JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **37**, 1999

STATISTICAL AND EQUIVALENT LINEARIZATION TECHNIQUES WITH PROBABILITY DENSITY CRITERIA

LESŁAW SOCHA

Institute of Transport, Silesian Technical University
e-mail: lsocha@polsl.katowice.pl

The concept of statistical and equivalent linearization with probability density criteria for dynamic systems under Gaussian excitations is considered in the paper. New criteria of linearization and two approximate approaches are proposed. In the first one (statistical linearization) in order to establish the linearization coefficients and response characteristics the output probability density functions of static nonlinear element and the corresponding static linearized element are used in an iterative procedure. In the second approach (equivalent linearization) the direct minimization of a criterion based on output probability density functions of dynamic nonlinear system and the corresponding dynamic linearized system is proposed. The detailed analysis and numerical results are given for the Duffing oscillator.

Key words: random vibration, linearization techniques, Duffing oscillator

1. Introduction

The statistical and equivalent linearization techniques are basic tools in the study of stochastic dynamic systems. The earliest work in the field of statistical linearization theory for control engineering were carried out independently by Botton (1954) and Kazakov (1956). The objective of this method is to replace the nonlinear elements in a model by linear forms, where the coefficients of linearization can be found basing on the specific criterion of linearization. The equivalent linearization was first proposed by Caughey (1959), (1960) considered the replacement of a nonlinear oscillator by a linear one for which the coefficients of linearization can be found from a mean square criterion. In both approaches the linearization coefficients depend on the first and second order moments of the response. Due to incorrect derivation of equivalent linearization both the techniques in practical calculations for dynamic systems with Gaussian external excitations and mean-square criterion gave the same results. The difference between both approaches was observed first by Socha and Pawleta (1994) and independently by Elishakof and Colajanni (1997), (1998). Statistical linearization has been developed in the field of control, mechanical and structural engineering and has been generalized by many authors. Numerous studies have been performed, in the context of this method and are summarized in the monograph by Roberts and Spanos (1990) and the review by Socha and Soong (1991). In almost all studies into different versions of linearization techniques the difference between variances of nonlinear and linearized systems has been taken as a measure of the accuracy of the considered version. Since the full information about random variable is contained in probability density function Socha (1995), (1998) proposed new criteria of linearization in probability density space and two approximate approaches. He considered criteria depending on the difference between probability densities of responses of nonlinear and linearized systems. Unfortunately, both the approaches required complicated numerical calculations particularly in the case of nonlinear multi-degree-of-freedom systems. Basic difficulties are associated with determination of the approximate probability density function of the response of nonlinear system and with numerical calculations of multiple integrals. To overcome these difficulties we propose another approach called "Statistical linearization with probability density criteria". The objective of this method is to replace the nonlinear elements in a model by linear forms, where the coefficients of linearization can be found based on the criterion of linearization which is a probabilistic metric in the probability density space. The elements of this space are found as probability density functions of random variables obtained by linear and nonlinear transformation of one-dimensional Gaussian variable. The objective of this paper is to show the difference between both these approaches; i.e. statistical linearizations and equivalent with criteria in the probability density space. The detailed analysis and numerical results are given for the Duffing oscillator. To compare the characteristics of responses obtained by the proposed methods and other equivalent linearization techniques the examples with exactly known stationary probability density functions have been chosen.

2. Statistical linearization

Consider a nonlinear stochastic model of dynamic system described by the Ito vector differential equation

$$d\boldsymbol{x}(t) = \boldsymbol{\Phi}(\boldsymbol{x}, t)dt + \sum_{k=1}^{M} \boldsymbol{G}_k(t)d\xi_k(t) \tag{2.1}$$

where

$\boldsymbol{x}$	–	state vector, $\boldsymbol{x} = [x_1, ..., x_n]^\top$
$\boldsymbol{\Phi}$	–	nonlinear vector function, $\boldsymbol{\Phi} = [\Phi_1, ..., \Phi_n]^\top$
$\boldsymbol{G}_k$	–	deterministic vectors, $\boldsymbol{G}_k = [G_{k1}, ..., G_{kn}]^\top$, $k = 1, ..., M$
ξ_k	–	independent standard Wiener processes.

