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NONLINEAR ELASTIC BRITTLE DAMAGE: NUMERICAL SOLUTION BY MEANS OF OPERATOR SPLIT METHODS

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To avoid the loss of well-posedness in the post-localization range, some continuum damage theories for elastic materials introduce higher order gradients of the damage variable into the constitutive model. Although such theories allow for mathematically correct modelling of the strain localization phenomena, they are usually considered to be very complex to handle from the numerical point of view. The present work deals with the numerical implementation of a gradient-enhanced damage theory for elastic materials. A simple numerical technique, based on the finite element method, is proposed to approximate the solution to the resulting nonlinear mathematical problems. The coupling between damage and strain variables is circumvented by means of a splitting technique.

Key words: damage mechanics, finite elements, splitting technique

1. Introduction

In the last few years, many different continuum damage theories have been proposed. Since the damage propagation generally leads to a local softening behaviour, the models based on a local approach (Kachanov, 1996; Lemaitre and Chaboche, 1990) may lead to a physically unrealistic description of strain localization phenomena when the hypothese of quasi-static and isothermal processes are considered. In general, due to the loss of ellipticity of the governing equations in the post-localization range, the resulting mathematical problems may present an infinite number of solutions with discontinuous fields of displacement gradients what leads to numerical difficulties of mesh-dependence (cf Knowles and Stenberg, 1978; Pietruszczak and Mróz, 1981; Needleman, 1987; Bazant and Pijandier Cabot, 1988; de Vree et al., Gils, 1995).

Recently, some alternative approaches to the local damage theories have been proposed (cf Saouridis and Mazars, 1988; Bazant and Cedolin, 1991; Costa Mattos et al., 1992; Frémond and Nedjar, 1996). The present paper deals with an alternative theory in which the continuum is supposed to possess a microstructure. Since damage results from microscopic movements, it is proposed a reformulation of kinematics and some basic governing principles of the classical continuum mechanics in order to account for such "micromovements". The constitutive equations are devired within the thermodynamic framework, the free energy is supposed to depend not only on the strain and the damage variable but on the damage gradient as well. Besides, to include microscopic effects, the power of the internal forces depends not only on the velocity and its gradient, but also on the damage velocity and its gradient.

The present contribution is focused on presentation of a numerical technique for approximating the resulting nonlinear mathematical problems. The coupling between damage and strain is circumvented by means of a splitting technique which allows one to study the nonlinear problem in terms of a sequence of simpler linear problems. This technique requires that at each time step the following two problems be solved: one similar to an equilibrium problem in linear elasticity and the other similar to a heat transfer problem in a rigid body. In order to assess main features of the numerical method, a number of examples is presented demonstrating a good performance of the algorithm and showing that the numerical computations are not mesh-dependent.

2. Modelling

A body is defined as a set of material points B which occupies, at a reference configuration, a region Ω of the Euclidean space. In the present theory, besides the classical variables that represent kinematics of a continuous medium (displacements and velocities of material points), an additional scalar variable $\beta \in [0,1]$ is introduced. This variable is related to the links between material points and can be interpreted as a measure of the local cohesion state of the material. If $\beta = 1$ all the links and the initial material properties are preserved. If $\beta = 0$ a local rupture is considered since all the links between material points have been broken. The variable β is associated with the damage variable D by the following relation: $\beta = 1 - D$. Since the degradation is an irreversible phenomenon, the rate $\dot{\beta}$ must be negative or equal to zero. A detailed presentation of the basic principles that govern the behaviour of the of continuum of this type can be found in Costa Mattos and Sampaio (1995) and Domingues (1996). A summary of the basic principles is presented in this section. For the sake of simplicity the hypothesis of quasi-static and isothermal processes is adopted throughout this work. Besides, that the hypothesis of small deformation it is also assumed and, consequently, the principle of conservation of mass holds automatically.

