf Kassir and Sih, 1968; Kaczyński, 1993).

Let us introduce and define the six harmonic functions $G_i(x_1,x_2,x_3)$, $H_i(x_1,x_2,x_3)$, $i=1,2,3$

$$
\begin{aligned}
&\widehat{G}_i(x_1,x_2,z_i)=G(x_1,x_2,z_i) && \widehat{H}_i(x_1,x_2,z_i)=H(x_1,x_2,z_i) \\
&F_\alpha(x_1,x_2,z_\alpha)=\widehat{G}_{\alpha,1}+\widehat{H}_{\alpha,2} && \alpha=1,2 \\
&F_3(x_1,x_2,z_3)=\widehat{G}_{3,2}-\widehat{H}_{3,1} && \widehat{g}=G_{,3} \qquad \widehat{h}=H_{,3}
\end{aligned}
\tag{3.10}
$$

Their relationships to $\widehat{\varphi}_i$ (see Eqs (2.5)) are

$$
\widehat{\varphi}_1=-\frac{m_2t_2}{m_2t_2-m_1t_1}F_1 \qquad \widehat{\varphi}_2=\frac{m_1t_1}{m_2t_2-m_1t_1}F_2 \qquad \widehat{\varphi}_3=F_3
\tag{3.11}
$$

The quantities that need to be specified are found in the limit as $x_3\to 0^+$ to reduce to

$$
\begin{aligned}
&w_1(x_1,x_2,0^+)=\Big[\widehat{g}_{,3}\Big]_{x_3=0^+} && \frac{\sigma_{31}}{2\widehat{\mu}}=\widehat{g}_{,33}+\widehat{\nu}(\widehat{g}_{,22}-\widehat{h}_{,12}) \\
&w_2(x_1,x_2,0^+)=\Big[\widehat{h}_{,3}\Big]_{x_3=0^+} && \frac{\sigma_{32}}{2\widehat{\mu}}=\widehat{h}_{,33}+\widehat{\nu}(\widehat{h}_{,11}-\widehat{g}_{,12}) \\
&w_3(x_1,x_2,0^+)=0
\end{aligned}
\tag{3.12}
$$

in which

$$
2\widehat{\mu}=c_{44}\frac{t_1t_2(m_2-m_1)}{m_2t_2-m_1t_1} \qquad \widehat{\nu}=1-\frac{t_3}{2\widehat{\mu}}
\tag{3.13}
$$

Thus, the conditions (3.3) involve the reduction of the inclusion problem to that of finding the two harmonic functions $\widehat{g}$, $\widehat{h}$ which satisfy the mixed conditions on the x_1x_2-plane

$$
\begin{aligned}
&\widehat{g}_{,3}=-w_{01} \qquad \widehat{h}_{,3}=-w_{02} && \forall(x_1,x_2)\in S \\
&\left.\begin{aligned}&\widehat{g}_{,33}+\widehat{\nu}(\widehat{g}_{,22}-\widehat{h}_{,12})=0\\ &\widehat{h}_{,33}+\widehat{\nu}(\widehat{h}_{,11}-\widehat{g}_{,12})=0\end{aligned}\right\} && \forall(x_1,x_2)\in Z-S
\end{aligned}
\tag{3.14}
$$

This form is dual to the well-known one obtained for the shear loading crack problem (cf Kassir and Sih, 1975).

Case 2, Problem (B)

Following the some procedure as in the previous case, introduce two harmonic functions $G(x_1,x_2,x_3)$, $H(x_1,x_2,x_3)$ and denote $g=G_{,3}$, $H=H_{,3}$.

The potentials appearing in the displacement equations (2.8) are selected according to

$$\varphi_1 = -(G_{,1} + H_{,2}) \qquad \varphi_3 = G_{,2} - H_{,1}$$
$$\varphi_2 = -\frac{B+\mu}{B+3\mu}(G_{,13} + H_{,23}) \tag{3.15}$$

It can be shown then that the conditions (3.3) now imply the problem of finding the harmonic potentials g and h taking a similar form as that stated in Eqs (3.14)

$$g_{,3} = -w_{01} \qquad h_{,3} = -w_{02} \qquad \forall (x_1, x_2) \in S$$
$$\left.\begin{array}{l} g_{,33} + \nu(g_{,22} - h_{,12}) = 0 \\ h_{,33} + \nu(h_{,11} - g_{,12}) = 0 \end{array}\right\} \qquad \forall (x_1, x_2) \in Z - S \tag{3.16}$$

with

$$\nu = \frac{B+\mu}{2(B+2\mu)}$$

4. Integral equations

In the previous section, the boundary-value problems of interface inclusions have been written in terms of harmonic potentials satisfying the mixed boundary conditions in the half-space with the inclusion region S of separation. The forms of these conditions are proved to be dual to those of well-known from the corresponding crack problems (cf Kassir and Sih, 1975) and are related to the typical punch problems for a semi-infinite elastic solid (see, for example, Galin, 1980; Gladwell, 1980). Explicit solutions to the resulting potential problems are possible to obtain only for circular and elliptical shapes of the surface S (see the general survey presented by Sneddon (1966) and the results achieved by Kassir and Sih (1968), (1975)).

