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APPLICATION OF THE REITERATED HOMOGENIZATION TO DETERMINATION OF EFFECTIVE MODULI OF A COMPACT BONE

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The aim of the paper is twofold. First, the available results of finding the effective macroscopic elastic moduli of a compact bone by using homogenization are surveyed. Secondly, it is shown that the proper framework for studying such organic materials with hierarchical microstructure is that of reiterated homogenization. Γ-convergence theory is applied to obtain the general formulae for the effective elastic moduli of a material with three structural levels.

Key words: compact bone, homogenization, reiterated homogenization

1. Introduction

Animal and human bones are porous materials of a complicated hierarchical structure. Bones occur in the two forms: as a dense solid (*compact bone*) and as a porous network of interconnected rods and plates (*cancellous or trabecular bone*). The most obvious difference between these two types of bones consists in their relative densities measured by volume fraction of solids, cf Fig.1 and Fig.2.

A bone with a volume fraction of less than 70% is classified as the cancellous one while that over 70% is compact (Gibson and Ashby, 1988).

Bone cells produce the two types of tissue, namely, highly organized lamellar bone and poorly organized woven bone. When the lamellar bone occurs in the midshaft of a long bone, it consists of concentrically arranged laminae as illustrated in Fig.3.

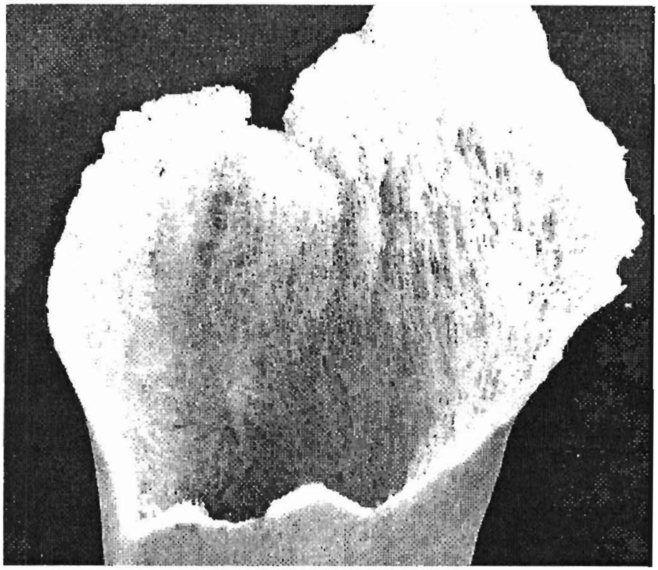

Fig. 1. Photograph of proximal part of the human femur

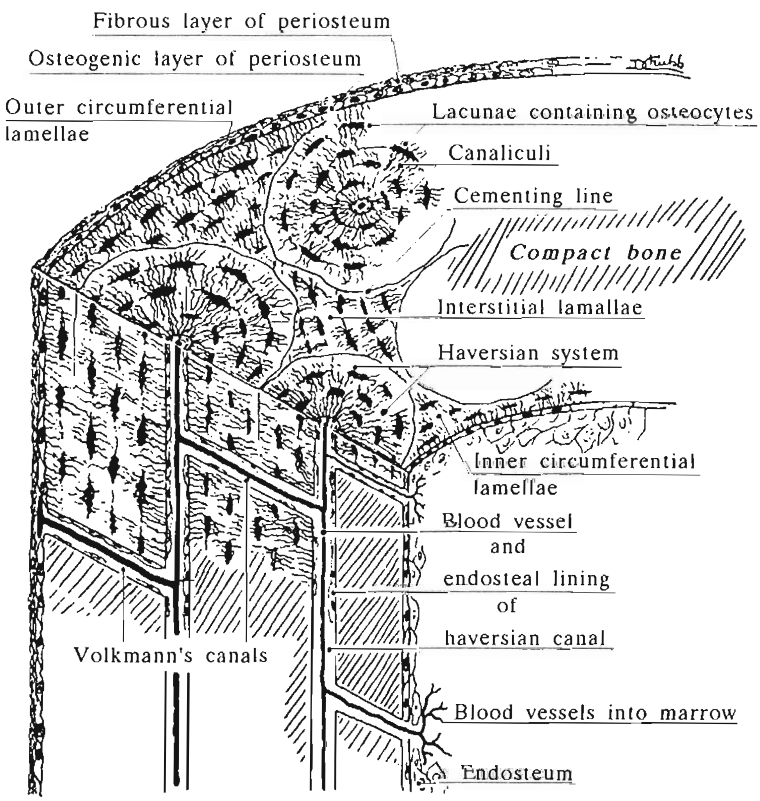

Fig. 2. The basic structure of compact bone, after Fung (1981)