We assume that the unique solution of Eq (2.1) exists.

As was mentioned in the Introduction the objective of statistical linearization is to find for the nonlinear vector $\boldsymbol{\Phi}(\boldsymbol{x}, t)$ an equivalent one "in some sense" but in a linear form; i.e. replacing

$$\boldsymbol{Y} = \boldsymbol{\Phi}(\boldsymbol{x}, t) \tag{2.2}$$

in Eq (2.1) by a linearized form

$$\boldsymbol{Y} = \boldsymbol{\Phi}^0\Big(g_I(\boldsymbol{x}, t)\Big) + \mathsf{K}\Big(g_I(\boldsymbol{x}, t)\Big)\boldsymbol{x}^0 \tag{2.3}$$

where $g_I(\boldsymbol{x}, t)$ is the probability density function of the input process exciting on static element, $\boldsymbol{x}^0 = \boldsymbol{x} - E[\boldsymbol{x}]$ is the centralized process, $\boldsymbol{\Phi}^0 = [\Phi_1^0, ..., \Phi_n^0]^\top$ is a nonlinear vector function and $\mathsf{K} = [k_{ij}]$ is the $n \times n$ matrix of statistical linearization coefficients.

2.1. One-dimensional case

First consider one-dimensional nonlinearity. $\boldsymbol{\Phi}^0$ and K in Eq (2.3) are now scalars and their determination depends upon the choice of the equivalence criterion. In what follows the two equivalence criteria in probability density space are presented (Socha, 1995):

— square probability metric

$$I_1(t) = \int_{-\infty}^{+\infty} [g_N(y, t) - g_L(y, t)]^2 \, dy \tag{2.4}$$

where $g_N(y,t)$ and $g_L(y,t)$ are the probability density functions of variables defined by Eqs (2.2) and (2.3), respectively;

— pseudo-moment metric

$$I_2(t) = \int_{-\infty}^{+\infty} |y|^{2l} |g_N(y,t) - g_L(y,t)| \, dy \qquad l = 1, 2, ... \tag{2.5}$$

If we assume that the input process is a Gaussian process with the probability density function

$$g_I(x,t) = \frac{1}{\sqrt{2\pi}\sigma_x(t)} \exp\Big\{-\frac{[x(t) - m_x(t)]^2}{2\sigma_x^2(t)}\Big\} \tag{2.6}$$

where $m_x = E[x]$, $\sigma_x^2 = E[(x - m_x)^2]$ then the output process from the static linear element defined by Eq (2.3) is also Gaussian. In particular case, when $m_x = 0$ then Eq (2.3) in one-dimensional case reduces to the form

$$Y = kx \tag{2.7}$$

and the corresponding probability density function of variable Y is given by

$$g_L(y,t) = \frac{1}{\sqrt{2\pi}k\sigma_x(t)} \exp\Big[-\frac{[y(t)]^2}{2k^2\sigma_x^2(t)}\Big] \tag{2.8}$$

To apply the proposed criteria (2.4) and (2.5) we have to find the probability density function $g_N(y,t)$. Unfortunately, except for some special cases it is impossible to find it in an analytical form. It is well known that one of these special cases is a scalar strictly monotonically increasing or decreasing function

$$Y = \Phi(x) \tag{2.9}$$

with the continuous derivative $\Phi'(x)$ for all $x \in R$. Then the probability density function of the output variable (2.9) is given by

$$g_Y(y) = g_I\Big(h(y)\Big)|h'(y)| \tag{2.10}$$

where $g_I(x)$ is the probability density function of the input variable and h is the inverse function to $\Phi(x)$ i.e

$$x = h(Y) = \Phi^{-1}(Y) \tag{2.11}$$

In a general case when the nonlinear function $\Phi(x)$ is not strongly monotonically increasing or decreasing or not differentiable everywhere the approximation methods have to be used.