This section will be devoted to formulation of the problems for an arbitrary shaped thin-sheet rigid inclusion in terms of integral equations by using the same line of reasoning as in Panasyuk et al. (1986) for the corresponding problems in the homogeneous bodies.

The starting point of the solution consists in two-dimensional Fourier integral representations of harmonic potentials $\widehat{f}$, $\widehat{g}$, $\widehat{h}$ (Case 1) and f, g, h

(Case 2) written in the following general form

$$\begin{bmatrix} \widehat{f}(\boldsymbol{x}) \\ \widehat{g}(\boldsymbol{x}) \\ \widehat{h}(\boldsymbol{x}) \end{bmatrix} = \int\!\!\int_{-\infty}^{+\infty} \frac{\exp\Big[-x_3\sqrt{\xi_1^2+\xi_2^2} + \mathrm{i}(x_1\xi_1 + x_2\xi_2)\Big]}{\xi_1^2+\xi_2^2} \begin{bmatrix} \widehat{A}_f(\xi_1,\xi_2) \\ \widehat{A}_g(\xi_1,\xi_2) \\ \widehat{A}_h(\xi_1,\xi_2) \end{bmatrix} d\xi_1 d\xi_2$$

$$\begin{bmatrix} f(\boldsymbol{x}) \\ g(\boldsymbol{x}) \\ h(\boldsymbol{x}) \end{bmatrix} = \int\!\!\int_{-\infty}^{+\infty} \frac{\exp\Big[-x_3\sqrt{\xi_1^2+\xi_2^2} + \mathrm{i}(x_1\xi_1 + x_2\xi_2)\Big]}{\xi_1^2+\xi_2^2} \begin{bmatrix} A_f(\xi_1,\xi_2) \\ A_g(\xi_1,\xi_2) \\ A_h(\xi_1,\xi_2) \end{bmatrix} d\xi_1 d\xi_2 \tag{4.1}$$

where $\widehat{A}_f$, $\widehat{A}_g$, $\widehat{A}_h$ and A_f, A_g, A_h are unknown functions to be determined.

Putting (see Eqs $(3.5)_2$, $(3.6)_2$ and $(3.8)_2$, $(3.9)_2$)

$$\int\!\!\int_{-\infty}^{+\infty} \begin{bmatrix} c_{44}(m_2 - m_1)\widehat{A}_f(\xi_1,\xi_2) \\ [-2\mu(B+2\mu)/(B+\mu)]A_f(\xi_1,\xi_2) \end{bmatrix} \exp[\mathrm{i}(x_1\xi_1 + x_2\xi_2)]\, d\xi_1 d\xi_2$$

$$= \begin{cases} \sigma_{33}(x_1,x_2) & \forall (x_1,x_2) \in S \\ 0 & \forall (x_1,x_2) \in Z - S \end{cases} \tag{4.2}$$

and upon application of the inverse Fourier transform theorem, we find

$$\begin{bmatrix} \widehat{A}_f(\xi_1,\xi_2) \\ A_f(\xi_1,\xi_2) \end{bmatrix} = \begin{bmatrix} [c_{44}(m_2 - m_1)]^{-1} \\ -(B+\mu)/[2\mu(B+2\mu)] \end{bmatrix} \cdot$$

$$\cdot \frac{1}{4\pi^2} \int\!\!\int_{-\infty}^{+\infty} \sigma_{33}(\eta_1,\eta_2) \exp[-\mathrm{i}(\xi_1\eta_1 + \xi_2\eta_2)]\, d\eta_1 d\eta_2 \tag{4.3}$$