2.1. Virtual power principle

Let a body B that occupies a region $\Omega \subset I\!R^3$ with a sufficiently regular boundary Γ be subjected at each instant t to the external forces $\boldsymbol{g}(t) : \Gamma_2 \subset \Gamma \to I\!R^3$ and $\boldsymbol{b}(t) : \Omega \to I\!R^3$, to the external microscopic forces $p(t) : \Omega \to I\!R$, $q(t) : \Gamma_2 \subset \Gamma \to I\!R$ and to the preset displacements $\boldsymbol{u}(t) = \boldsymbol{0} \in \Gamma_1 \subset \Gamma$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \Gamma$. By accepting the hypothesis of slow deformations, the inertial effects can be neglected and the virtual power principle can be expressed as

$$\pi_{int} + \pi_{ext} = 0 \tag{2.1}$$

for any admissible variations of the fields $\boldsymbol{u}$ and β that characterize the kinematics of the medium. The power π_{int} of the internal generalized forces $\boldsymbol{\sigma}$ (the stress tensor), F and $\boldsymbol{H}$ (thermodynamical forces related to the degradation process) can be written as

$$\pi_{int} = -\int\limits_{\Omega} \boldsymbol{\sigma} \cdot \nabla \widehat{\boldsymbol{u}}\, d\Omega - \int\limits_{\Omega} (F\widehat{\beta} + \boldsymbol{H} \cdot \nabla \widehat{\beta})\, d\Omega \tag{2.2}$$

Here, $\widehat{\boldsymbol{u}} : \Omega \to I\!R^3$ is an element of the set V_v of the virtual velocities such that $\widehat{\boldsymbol{u}}\,|_{\Gamma+1}= \boldsymbol{0}$ and $\widehat{\beta} : \Omega \to I\!R$ is an element of the set V_β of the virtual variations of β. The corresponding power π_{ext} of the external generalized forces $\boldsymbol{b}$, $\boldsymbol{g}$, p and q assumes the representation

$$\pi_{ext} = \int\limits_{\Omega} \boldsymbol{b} \cdot \widehat{\boldsymbol{u}}\, d\Omega + \int\limits_{\Gamma_2} \boldsymbol{g} \cdot \widehat{\boldsymbol{u}}\, dA + \int\limits_{\Omega} p\widehat{\beta}\, d\Omega + \int\limits_{\Gamma} q\widehat{\beta}\, dA \tag{2.3}$$

where $p : \Omega \to I\!R$ is defined as a microscopic distance force while $q : \Gamma \to I\!R$ is a microscopic contact force, both in duality with β. The microscopic forces are associated with non-mechanical actions (chemical and electromagnetic, for instance) that can cause the damage propagation. Under the above assumptions the principle of virtual power can be expressed as

$$\int_{\Omega} (\boldsymbol{\sigma} \cdot \nabla\widehat{\boldsymbol{u}} - \boldsymbol{b} \cdot \widehat{\boldsymbol{u}})\, d\Omega - \int_{\Gamma_2} \boldsymbol{g} \cdot \widehat{\boldsymbol{u}}\, dA + \\ + \int_{\Omega} (\boldsymbol{H} \cdot \nabla\widehat{\beta} + F\widehat{\beta} - p\widehat{\beta})\, d\Omega - \int_{\Gamma} q\widehat{\beta}\, dA = 0 \qquad \forall \widehat{\boldsymbol{u}} \in V_v \quad \forall \widehat{\beta} \in V_\beta \tag{2.4}$$