To obtain an approximate probability density function of nonlinear random variable $Y = \Phi(x)$ one can use for instance the Gram-Charlier expansion. In particular case for a scalar function Φ of a scalar random variable x the nonlinear variable has the probability density function

$$g_Y(y) = g_G(y)\Big[1 + \sum_{\nu=3}^{N} \frac{c_\nu}{\nu!} H_\nu\Big(\frac{y - m_y}{\sigma_y}\Big)\Big] \tag{2.12}$$

where

$$g_G(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\Big(-\frac{(y - m_y)^2}{2\sigma_y^2}\Big) \tag{2.13}$$

$m_y = E[y]$, $\sigma_y^2 = E[(y - m_y)^2]$, $c_\nu = E[G_\nu(y - m_y)]$, $\nu = 3, 4, ..., N$ are quasi-moments, $H_k(x)$ and $G_k(x)$ are the one-dimensional Hermite polynomials defined by

$$\begin{aligned} H_\nu(x) &= (-1)^\nu \exp\Big(\frac{x^2}{2\sigma^2}\Big) \frac{d^\nu}{dx^\nu} \exp\Big(-\frac{x^2}{2\sigma^2}\Big) \\ G_\nu(x) &= (-1)^\nu \exp\Big(\frac{x^2}{2\sigma^2}\Big) \Big[\frac{d^\nu}{dy^\nu} \exp\Big(-\frac{\sigma^2 y^2}{2}\Big)\Big]_{y=\frac{x}{\sigma^2}} \end{aligned} \tag{2.14}$$

In contrast to the standard statistical linearization with criteria in a state space one can not find the formulae for linearization coefficients in an analytical form. However, in some particular cases some analytical study can be done. For instance, for criterion I_1 defined by Eq (2.4) and for an input Gaussian process with mean equal to zero the necessary condition for minimum can be derived in the following form

$$\frac{\partial I_1(t)}{\partial k} = \int_{-\infty}^{+\infty} [g_N(y,t) - g_L(y,t)] \frac{1}{k} \Big(1 - \frac{y^2}{k^2 q^2}\Big) g_L(y,t) \, dy \tag{2.15}$$

To use the proposed linearization technique in determination of the linearization coefficient k and response characteristics one apply use an iterative procedure involving minimization of one of the proposed criteria and the solution of a Lyapunov differential equation. Such a procedure will be proposed in Section 4.

3. Equivalent linearization

As was mentioned in the Introduction the objective of equivalent linearization with criterion in probability density space is to find for the nonlinear dynamic system (2.1) an equivalent linear dynamic system in the form

$$d\boldsymbol{x}(t) = [\mathsf{A}(t)\boldsymbol{x}(t) + \boldsymbol{C}(t)]dt + \sum_{k=1}^{M} \boldsymbol{G}_k(t)d\xi_k(t) \tag{3.1}$$

where $\mathsf{A} = [a_{ij}]$ is a matrix and $\boldsymbol{C} = [C_1, ..., C_n]^\top$ is a vector of linearization coefficients, respectively.

The following criteria of linearization are proposed:

— square probability metric

$$I_3(t) = \int_{-\infty}^{+\infty} [g_N(y,t) - g_L(y,t)]^2 \, dy \tag{3.2}$$

— pseudo-moment metric

$$I_4(t) = \int_{-\infty}^{+\infty} |y|^{[2l]} |g_N(y,t) - g_L(y,t)| \, dy \qquad l = 1, 2, ... \tag{3.3}$$

where

$$y^{[2l]} = y_1^{q_1} y_2^{q_2} ... y_n^{q_n} \qquad \sum_{i=1}^{n} q_i = 2l \qquad l = 1, 2, ...$$

are the probability density functions of solutions of the nonlinear equation (2.1) and linearized system (3.1), respectively.

It can the noticed that the criteria for linearization (3.2) and (3.3) have the same form as those for statistical linearization but the quantities $g_N(y,t)$ and $g_L(y,t)$ have different meaning and the dimension of the variable y is different (1 for the statistical linearization and n for the equivalent linearization, respectively).