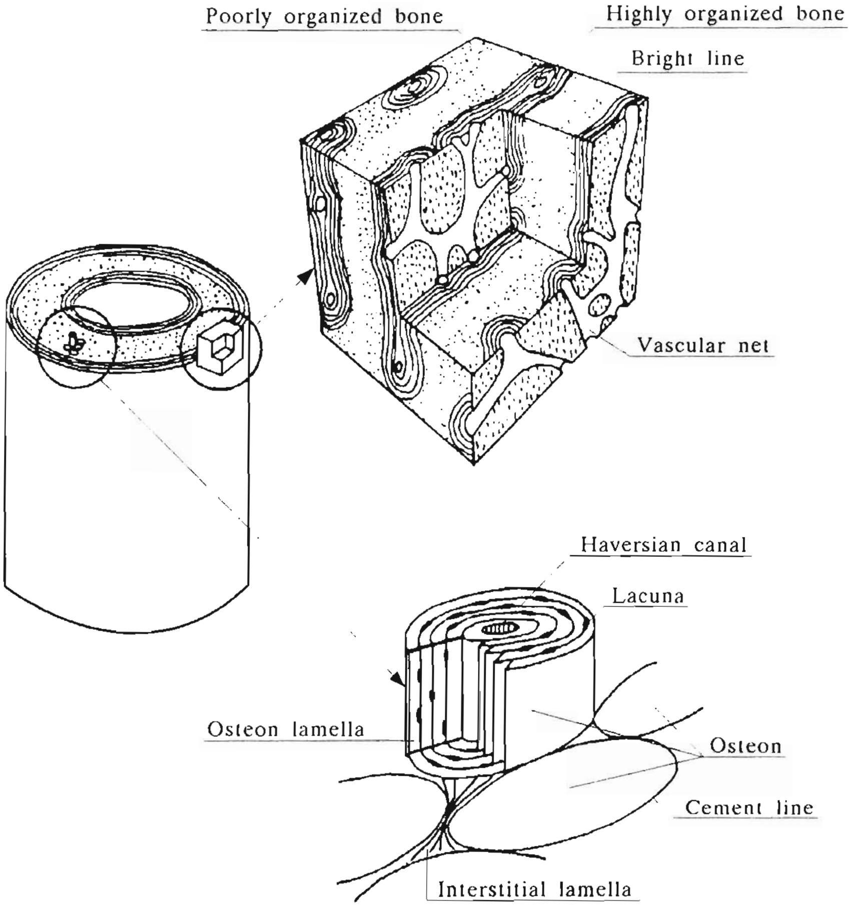

Fig. 3. Typical bone structure in the diaphysis of the femur, after Cowin (1989a)

The thickness of the lamina is about 200 μm. Between each lamina and the next there is a net-like system of blood vessels which is essentially a surface. Occasional large radial vessels through a lamina connect the surface nets. Each lamina is divided into the three zones shown in Fig.3. The first zone, which extends from the surface of vascular network to about one third of the way across the lamina is composed of highly organized dense bone. The second zone, which extends the next one third of the distance, is composed of poorly organized tissue. This zone is breaked in the middle by a line that, under an ordinary microscope, appears to be bright. This bright line is the boundary between the two blood supply networks bounding the lamina.

Cortical haversian bone is also ilustrated in Fig.3 and its structure is further detailed in Fig.4. It consists of quasi-cylindrically shaped elements called *osteons* or *haversian systems*. The individual haversian systems are composed of concentric lamellae about $3 \div 7\,\mu$m thick. These thin lamellae, in turn, are constructed from wrapped collagen fibers impregnated at regularly spaced sites with hydroxyapatite and other mineral crystals about $20 \div 40$ mm long.

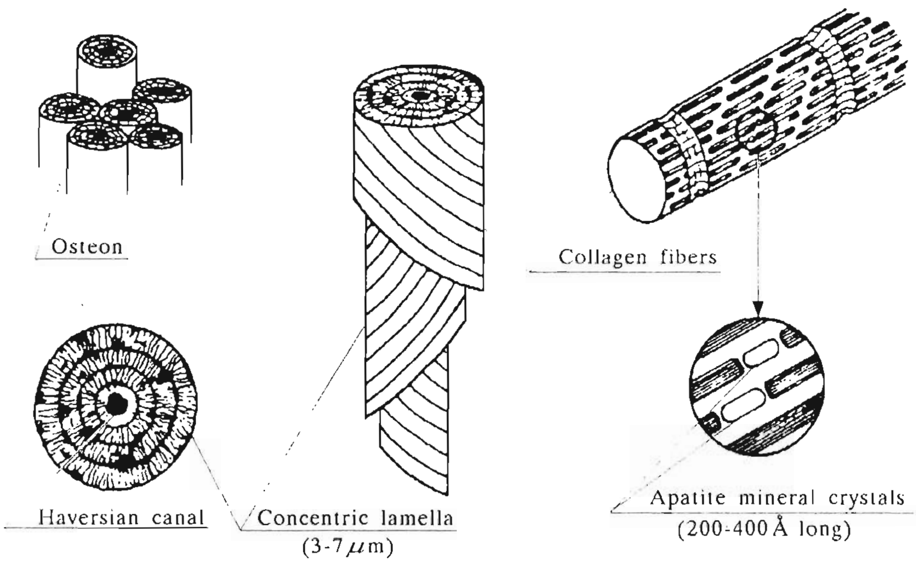

Fig. 4. The detailed structure of an osteon, after Cowin (1989a)

This structure is ilustrated in Fig.4. Osteons are typically about $200\,\mu$m in diameter, the same thickness as the laminae in a laminar bone, and about 10 to 20 mm long. The thickness is the same because the blood supply for the haversian system is a central lumen containing a blood vessel, and thus every point in the haversian system is no more than $100\,\mu$m away from the blood supply, as in the case of laminar bone. Haversian bone is organized to accomodate small arteries, arterioles, capillaries and venules of the microcirculating system. Haversian bone is never formed as a primary effect, but forms as the result of the vascular invasion of bone. In young animals, woven bone is formed initially, the endosteal capillaries invade the avascular bone forming haversian systems.