In a similar way, using the conditions $(3.14)_2$ and $(3.16)_2$ yields the following system

$$\begin{bmatrix} 1 - \dfrac{\widehat{\nu}\xi_2^2}{\xi_1^2+\xi_2^2} & \dfrac{\widehat{\nu}\xi_1\xi_2}{\xi_1^2+\xi_2^2} \\ \dfrac{\widehat{\nu}\xi_1\xi_2}{\xi_1^2+\xi_2^2} & 1 - \dfrac{\widehat{\nu}\xi_1^2}{\xi_1^2+\xi_2^2} \end{bmatrix} \begin{bmatrix} \widehat{A}_g(\xi_1,\xi_2) \\ \widehat{A}_h(\xi_1,\xi_2) \end{bmatrix} =$$

$$= \frac{1}{8\pi^2\widehat{\mu}} \int\!\!\int_S \begin{bmatrix} \sigma_{31}(\eta_1,\eta_2) \\ \sigma_{32}(\eta_1,\eta_2) \end{bmatrix} \exp[-\mathrm{i}(\eta_1\xi_1 + \eta_2\xi_2)]\, d\eta_1 d\eta_2 \tag{4.4}$$

which has the solution

$$
\begin{aligned}
(1-\widehat{\nu})\widehat{A}_g(\xi_1,\xi_2) &= \Big(1-\frac{\widehat{\nu}\xi_1^2}{\xi_1^2+\xi_2^2}\Big)\frac{1}{8\pi^2\widehat{\mu}}\cdot \\
&\cdot\iint\limits_S \sigma_{31}(\eta_1,\eta_2)\exp[-\mathrm{i}(\xi_1\eta_1+\xi_2\eta_2)]\,d\eta_1 d\eta_2 + \\
&-\frac{\widehat{\nu}\xi_1\xi_2}{\xi_1^2+\xi_2^2}\iint\limits_S \sigma_{32}(\eta_1,\eta_2)\exp[-\mathrm{i}(\xi_1\eta_1+\xi_2\eta_2)]\,d\eta_1 d\eta_2 \\
(1-\widehat{\nu})\widehat{A}_h(\xi_1,\xi_2) &= -\frac{\widehat{\nu}\xi_1\xi_2}{\xi_1^2+\xi_2^2}\frac{1}{8\pi^2\widehat{\mu}}\cdot \\
&\cdot\iint\limits_S \sigma_{31}(\eta_1,\eta_2)\exp[-\mathrm{i}(\xi_1\eta_1+\xi_2\eta_2)]\,d\eta_1 d\eta_2 + \\
&+\Big(1-\frac{\widehat{\nu}\xi_2^2}{\xi_1^2+\xi_2^2}\Big)\frac{1}{8\pi^2\widehat{\mu}}\iint\limits_S \sigma_{32}(\eta_1,\eta_2)\exp[-\mathrm{i}(\xi_1\eta_1+\xi_2\eta_2)]\,d\eta_1 d\eta_2
\end{aligned}
\tag{4.5}
$$

The same system and its solution is obtained for $A_g(\xi_1,\xi_2)$, $A_h(\xi_1,\xi_2)$ with the evident change $\widehat{\mu}$, $\widehat{\nu}$ into μ, ν, respectively.

Substituting Eqs (4.3) and (4.5) into Eqs (4.1) and making use of some standard integrals from Gradsteyn and Ryzhik (1965), we find that the remaining conditions $(3.6)_1$, $(3.9)_1$ and $(3.14)_1$, $(3.16)_1$ lead to the following integral equations for the stresses $\sigma_{3i}(x_1,x_2)$ in S

$$
\iint\limits_S \frac{\sigma_{33}(\eta_1,\eta_2)\,d\eta_1 d\eta_2}{\sqrt{(x_1-\eta_1)^2+(x_2-\eta_2)^2}} = w_{03}(x_1,x_2)\begin{cases} \dfrac{2\pi c_{44}(m_2-m_1)}{m_2t_2-m_1t_1} & \text{Case 1} \\[2mm] \dfrac{4\pi\mu(B+2\mu)}{B+3\mu} & \text{Case 2} \end{cases}
$$

$$
\begin{aligned}
&\iint\limits_S \Big\{\frac{\sigma_{31}(\eta_1,\eta_2)\,d\eta_1 d\eta_2}{\sqrt{(x_1-\eta_1)^2+(x_2-\eta_2)^2}}\Big[1-\frac{\widehat{\nu}(x_2-\eta_2)^2}{(x_1-\eta_1)^2+(x_2-\eta_2)^2}\Big] + \\
&+\frac{\widehat{\nu}\sigma_{32}(\eta_1,\eta_2)(x_1-\eta_1)(x_2-\eta_2)}{\sqrt{[(x_1-\eta_1)^2+(x_2-\eta_2)^2]^3}}\Big\}\,d\eta_1 d\eta_2 = 4\pi\widehat{\mu}(1-\widehat{\nu})w_{01}(x_1,x_2) \\
&\iint\limits_S \Big\{\frac{\sigma_{32}(\eta_1,\eta_2)\,d\eta_1 d\eta_2}{\sqrt{(x_1-\eta_1)^2+(x_2-\eta_2)^2}}\Big[1-\frac{\widehat{\nu}(x_1-\eta_1)^2}{(x_1-\eta_1)^2+(x_2-\eta_2)^2}\Big] + \\
&+\frac{\widehat{\nu}\sigma_{31}(\eta_1,\eta_2)(x_1-\eta_1)(x_2-\eta_2)}{\sqrt{[(x_1-\eta_1)^2+(x_2-\eta_2)^2]^3}}\Big\}\,d\eta_1 d\eta_2 = 4\pi\widehat{\mu}(1-\widehat{\nu})w_{02}(x_1,x_2)
\end{aligned}
\tag{4.6}
$$

(in Case 2 the replacement $\widehat{\mu}\to\mu$, $\widehat{\nu}\to\nu$ has to be made).