In the case of linearized system the probability density of the solution of Eq (3.1) is known and can be expressed as follows

$$g_L(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathsf{K}_L|}} \exp\Big[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m})^\top \mathsf{K}_L^{-1} (\boldsymbol{x} - \boldsymbol{m})\Big] \tag{3.4}$$

where $\boldsymbol{m} = \boldsymbol{m}(t) = E[\boldsymbol{x}(t)]$ and $\mathsf{K}_L = \mathsf{K}_L(t) = E[\boldsymbol{x}(t)\boldsymbol{x}^\top(t)] - \boldsymbol{m}(t)\boldsymbol{m}^\top(t)$ are the mean value and the covariance matrices, respectively, of the solution $\boldsymbol{x} = \boldsymbol{x}(t)$ of Eq (3.1), respectively, $|\mathsf{K}_L|$ denotes the determinant of the matrix K_L.

The vector $\boldsymbol{m}$ and matrix K_L satisfy the following equations

$$\begin{aligned} \frac{d\boldsymbol{m}}{dt} &= \mathsf{A}(t)\boldsymbol{m} + \boldsymbol{C}(t) \\ \frac{d\mathsf{K}_L}{dt} &= \mathsf{K}_L\mathsf{A}^\top(t) + \mathsf{A}(t)\mathsf{K}_L + \sum_{k=1}^{M} \boldsymbol{G}_k(t)\boldsymbol{G}_k^\top(t) \end{aligned} \tag{3.5}$$

To apply the proposed criterion (3.2) or (3.3) we have to find the probability density $g_N(\boldsymbol{x})$. Unfortunately, except for some special cases it is impossible to find the function $g_N(\boldsymbol{x})$ in the analytical form. However, it can be done by approximation methods or by simulations.

To obtain approximate probability density function of the stationary solution of a nonlinear dynamic system one can use, for instance, the Gram-Charlier expansion. For n-dimensional system the one-dimensional probability density function has the following truncated form (see Pugacev and Sinicin, 1987)

$$g_N(\boldsymbol{x}) \simeq g_{GC}(\boldsymbol{x}) = g_G(\boldsymbol{x})\Big[1 + \sum_{k=3}^{N} \sum_{\sigma(\boldsymbol{\nu})=k} \frac{c_{\boldsymbol{\nu}} H_{\boldsymbol{\nu}}(\boldsymbol{x}-\boldsymbol{m})}{\nu_1!...\nu_n!}\Big] \tag{3.6}$$

where $g_G(\boldsymbol{x})$ is the probability density of a vector Gaussian random variable $\boldsymbol{x} \in R^n$

$$g_G(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathsf{K}_G|}} \exp\Big[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^\top \mathsf{K}_G^{-1}(\boldsymbol{x}-\boldsymbol{m})\Big] \tag{3.7}$$

where

- $\boldsymbol{m}, \mathsf{K}_G$ – mean value and covariance matrix of vector variable $\boldsymbol{x}$, respectively
- $\boldsymbol{\nu}$ – multiindex, $\boldsymbol{\nu} = [\nu_1, ..., \nu_n]^\top$ and $\sigma(\boldsymbol{\nu}) = \sum_{i=1}^{n} \nu_i$
- N – number of elements in truncated
- $c_{\boldsymbol{\nu}}$ – quasi-moments, $c_{\boldsymbol{\nu}} = E[G_{\boldsymbol{\nu}}(\boldsymbol{x}-\boldsymbol{m})]$
- $H_{\boldsymbol{\nu}}, G_{\boldsymbol{\nu}}$ – Hermite polynomials defined by