The osteons of haversian bone and the laminae of laminar bone are basically just different geometric configurations of the same material. In both geometric configurations no point in the tissue is more than $100\,\mu$m away from the blood supply. The interfaces between the laminae in both haversian and lamina bones contain an array of roughly ellipsoidal-shaped cavities called the *lacunae* which contain bone cells, and from which extend numerous fine canals called the canaliculi. The thin layer between adjacent osteons is called the *cement line* and the three-dimensional region between osteons is filled with irregular pieces of lamellar bone. The canaliculi do not cross the cement line nor do they cross the bright lines between laminae in laminar bone.

Both haversian and laminar bones occur simultaneously in the long human bones and in many animal bones, including cattle. In very young, the long bones are composed of woven bone with a few osteons, called *primary osteons*. In the process of maturation the woven bone is being converted to a laminar bone and, at maturity, there is a partial conversion to haversian bone. According to Cowin (1989a), the conversion from laminar to haversian bone is something of a biological enigma. Haversian bone is known to have a less efficient local circulation system and to have less mechanical strength compared to a laminar bone, yet the percentage of haversian bone generaly increases with age.

The bone tissue is composed of roughly speaking, equal thirds by volume of minerals, water, and the extracellular collagenous matrix. If one tries to be more more precise about the bone composition, then one must specify species, age, sex, specific bone in question, type of bone tissue (cancellous or cortical), and whether the individual is experiencing a bone disease or not.

Smith (1960) proved the existence of several types of osteons composed of concentric lamellae. Ascenzi and Bonucci (1967, 1968, 1972, 1976) and Ascenzi et al. (1966, 1973) described the structure of bone consisting of three types of osteons with lamellae and fibers within these lamellae. Frasca (1974) and Katz (1976) described the fourth type of osteon, cf also Frasca and Harper (1977). For further results of investigations into properties of osteons and lamellae the reader is referred to Ascenzi et al. (1966, 1982, 1983, 1985a,b, 1986, 1987, 1990, 1994, 1997, 1998).

The properies of single osteonic lamellae were studied by Ascenzi et al. (1982, 1983), Portigliatti Barbos et al. (1984) and Frasca and Harper (1977).

Both types can be found in most bones in the body, the dense compact bone forming an outer shell surrounding a core of spongy cancellous bone, see Gibson and Ashby (1988), Lowet et al. (1997). An idealization of compact bone structure is shown in Fig.2 ÷ Fig.4.

Bone may be viewed as a structurally hierarchical porous material. It is then possible to use the reiterated homogenization (Bensousan et al., 1978) to derive the formulae for the macroscopic elastic moduli, cf Aoubiza (1991), Aoubiza et al. (1996), Crolet (1990), Crolet et al. (1993). Optimal design of structures often involves homogenization and relaxation methods (Bendsøe, 1995; Bendsøe and Kikuchi, 1988; Kohn and Strang, 1986; Lewiński and Telega, 1999; Lurie et al., 1982). Such an approach may be used to model bone microstructure via adaptive elasticity. Payten et al. (1998) presented an optimisation process that has, as its basis, an algorithm originally developed for predicting anatomical density distributions in natural human bone.

The microstructure of bone is such that at the macroscopic level its behaviour is anisotropic. To model bone anisotropy one can use Cowin's fabric tensor, see Cowin (1989b), Jemioło and Telega (1998); Lowet et al. (1997) and the references cited therein. Jemioło and Telega (1998) showed that the compact bone is close to transverse isotropy whilst trabecular bone is approximately orthotropic, cf also Zysset et al. (1998). The approach employed by Jemioło and Telega (1998) exploits Cowin's fabric tensor. In Zysset et al. (1998) the authors claimed to use homogenization method for finding orthotropic elastic constants of trabecular bone yet no precise formulation was given. In the papers by Tokarzewski et al. (1998, 1999), Gałka et al. (1999) the problem of finding effective elastic moduli of trabecular bone was investigated. In the last three papers numerical solving of the local problem was avoided.

The aim of the present contribution is twofold. First, in Section 2 we develop a general scheme of reiterated homogenization within the framework of the theory of Γ-convergence, see Dal Maso (1993). We observe that the reiterated homogenization developed by Bensoussan et al. (1978) is limited to scalar problems and asymptotic developments. In the standard book on homogenization by Sanchez-Palencia (1980) the reiterated homogenization was not discussed. Second, in Section 3 the available results of finding the macroscopic elastic moduli of compact bone are reviewed. We mean here the results obtained by Aoubiza (1991), Aoubiza et al. (1996), Crolet (1990) and Crolet et al. (1993). In fact, the macroscopic moduli are derived provided that the microscopic organization of bone is specified by the elasticity tensor

$$C^{\varepsilon}_{ijkl}(\boldsymbol{x}) = C_{ijkl}\Big(\frac{\boldsymbol{x}}{\varepsilon}, \frac{\boldsymbol{x}}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^3}\Big) \qquad i,j,k,l = 1,2,3 \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter. There are thus the three microscopic levels specified by $\boldsymbol{x}/\varepsilon$, $\boldsymbol{x}/\varepsilon^2$, $\boldsymbol{x}/\varepsilon^3$. The determination of the macroscopic moduli C^h_{ijkl} means passing to zero with ε and leads to the so-called *reiterated homogenization.*