Once the stresses $\sigma_{3i}(x_1, x_2)$ acting on the plane S of the inclusion are known from the solution of the above integral equations, the stresses and displacements at any point of the composite can be found from the harmonic potentials $\widehat{f}$, $\widehat{g}$, $\widehat{h}$ (Case 1) and f, g, h (Case 2), determined from Eqs (4.1) by virtue of (4.3) and (4.5).

It is worth noting that the form of Eqs (4.6) is similar to that given for the corresponding homogeneous isotropic problem. Moreover, if we assume that the body under consideration is homogeneous, then in this case the derived integral equations are in agreement with those obtained by Silovanyuk (1984).

Analytical solutions of integral equations obtained for an inclusion of arbitrary shape appear to be beyond the capabilities of present mathematical techniques. For this reason, numerical methods must be used to obtain specific results (see, for instance, the treatment presented by Goldstein (1978)). Special attention is given to the investigation of stress intensification in the close neighborhood of the inclusion border (see the results closely related to the corresponding crack problems, obtained by Kassir and Sih (1968)).

A. Appendix

1. Denoting by $\eta = \delta_1/\delta$, $b_l = \lambda_l + 2\nu_l$ $(l = 1, 2)$, $b = (1-\eta)b_1 + \eta b_2$, the positive coefficients in the governing equations (2.3) and (2.4) are given by the following formulae

$$c_{33} = \frac{b_1 b_2}{b} \qquad c_{13} = \frac{(1-\eta)\lambda_2 b_1 + \eta\lambda_1 b_2}{b}$$

$$c_{11} = c_{33} + \frac{4\eta(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{b}$$

$$c_{12} = \frac{\lambda_1\lambda_2 + 2[\eta\mu_2 + (1-\eta)\mu_1][\eta\lambda_1 + (1-\eta)\lambda_2]}{b}$$

$$c_{44} = \frac{\mu_1\mu_2}{(1-\eta)\mu_1 + \eta\mu_2} \qquad d_{11}^{(l)} = \frac{4\mu_l(\lambda_l+\mu_l) + \lambda_l c_{13}}{b_l}$$

$$d_{12}^{(l)} = \frac{2\mu_l\lambda_l + \lambda_l c_{13}}{b_l} \qquad d_{13}^{(l)} = \frac{\lambda_l c_{33}}{b_l}$$

2. The constants appearing in Eqs (2.5) and (2.6) are given as follows

$$t_1 = \frac{1}{2}(t_+ - t_-) \qquad t_2 = \frac{1}{2}(t_+ + t_-)$$

$$t_3 = \sqrt{\frac{\eta\mu_1 + (1-\eta)\mu_2}{c_{44}}} \qquad m_\alpha = \frac{c_{11}t_\alpha^{-2} - c_{44}}{c_{13} + c_{44}} \qquad \forall\alpha \in \{1,2\}$$

provided

$$t_\pm = \sqrt{\frac{(A_\pm \pm 2c_{44})A_\mp}{c_{33}c_{44}}} \qquad A_\pm = \sqrt{c_{11}c_{33}} \pm c_{13}$$

Note that $t_1t_2 = \sqrt{c_{11}/c_{33}}$, $m_1m_2 = 1$.

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Sztywna cienka inkluzja miedzywarstwowa w nieskończonym, periodycznie dwuwarstwowym kompozycie

Streszczenie

W niniejszej pracy przedstawia się metodę potencjałów zastosowaną do rozwiązywania trójwymiarowych zagadnień inkluzji międzywarstwowej w periodycznie dwuwarstwowej przestrzeni, w ramach liniowej teorii sprężystości z parametrami mikrolokalnymi. Dobierając odpowiednie funkcje harmoniczne, zagadnienia brzegowe stawiane dla cienkiej (lamelkowej) inkluzji sprowadzają się do klasycznych mieszanych zagadnień teorii potencjału. Dla inkluzji o dowolnym kształcie podaje się następnie sformułowanie w postaci równań całkowych, możliwe do wykorzystania w obliczeniach numerycznych.

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