$$\begin{aligned} H_{\boldsymbol{m}}(\boldsymbol{x}) &= (-1)^{\sigma(\boldsymbol{m})} \exp\Big(\frac{1}{2}\boldsymbol{x}^\top \mathsf{K}^{-1}\boldsymbol{x}\Big) \frac{\partial^{\sigma(\boldsymbol{m})}}{\partial x_1^{m_1}...\partial x_N^{m_n}} \exp\Big(-\frac{1}{2}\boldsymbol{x}^\top \mathsf{K}^{-1}\boldsymbol{x}\Big) \\ G_{\boldsymbol{m}}(\boldsymbol{x}) &= (-1)^{\sigma(\boldsymbol{m})} \exp\Big(\frac{1}{2}\boldsymbol{x}^\top \mathsf{K}^{-1}\boldsymbol{x}\Big) \Big[\frac{\partial^{\sigma(\boldsymbol{m})}}{\partial y_1^{m_1}...\partial y_N^{m_n}} \exp\Big(-\frac{1}{2}\boldsymbol{y}^\top \mathsf{K}^{-1}\boldsymbol{y}\Big)\Big]_{\boldsymbol{y}=\mathsf{K}^{-1}\boldsymbol{x}} \end{aligned} \tag{3.8}$$

where K is a real positive definite matrix.

To obtain the quasi-moments $c_{\boldsymbol{\nu}}$ we derive first, the moment equations for the system (2.1) which can be closed, for instance, by the cumulant closure technique and next we use algebraic relationships between quasi-moments and moments. As in the case of statistical linearization one can not find expressions for linearization coefficients in an analytical form and only in the case of criterion, for instance, I_3 established by Eq (3.2) the necessary condition of minimum can be derived in the following form

$$\begin{aligned} \frac{\partial I_1}{\partial a_{ij}} &= 2\int_{-\infty}^{+\infty} w(\boldsymbol{x})\frac{\partial \Psi(g_N,g_L)}{\partial g_L}\frac{\partial g_L(\boldsymbol{x})}{\partial a_{ij}}\,d\boldsymbol{x} = 0 \\ \frac{\partial I_1}{\partial C_i} &= 2\int_{-\infty}^{+\infty} w(\boldsymbol{x})\frac{\partial \Psi(g_N,g_L)}{\partial g_L}\frac{\partial g_L(\boldsymbol{x})}{\partial C_i}\,d\boldsymbol{x} = 0 \end{aligned} \tag{3.9}$$

4. Duffing oscillator

Consider the nonlinear Duffing oscillator

$$dx_1 = x_2 dt \qquad dx_2 = [-2\eta\omega_0 x_2 - f(x_1)]dt + q d\xi \tag{4.1}$$

where

$$f(x_1) = Y = \omega_0^2 x_1 + \varepsilon x_1^3 \tag{4.2}$$

$\omega_0, \eta, \varepsilon, q$ – positive constant parameters
ξ – standard Wiener process.

For simplicity we limit restrict our consideration to the stationary case. When we apply the statistical linearization technique to the nonlinear function (4.2) then one can show that the probability density function of the output variable Y is given by

$$g_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_L}\exp\Big[\frac{(v_1+v_2)^2}{2\sigma_L^2}\Big]\frac{1}{6a\varepsilon}\Big(\frac{a+2y}{v_1^2}+\frac{a-2y}{v_2^2}\Big) \tag{4.3}$$

where

$$\begin{aligned} v_1 &= \sqrt[3]{\frac{y}{2\varepsilon}+\sqrt{\frac{y^2}{\varepsilon^2}+\frac{\omega_0^6}{27\varepsilon^3}}} \qquad a = \sqrt{y^2+\frac{\omega_0^6}{27\varepsilon}} \\ v_2 &= \sqrt[3]{\frac{y}{2\varepsilon}-\sqrt{\frac{y^2}{\varepsilon^2}+\frac{\omega_0^6}{27\varepsilon^3}}} \end{aligned} \tag{4.4}$$

The probability density of the linearized variable

$$y = kx_1 \tag{4.5}$$

takes the form

$$g_L(y) = \frac{1}{\sqrt{2\pi}k\sigma_L} \exp\left(-\frac{y^2}{2k^2\sigma_L^2}\right) \tag{4.6}$$

where $\sigma_L^2 = E[\boldsymbol{x}]$ is the variance of the input Gaussian variable.

To find characteristics of stationary solution of the linearized system we propose the following iterative procedure.