2. Reiterated homogenization via Γ-convergence

In the next section the compact bone is modelled as a material with a hierarchical structure. Only three structural levels are considered. It is thus reasonable to assume that the elasticity tensor is given by

$$C^{\varepsilon}_{ijkl}(\boldsymbol{x}) = C_{ijkl}\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}, \frac{\boldsymbol{x}}{\varepsilon^2}, \frac{\boldsymbol{x}}{\varepsilon^3}\Big) \tag{2.1}$$

and $C_{ijkl}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ is $Y_1 \times Y_2 \times Y_3$-periodic in the second, third and fourth variables. Here $\boldsymbol{y}_i = \boldsymbol{x}/\varepsilon^i$, $\boldsymbol{y}_i \in Y_i$. Particularly, it may happen that $Y_1 = Y_2 = Y_3$, cf Allaire and Briane (1996). We make the following assumptions, cf Bensoussan et al. (1978), Allaire and Briane (1996)

(i) $C^{\varepsilon}_{ijkl} \in L^{\infty}(\Omega)$

(ii) $\exists\ c_1 \geqslant c_0$ such that $\forall \mathbf{E} \in \mathbb{E}^3_s$, $c_0 E_{ij}E_{ij} \leqslant C_{ijkl}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)E_{ij}E_{kl} \leqslant c_1 E_{ij}E_{ij}$ almost everywhere in $\Omega \times Y_1 \times Y_2 \times Y_3$.

Here $\mathbb{E}^3_s$ denotes the space of symmetric 3×3 matrices.

Let $\Omega \subset \mathbb{R}^3$ be a bounded, sufficiently regular domain representing the linear elastic body in its undeformed configuration. For a fixed $\varepsilon > 0$ the functional of the total potential energy is given by

$$J_{\varepsilon}(\boldsymbol{u}) = G_{\varepsilon}(\boldsymbol{u}) - L(\boldsymbol{u}) \tag{2.2}$$

where

$$G_{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\int\limits_{\Omega} C^{\varepsilon}_{ijkl}(\boldsymbol{x}) e_{ij}(\boldsymbol{u}) e_{kl}(\boldsymbol{u})\ d\boldsymbol{x} \tag{2.3}$$

and $L(\boldsymbol{u})$ stands for the functional of the external loading. For instance, if the body is subjected to body forces $\boldsymbol{f} = (f_i)$ only, then

$$L(\boldsymbol{u}) = \int\limits_{\Omega} f_i u_i\ d\boldsymbol{x} \tag{2.4}$$

The strain tensor $\mathbf{e}(\boldsymbol{u})$ is linear, i.e., one has

$$e_{ij}(\boldsymbol{u}) = u_{(i,j)} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{2.5}$$

To perform homogenization when $\varepsilon \to 0$ the precise form of L is not required. It is sufficient to assume that L is a so called perturbation functional, continuous in the weak topology of $H^1(\Omega)^3 = [H^1(\Omega)]^3$.

Applying the Γ-convergence theory we conclude that the homogenized functional J_h is given by

$$J_h(\boldsymbol{u}) = \frac{1}{2}\int\limits_{\Omega} C^h_{ijkl}(\boldsymbol{x}) e_{ij}(\boldsymbol{u}) e_{kl}(\boldsymbol{u})\, d\boldsymbol{x} - L(\boldsymbol{u}) \tag{2.6}$$

where the macroscopic elasticity tensor $\mathbf{C}^h$ is defined by the inductive homogenization formula:

(a) $\mathbf{C}^{(3)} = \mathbf{C}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$

(b) $\mathbf{C}^{(2)} = \mathbf{C}^{(2)}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2)$ is obtained by using periodic homogenization of $\mathbf{C}^{(3)}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{z}/\varepsilon)$

(c) $\mathbf{C}^{(1)} = \mathbf{C}^{(1)}(\boldsymbol{x}, \boldsymbol{y}_1)$ is obtained by using periodic homogenization of $\mathbf{C}^{(2)}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{z}/\varepsilon)$

(d) $\mathbf{C}^h = \mathbf{C}^{(0)}(\boldsymbol{x})$ is obtained by using periodic homogenization of $\mathbf{C}^{(1)}(\boldsymbol{x}, \boldsymbol{z}/\varepsilon)$.