2.2. Constitutive equations

According to the hypothesis of small deformations and isothermal processes, the free energy is a function of the deformation $\boldsymbol{\varepsilon}$, temperature Θ, damage variable β and its gradient $\nabla\beta$. For the sake of clarity, clear the thermodynamic framework used to derive the constitutive equations is not presented in this paper, for the details see Costa Mattos and Sampaio (1995). The final relations read

$$\boldsymbol{\sigma} = \frac{\beta E}{1+\nu}\Big[\frac{\nu}{1-2\nu}\operatorname{tr}(\boldsymbol{\varepsilon})\mathsf{I} + \boldsymbol{\varepsilon}\Big] = \beta[\lambda \operatorname{tr}(\boldsymbol{\varepsilon})\mathsf{I} + 2\mu\boldsymbol{\varepsilon}) \tag{2.5}$$

$$F = \frac{E}{2(1+\nu)}\Big[\frac{\nu}{1-2\nu}\operatorname{tr}(\boldsymbol{\varepsilon})^2 + \boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\Big] - w + \lambda_\beta + C\dot{\beta} + \lambda_{\dot{\beta}} = \\ = \frac{1}{2}\lambda \operatorname{tr}(\boldsymbol{\varepsilon})^2 + \mu\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} - w + \lambda_\beta + C\dot{\beta} + \lambda_{\dot{\beta}} \tag{2.6}$$

$$\boldsymbol{H} = k\nabla\beta \tag{2.7}$$

where E is the Young modulus, ν is the Poisson ratio and λ and μ are the Lamé constants. The therms λ_β and $\lambda_{\dot{\beta}}$ are Lagrange multipliers associated with the constraints $\beta \geqslant 0$ and $\dot{\beta} \leqslant 0$, respectively, ensuring that the following complementary conditions are satisfied: $\lambda_\beta \leqslant 0$, $\beta\lambda_\beta = 0$ and $\dot{\beta}\lambda_{\dot{\beta}} = 0$. The material parameters w, C and k are also introduced, which represent the minimum energy required to start the damage process and viscosity and difusion constants related with the damage distribution, respectively.

3. Mechanical problem

Introducing the constitutive equations (2.5), (2.6) in (2.4), neglecting the external microscopic forces (which are related to chemical or electromagnetic actions) and assuming the initial conditions: $\beta(x,t=0)=1 \ \ \forall x \in \Omega$, the following mathematical problem defined over the time interval $I=[0,\tau]$ is obtained:

Find $(\boldsymbol{u}(x,t),\beta(x,t))$, the displacement field $\boldsymbol{u}: \ \Omega \times I \rightarrow I\!R^3$ such that $\boldsymbol{u}\,|_{\Gamma_1} = \overline{\boldsymbol{u}}(t)$ and the field $\beta: \ \Omega \times I \rightarrow I\!R$ such that

$$\int\limits_{\Omega} \beta[\lambda \operatorname{div} \boldsymbol{u} \operatorname{div} \widehat{\boldsymbol{u}} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\widehat{\boldsymbol{u}}) \; d\Omega - \int\limits_{\Omega} \boldsymbol{b} \cdot \widehat{\boldsymbol{u}} \; d\Omega - \int\limits_{\Gamma_2} \boldsymbol{g} \cdot \widehat{\boldsymbol{u}} \; dA = 0$$

$$\int\limits_{\Omega} k\nabla\beta \cdot \nabla\widehat{\beta} \; d\Omega - \int\limits_{\Omega} \Big[\frac{1}{2}\lambda(\operatorname{div} u)^2 + \mu\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} - w\Big]\widehat{\beta} \; d\Omega - \int\limits_{\Omega} C\dot{\beta}\widehat{\beta} \; d\Omega = 0 \tag{3.1}$$

$\forall \widehat{\boldsymbol{u}} \in V_v$ and $\forall \widehat{\beta} \in V_\beta$.
Subjected to the following constraints

$$\beta \geqslant 0 \qquad \text{and} \qquad \dot{\beta} \leqslant 0$$

and with the following initial condition

$$\beta(x,t=0)=1$$

4. Numerical approximation

The nonlinear mathematic damage propagation problem resulting from the model, including the coupling between damage and displacement fields, can be solved through a staggered algorithm, in which the coupled system is divided, often according to different coupled fields, and each division can be treated by a different step-by-step time algorithm. The approach proposed in this work is based on the concept that divisio of the coupled system defines only an operator split of the propagation problem. In this context, a staggered scheme is interpreted as a product formula algorithm determined by a specific operator split, exactly as in the classical method of fractional steps (Yanenko, 1980). This point of view wass also adopted by Simo and Miehe (1992), where standard staggered algorithms for coupled thermomechanical problems, consisting of an isothermal phase followed by a heat conduction phase at a fixed configuration, were cast into the format of fractional step method.