Step 1. Substitute $\varepsilon = 0$ i.e. $k = \omega_0^2$, calculate the probability density of linearized element (one-dimensional) (4.6) for $\sigma_L^2 = q^2/(4h\omega_0^2)$.

Step 2. Consider a criterion, e.g., I_1

$$I_1 = \int_{-\infty}^{+\infty} [g_Y(y) - g_L(y)]^2 \, dy \tag{4.7}$$

where $g_Y(y)$ and $g_L(y)$ are the probability density functions (4.3) and (4.6), respectively, and find the coefficient $k_{\min}$ which minimizes criterion (4.7). Next, substitute $k = k_{\min}$.

Step 3. Calculate the stationary characteristics of the linearized oscillator response

$$\sigma_L^2 = \sigma_{x_1}^2 = E[x_1^2] = \frac{q^2}{4hk} \tag{4.8}$$

Step 4. Redefine the probability density functions for linearized and nonlinear elements, by substituting σ_L into Eqs (4.3) and (4.6), respectively.

Step 5. Repeat steps $2 \div 4$ until k and σ_L converge.

When we apply the equivalent linearization technique with probability density criteria to Eqs (4.1) and (4.2) then the equivalent linearized oscillator has the form

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= (-2\eta\omega_0 x_2 - kx_1)dt + q d\xi \end{aligned} \tag{4.9}$$

where k is a linearization coefficient.

To apply one of the considered criteria (Eq (3.2) or (3.3)) we use the probability density functions of stationary solutions of nonlinear and linearized oscillator which are known in the exact form

$$g_N(\boldsymbol{x}) = \frac{1}{c_N} \exp\Big[-\frac{2\eta\omega_0}{q^2}\Big(\omega_0^2 x_1^2 + \varepsilon\frac{x_1^4}{2} + x_2^2\Big)\Big]$$
$$g_L(\boldsymbol{x}) = \frac{1}{c_L}\frac{4\eta\omega_0\sqrt{k}}{q^2} \exp\Big[-\frac{2\eta\omega_0}{q^2}(kx_1^2 + x_2^2)\Big] \tag{4.10}$$

where c_N and c_L are normalized constants.

The necessary condition for minimum of criterion I_3 defined by (3.2) has the form

$$\begin{aligned}\frac{\partial I_3}{\partial k} = 2\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\Big\{\frac{1}{c_N}\exp\Big[-\frac{2\eta\omega_0}{q^2}\Big(\omega_0^2x_1^2 + \varepsilon\frac{x_1^4}{2} + x_2^2\Big)\Big] + \\ -\frac{1}{c_L}\frac{4\eta\omega_0\sqrt{k}}{q^2}\exp\Big[-\frac{2\eta\omega_0}{q^2}(kx_1^2 + x_2^2)\Big]\Big\}\cdot \\ \cdot\Big(\frac{1}{2k} - \frac{2\eta\omega_0 x_1^2}{q^2}\Big)\frac{1}{c_L}\frac{4\eta\omega_0\sqrt{k}}{q^2}\exp\Big[-\frac{2\eta\omega_0}{q^2}(kx_1^2 + x_2^2)\Big]\;dx_1dx_2 = 0\end{aligned} \tag{4.11}$$

In the case of pseudo-moment equivalent linearization the linearization coefficient we find by minimization of the following criterion

$$I_4 = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} x_1^{2l}|g_N(x_1,x_2) - g_L(x_1,x_2)|\;dx_1dx_2 \qquad l = 1,3 \tag{4.12}$$

The linearization coefficient k can be calculated numerically in the two considered cases from Eq (4.11) and directly from Eq (4.12).

To illustrate all the discussed methods we compare the stationary mean-square displacements of linearized systems obtained by applying the statistical and equivalent linearization techniques with criteria in probability density functions space for the Duffing oscillator. The numerical results for parameters $q^2 = 0.2$, $h = 0.5$, $\omega_0^2 = 1.85$, $\varepsilon = 1.85 \times i$, $i = 1, ..., 10$, are presented in Fig.1.