More precisely, to obtain the moduli $\mathbf{C}^{(2)}$, $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(0)}$ we proceed as follows:

(1)

$$\begin{aligned} C^{(2)}_{mnpq}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2) &= \frac{\partial^2 W_2}{\partial E_{pq}\partial E_{mn}} = \\ &= \frac{1}{|Y_3|}\int\limits_{Y_3} C_{ijpq}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y})\Big(e^y_{ij}(\boldsymbol{\chi}^{(mn)}) + \delta_{im}\delta_{jn}\Big)\, d\boldsymbol{y} \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} W_2(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \mathbf{E}) = \inf\Big\{ &\frac{1}{|Y_3|}\int\limits_{Y_3} C_{ijkl}(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}) \cdot \\ &\cdot\Big(e^y_{ij}(\boldsymbol{v}) + E_{ij}\Big)\Big(e^y_{kl}(\boldsymbol{v}) + E_{kl}\Big)\, d\boldsymbol{y} \Big| \boldsymbol{v} \in \widetilde{H}^1_{per}(Y_3)^3 \Big\} \end{aligned} \tag{2.8}$$

$\mathbf{E} \in \mathbb{E}^3_s$ and $\widetilde{H}^1_{per}(Y_3) = \Big\{ v \in H^1(Y_3)\ v$ assumes equal values on the opposite faces of Y_3, $\langle v\rangle_{Y_3} = 0\Big\}$.

Here

$$e^y_{ij}(\boldsymbol{v}) = \frac{1}{2}\Big(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\Big) \qquad \langle v\rangle_{Y_k} = \frac{1}{|Y_k|}\int\limits_{Y_k} v\, dy \tag{2.9}$$

The function $\widetilde{\boldsymbol{v}}$, the solution to the minization problem on the r.h.s. of Eq (2.8) depends linearly on E, i.e., $\widetilde{\boldsymbol{v}} = \boldsymbol{\chi}^{(mn)}E_{mn}$. The functions $\boldsymbol{\chi}^{(mn)}$ are solutions to the following local problem

$$\begin{gathered} \boldsymbol{\chi}^{(mn)} \in \widetilde{H}^1_{per}(Y_3)^3 \ : \\ \int\limits_{Y_3} C_{ijkl}(\boldsymbol{x},\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)\Big(e^y_{ij}(\boldsymbol{\chi}^{(mn)}) + \delta_{im}\delta_{jn}\Big)e^y_{kl}(\boldsymbol{w})\, d\boldsymbol{y}_3 = 0 \end{gathered} \tag{2.10}$$

for any $\boldsymbol{w} \in \widetilde{H}^1_{per}(Y_3)^3$. Obviously, the functions $\boldsymbol{\chi}^{(mn)}(\boldsymbol{y}_3)$ depend also on $\boldsymbol{x}, \boldsymbol{y}_1$ and $\boldsymbol{y}_2$.

We observe that with $\mathsf{C}^{(2)}$ one can associate the *microstresses* of the *second order*

$$\sigma^{(2)}_{ij} = C^{(2)}_{ijkl}e_{kl}(\boldsymbol{u})$$

Similarly, with $\mathsf{C}^{(1)}$ the *microstresses* of the *first order* $\boldsymbol{\sigma}^{(1)}$ are associated. Thus, it seems that the reiterated homogenization opens a new and rigorous way to the study of microstresses, much discussed in solid mechanics.

(2) The moduli $C^{(1)}_{ijkl}(\boldsymbol{x},\boldsymbol{y}_1)$ are found similarly.

(3) Finally, the macroscopic elastic moduli $C^h_{ijkl}(\boldsymbol{x}) = C^{(0)}_{ijkl}(\boldsymbol{x})$ are given by

$$\begin{aligned} C^h_{ijkl}(\boldsymbol{x}) &= \frac{\partial^2 W_h}{\partial E_{pq}\partial E_{mn}} = \\ &= \frac{1}{|Y_1|}\int\limits_{Y_1} C^{(1)}_{ijpq}(\boldsymbol{x},\boldsymbol{y})\Big(e^y_{ij}(\boldsymbol{\Phi}^{(mn)}) + \delta_{im}\delta_{jn}\Big)e^y_{kl}(\boldsymbol{w})\, d\boldsymbol{y} \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &W_h(\boldsymbol{x},\mathsf{E}) = \\ &= \inf\Big\{\frac{1}{|Y_1|}\int\limits_{Y_1} C^{(1)}_{ijkl}(\boldsymbol{x},\boldsymbol{y})\Big(e^y_{ij}(\boldsymbol{\xi}) + E_{ij}\Big)\Big(e^y_{kl}(\boldsymbol{\xi}) + E_{kl}\Big)\, d\boldsymbol{y}\Big|\boldsymbol{\xi} \in \widetilde{H}^1_{per}(Y_1)^3\Big\} \end{aligned} \tag{2.12}$$

$\mathsf{E} \in \mathbb{E}^3_s$, and

$$\boldsymbol{\Phi}^{(mn)} \in \widetilde{H}^1_{per}(Y_1)^3 : \int\limits_{Y_1} C^{(1)}_{ijkl}(\boldsymbol{x},\boldsymbol{y})\Big(e^y_{ij}(\boldsymbol{\Phi}^{(mn)}) + \delta_{im}\delta_{jn}\Big)e^y_{kl}(\boldsymbol{\phi})\, dy = 0 \quad (2.13)$$

for each $\boldsymbol{\phi} \in \widetilde{H}^1_{per}(Y_1)^3$.