Fig.1 shows that for the second order moments of the displacement the relative errors obtained by (SPD-SL) and (SPD-EL) are almost the same while the errors obtained for $E[x_1^6]$ are quite different. The relative errors for both moments i.e. $E[x_1^2]$ and $E[x_1^6]$ for pseudo-moment metrics for the statistical linearization are significantly greater than the corresponding errors obtained by the equivalent linearization.

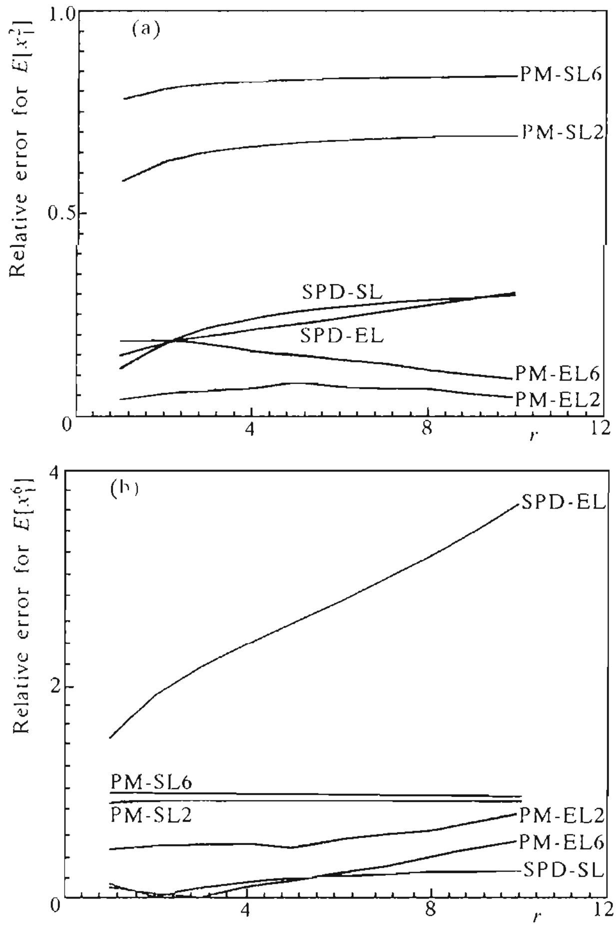

Fig. 1. The comparison between the relative errors of (a) – displacement variance $E[x_1^2]$, (b) – six order moments of the displacement $E[x_1^6]$, versus the ratio of parameters $r = \varepsilon/\omega_0^2$ for $\omega_0^2 = 1.85$, $\varepsilon = 1.85 \times i$, $i = 1, ..., 10$ and $q^2 = 0.2$, $h = 0.5$ (square metric probability density statistical linearization (SPD-SL); second order pseudo-moment statistical linearization (PM-SL2); six order pseudo-moment statistical linearization (PM-SL6); square metric probability density equivalent linearization (SPD-EL); second order pseudo-moment equivalent linearization (PM-EL2); six order pseudo-moment equivalent linearization (PM-EL6))

5. Conclusions and generalizations

The application of statistical and equivalent linearization techniques with criteria defined in the space of probability density functions to dynamic systems subjected to external Gaussian excitations has been considered. The differences between statistical and equivalent linearizations for two types of criteria have been discussed. The detailed analysis and numerical results have been obtained for the Duffing oscillator. The comparison has been shown for second and six order moments of the displacement. From the numerical results it follows that in the case of square metric the application of statistical linearization gives smaller relative errors than equivalent linearization for the six order moments and almost the same for the second order moments. In the case of pseudo-moment metrics for $l = 2$ and $l = 6$ and both moments i.e. $E[x_1^2]$ and $E[x_1^6]$ the relative errors obtained by the equivalent linearization are smaller than the corresponding errors obtained by the statistical linearization. The numerical results confirm the significant difference between both linearization techniques with criteria in the space of probability density functions which has been earlier observed for mean-square criterion in the state space (Soch and Pawleta, 1994; Elishakoff and Colojanni, 1997, 1998). Also from numerical it follows that statistical linearization could be recommended as a good mathematical tool with the criterion defined by square probabilistic metric.