Remark 2.1. More general scaling than that described by ε, ε^2 and ε^3 is possible. The elasticity tensor C^ε can be given by (cf Allaire and Briane, 1996)

$$C^\varepsilon_{ijkl}(\boldsymbol{x}) = C_{ijkl}\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon_1}, \frac{\boldsymbol{x}}{\varepsilon_2}, \frac{\boldsymbol{x}}{\varepsilon_3}\Big) \quad (2.14)$$

provided that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon_3}{\varepsilon_2} = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\frac{\varepsilon_2}{\varepsilon_1} = 0 \quad (2.15)$$

This means that each scale can be distinguished from the others, i.e., they are not of the same order of magnitude. Obviously, in (2.1) we have

$$\varepsilon_k = \varepsilon^k \qquad k = 1,2,3 \quad (2.16)$$

Remark 2.2. The reiterated homogenization procedure just outlined can be extended as follows to cover perforated domains. For each $k = 1,2,3$ the basic cell Y_k is divided into a material part Y^*_k and a hole T_k. The case, where T_k is empty is not precluded. Now the integrals in Eqs (2.3) and (2.4) are over the domain Ω_ε, being the multiscale perforated domain, cf Allaire and Briane (1996).

To derive the homogenized moduli we proceed similarly as previously, except for replacing the integrals over Y_k, $k = 1,2,3$, by the integrals over Y^*_k and the spaces $\widetilde{H}^1_{per}(Y_k)$ by the spaces $\widetilde{H}^1_{per}(Y^*_k)$. Moreover, if L is given by

$$L_\varepsilon(\boldsymbol{u}) = \int\limits_{\Omega_\varepsilon} f_i u_i\, d\boldsymbol{x} + L_1(\boldsymbol{u}) \quad (2.17)$$

then

$$L_h(\boldsymbol{u}) = \int\limits_{\Omega_\varepsilon} \theta f_i u_i\, d\boldsymbol{x} + L_1(\boldsymbol{u}) \quad (2.18)$$

where $\theta = \theta_1\theta_2\theta_3$ is the overall volume fraction of material, $\theta_k = |Y^*_k|$.

Remark 2.3. Proceeding similarly, one can easily extend the reiterated homogenization to nonlinear problems, with nonquadratic stored energy function. Fore instance, such a function can be given by

$$W_\varepsilon(\boldsymbol{x},\mathsf{E}) = W\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},\frac{\boldsymbol{x}}{\varepsilon^2},\dots,\frac{\boldsymbol{x}}{\varepsilon^n},\mathsf{E}\Big) \qquad \mathsf{E}\in \mathbb{E}^3_s \tag{2.19}$$

To find the homogenized potential $W_h(\boldsymbol{x},\mathsf{E})$ it is sufficient to generalize the procedure outlined above for the linear case. Obviously, one has to impose appriopriate conditions on $W(\boldsymbol{x},\boldsymbol{y}_1,\boldsymbol{y}_2,\dots,\boldsymbol{y}_n;\mathsf{E})$.

3. Application of the reiterated homogenization to determination of effective elastic moduli of compact bone

As we already know, compact bones are characterized by many structural levels. Here we are going to consider three of them, most important from the point of view of finding the macroscopic elastic moduli. We follow Aoubiza (1991), Crolet et al. (1993), and Aoubiza et al. (1996).

At the *lowest level*, the lamellar structure is considered: collagen fibres are embedded in hydroxyapatite crystals. In a single lamella, all the collagen fibres have the same orientation, but the orientation of these fibres can differ between two adjacent lamellae.

The *second level* corresponds to the structural definition of a single osteon and a part of the interstitial system, an osteon being a set of concentric lamellae, which surround a haversian canal.

At the *highest level*, a representative volume of compact bone is examined. This volume consists of a sufficiently large number of osteons embedded in the interstitial system. The osteons are packed tightly together, mutually parallel and oriented in the direction of the long axis of the bone.

3.1. Modelling of the lamellar structure

The simulation of the characteristics of a single lamella is performed in two steps. First, a lamella is divided into a finite number of identical cylindrical sectors, cf Fig.5.

Obviously, by knowing the elasticity tensor of one sector, the elasticity tensor of any other sector can be calculated by performing a rotation. Secondly, the cylindrical sector is geometrically approximated by a parallelepiped sector and, through a change of axis, the directions of fibres is assumed to be

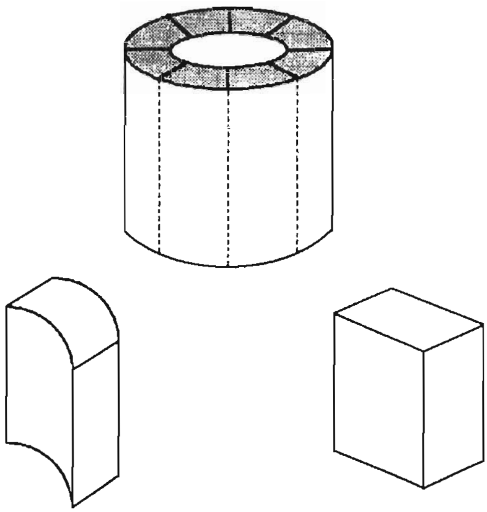

Fig. 5. Decomposition of a lamella and approximation of a sector, after Crolet et al. (1993)

parallel to one side of this cubic sector. In the case of cubic sector (fibrous unidirectional composite), the basic cell $Y_3 = Y_{12}$ is chosen to be a collagen fiber and a hydroxyapatite matrix, see Fig.6.