We note that, similarly to the generalization obtained for the standard statistical linearization technique several new approaches of statistical linearization with criteria in probability density function space can be considered. It includes the cases of criteria depending on the probability density of energy of the response of nonlinear and linearized elements and linearization of stochastic dynamic systems under parametric excitations. Also other probabilistic measures (metrics) discussed in mathematical literature (cf Zolotarev, 1986) can be analyzed.

Acknowledgments

This study was partially supported by the State Committee for Scientific Research (KBN) under Grant No. 7 TO7A 074 08. The computer programming assistance given by M.Sc. A.Trzaska is gratefully acknowledged.

References

1. BOTTON R.C., 1954, Nonlinear Control Systems with Random Inputs, *IRE*

Transactions Circuit Theory, 1, 32-34

2. CAUGHEY T.K., 1959, Response of a Nonlinear String to Random Loading, *Trans. ASME J. of Appl. Mech.*, **81**, 345-348
3. CAUGHEY T.K., 1960, Random Excitation of a Loaded Nonlinear String, *Trans. ASME J. of Appl. Mech.*, **27**, 575-578
4. ELISHAKOFF I., COLAJANNI P., 1997, Stochastic Linearization Critically Re-Examined, *Chaos, Solitons and Fractals*, **8**, 1957-1972
5. ELISHAKOFF I., COLAJANNI P., 1998, Booton's Problem Re-Examined, *J. of Sound and Vibr.*, **210**, 683-691
6. KAZAKOV I.E., 1956, Approximate Probabilistic Analysis of the Accuracy of Operation of Essentially Nonlinear Systems, *Avtomatika i Telemekhanika*, **17**, 423-450
7. PUGACEV V.S., SINICYN I.N., 1987, *Stochastic Differential Systems*, Chichester: Willey
8. ROBERTS J.B., SPANOS P.D., 1990, *Random Vibration and Statistical Linearization*, J.Willey and Sons, Chichester
9. SOCHA L., 1995, Application of Probability Metrics to the Linearization and Sensitivity Analysis of Stochastic Dynamic Systems, *Proc. International Conference on Nonlinear Stochastic Dynamics*, Hanoi, Vietnam, 193-202
10. SOCHA L., PAWLETA M., 1994, Corrected Equivalent Linearization, *Machine Dynamics Problems*, **7**, 149-161
11. SOCHA L., SOONG T.T., 1991, Linearization in Analysis of Nonlinear Stochastic Systems, *Appl. Mech. Reviews*, **10**, 399-422
12. SOCHA L., 1998, Probability Density Equivalent Linearization Technique for Nonlinear Oscillator with Stochastic Excitations, *Z. Angew. Math. Mech.*, **78**, 1087-1088
13. ZOLOTAREV W.M., 1986, *Modern Theory of Summation Independent Random Quantities*, Moscau: Nauka, (in Russian)

Statystyczna i równoważna linearyzacja z kryteriami w przestrzeni funkcji gęstości prawdopodobieństw

Streszczenie

W pracy zaproponowano metodę statystycznej i równoważnej linearyzacji dla układów dynamicznych z wymuszeniami o charakterze białych szumów Gaussowskich i kryteriami w przestrzeni funkcji gęstości prawdopodobieństw. Przy wyznaczaniu współczynników linearyzacji i charakterystyk rozwiązań układu dynamicznego

za pomocą metody statystycznej linearyzacji korzysta się z minimalizacji kryterium uwzględniającego różnicę wyjściowych funkcji gęstości prawdopodobieństw odpowiednio elementu nieliniowego i zlinearyzowanego oraz pewnej procedury iteracyjnej. W przypadku równoważnej linearyzacji przeprowadza się bezpośrednią minimalizację kryterium uwzględniającego różnicę wyjściowych funkcji gęstości prawdopodobieństw odpowiednio układów dynamicznych nieliniowego i zlinearyzowanego. Szczegółową analizę i obliczenia numeryczne przeprowadzono dla oscylatora Duffinga.

Manuscript received November 26, 1998; accepted for print December 18, 1998