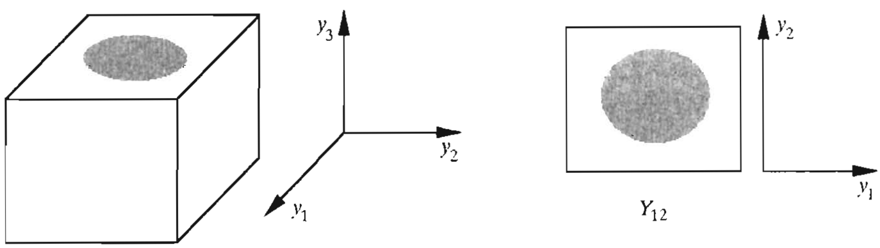

Fig. 6. Basic cell for the homogenization in a sector, after Crolet et al. (1993)

In this case the homogenization is two-dimensional. It means that the homogenized coefficients $C^{(2)}_{ijkl}$ are calculated from Eq (2.7) with Y_3 replaced by Y_{12}. To solve this two-dimensional homogenization problem one can use the FEM. In this way the homogenized moduli of a lamella sector are obtained. The direction of fibres was assumed to be parallel to the longitudinal axis of the lamella. In a more general case, where the fibres do not coincide with the longitudinal axis (see Fig.7) one can use the transformation formula.

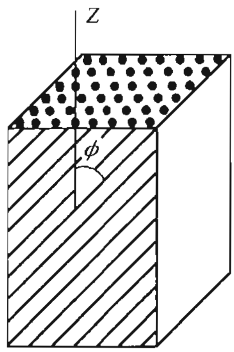

Fig. 7. Approximation of a lamella sector

Crolet et al. (1993) and Aoubiza et al. (1996) assumed that the collagen and hydroapatite are homogeneous, isotropic and perfectly bonded.

3.2. Modelling of the osteonic structure

Each osteon is considered as cylindrical in shape, with all lamella having the same thickness and two adjacent lamellae are perfectly bonded. The results from the previous level are used as the data for this second level. This simulation was also used to obtain the moduli of the interstitial system. The interstitial system is assumed to be a set of fragment of "old" osteons. In this case, the degree of mineralization is assumed to be more elevated.

As previously, the osteon is divided into cylindrical sectors, each sector being approximated by a parallelepiped made of a superposition of plates or lamellae, see Fig.8.

Now the basic cell Y_2 is one- dimensional and the homogenized moduli $C^{(1)}_{ijkl}$ can be calculated explicitly.

3.3. Modelling of the macrostructure

At this level several schemes of analysis can be defined corresponding to different types of osteons, see Section 1. The numerical simulation is based on two points: the simulation of the haversian canal and the application of the homogenization theory (calculation of $\mathbf{C}^h$). Crolet et al. (1993) used two strategies to deal with the haversian canal (Volkmann's canals were not simulated). First, the fluid was replaced with a homogeneous, isotropic and linearly elastic material characterized by the tensor $\mathbf{C}_H$ with an extremely low rigidity. Secondly, instead of the tensor $\mathbf{C}_H$ the tensor $\eta\mathbf{C}_H$ is used, where η is

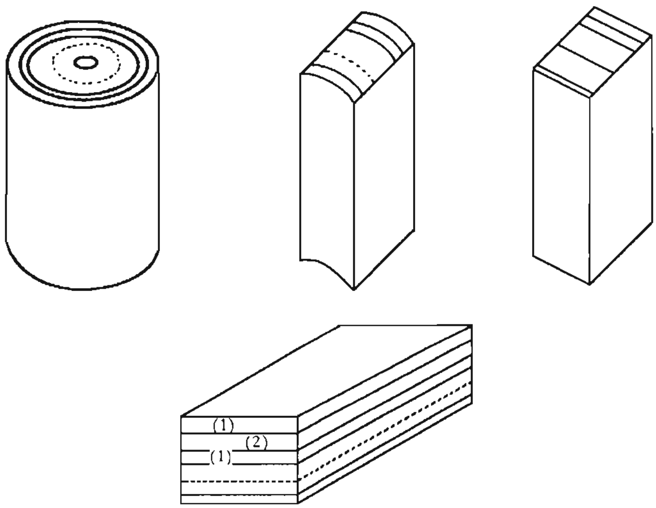

Fig. 8. Decomposition and approximation in an osteon, after Crolet et al. (1993)

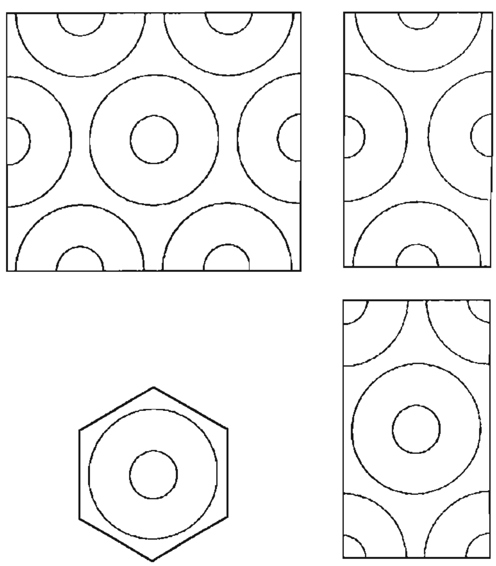

Fig. 9. Sections of basic cells, after Crolet et al. (1993)

a parameter which tends to zero.

In order to simulate different types of osteons (see Section 1), various basic cells can be defined, cf Fig.9.

We observe, however, that whatever the situation, the components of the period are monoclinic and the homogenization is two-dimensional i.e., the local function $\boldsymbol{\Phi}^{(ij)}$ in Eq (2.13) depends on two local variables. As in Section 3.1, the homogenized moduli C^h_{ijkl} can be determined by using a specially developed finite element code, which takes into account periodic boundary conditions.

Crolet et al. (1993) studied six architectures of compact bones:

- architecture No. 1: one type of osteons (type I)
- architecture No. 2: one type of osteons (type II)
- architecture No. 3: one type of osteons (type IV)
- architecture No. 4: two types of osteons (I and II) in the same proportion
- architecture No. 5: three types of osteons (I, II and III) in the following proportions 25, 25 and 50%
- architecture No. 6: four types of osteons (I $\div$ IV) in the proportions: 25, 25, 25 and 25%.

Obviously one can also envisage other architectures.

We recall that the four types of osteons differ in the collagen fiber orientations. More precisely, these four types are described as follows:

- type I: fiber orientation is transverse in all lamellae
- type II: fiber orientation is longitudinal in all lamellae
- type III: fiber orientation is either longitudinal or transverse in two consecutive lamellae
- type IV: fiber orientation is either $45°$ or $-45°$ in two consecutive lamellae.

Aoubiza (1991), Crolet et al. (1993) and Aoubiza et al. (1996) presented specific results of calculations, including comparisons with other methods. One of such results is presented in Fig.10, where C_{ij} are components of the matrix $\mathbf{C}^h$ in Voigt's notation.

The subscripts c, H in Fig.10 stand for the collagen and hydroxyapatite (mineral part), respectively.

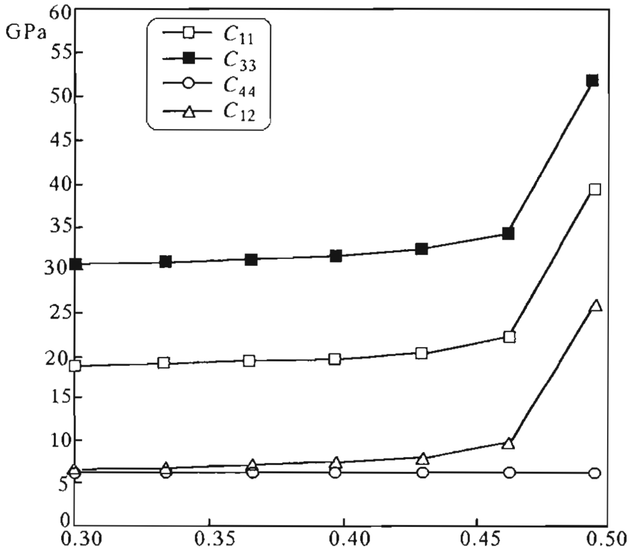

Fig. 10. Effective moduli of compact bone as a function of the Poisson ratio of collagen. Data: $E_c = 1.2\,\mathrm{GPa}$, $E_H = 118\,\mathrm{GPa}$, $\nu_H = 0.28$; the osteon proportions: 25% for each of the four types; porosity 10%, mineral content in osteons 63%; interstitial matrix : volume fraction 25%; fraction of water 6%, after Aoubiza et al. (1996)

4. Concluding remarks

The reiterated homogenization offers new possibilities of finding the effective macroscopic moduli. One can obviously envisage more than three levels. However, one should be aware that the fundamental notions such as those of stresses and strains must remain meaningful. Though we considered only the elastic behaviour it seems to be possible to extend the reiterated homogenization to cover inelastic materials.

The periodic reiterated homogenization can be extended to *stochastic reiterated homogenization*. Each level is then characterized by a probability space. A combination of periodic-stochastic reiterated homogenization is also possible. The stochastic reiterated homogenization enables one to model real materials with a hierarchical microstructure, like biological ones, more realistically.

Living bone is a porous organic material with extremely complex hierarchical structure. Collagen reveals piezoelectric properties and streaming potentials play an important role. Future research should be directed to better modelling of the porous structure of compact bone.

Trabecular bone with a plate-like structure is studied by Gałka et al. (1999) whilst rod-like trabecular bone by Tokarzewski et al. (1998, 1999), cf also the references cited therein.

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Zastosowanie homogenizacji reiterowanej do wyznaczania modułów efektywnych kości zbitej

Streszczenie

Cel pracy jest dwojaki: po pierwsze, przedstawiono podsumowanie dotychczasowych badań dotyczących wyznaczania współczynników sprężystości kości zbitej przy zastosowaniu metod homogenizacji. Po drugie, wykazano, że homogenizacja reiterowana stanowi odpowiednie narzędzie do badania takich materiałów organicznych o hierarchicznej mikrostrukturze. Zastosowano teorię Γ-zbieżności do wyprowadzenia ogólnych zależności opisujących efektywne współczynniki sprężyste materiału o trzech poziomach strukturalnych.

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