4.1. Semi-discrete problem: the finite element method

The solution to the damage propagation problem is based on a spatial discretization using the Finite Element Method (FEM) leading to a semi-discrete approach in terms of a nonlinear system of Ordinary Differential Equations (EDO). This system is derived using of a splitting scheme, which yields a sequence of simpler evolution problems, which are in turn solved by standard techniques; like, backward and forward Euler and trapezoidal rules.

Let $N_i \in V_v^h$ the basic function provided by the FEM (Hughes, 1987), where V_v^h is a finite sub-space of the space V_v, and, $\varphi_i \in V_\beta^h$ where V_β^h is a finite sub-space of V_β. Those functions allow for construction of the following approximations

$$
\begin{aligned}
u_h(x,t) &= \sum_{i=1}^{m_h} u_i(t) N_i(x) \qquad i = 1, ..., mh \\
\beta_h(x,t) &= \sum_{i=1}^{m_h} \beta_i(t) \varphi_i(x) \qquad i = 1, ..., mh
\end{aligned}
\tag{4.1}
$$

where m_h is the number of nodal points and h is the mesh parameter (associated with the mesh refinement). The semi-discrete problem is obtained by replacing the continuous fields $\boldsymbol{u}$ and β in Eqs (3.1) by their finite element approximations defined in Eqs (4.1). The semi-discrete problem is representation by the follows nonlinear system of ordinary differencial equations:

$$
\begin{aligned}
&\mathsf{K}(\beta_h)\boldsymbol{u} = \boldsymbol{R} \\
&\mathsf{C}\dot{\boldsymbol{\beta}} + \mathsf{A}\boldsymbol{\beta} + \boldsymbol{F}(\boldsymbol{u}) = \mathbf{0}
\end{aligned}
\tag{4.2}
$$

with the following initial condition

$$\beta_h(x,0) = 1 \qquad \text{and} \qquad u_h(x,0) = 0$$

and the following constraints imposed

$$0 \leqslant \beta_h(x,t) \leqslant 1 \qquad \text{and} \qquad \dot{\beta}_h(x,t) \leqslant 0$$

where $\boldsymbol{u}$ and $\boldsymbol{\beta}$ are the vectors of the nodal values defined in Eqs (4.1). The other vectors and matrices appearing in the above problem have the following components

$$
\begin{aligned}
[\mathbf{K}(\beta_h)]_{ij} &= \int_\Omega \beta_h [\mathbf{B}^\top \mathbf{D} \mathbf{B}]_{ij} \, d\Omega && i,j = 1,...,3m_h \\
[\boldsymbol{R}]_i &= \int_\Omega [\boldsymbol{b}]_k N_i \, d\Omega - \int_\Gamma [\boldsymbol{g}]_k N_i \, dA && k = int\Big(\frac{i}{3}\Big) \\
[\mathbf{C}]_{ij} &= \int_\Omega C \varphi_i \varphi_j \, d\Omega && i,j = 1,...,m_h \\
[\mathbf{A}]_{ij} &= \int_\Omega k \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega && i,j = 1,...,m_h \\
[\boldsymbol{F}(\boldsymbol{u})]_i &= \int_\Omega \frac{1}{2} [(\mathbf{B}^\top \mathbf{D} \mathbf{B} \boldsymbol{u}) \cdot \boldsymbol{u} - w] \varphi_i \, d\Omega && i,j = 1,...,m_h
\end{aligned}
\tag{4.3}
$$

and $\mathbf{B}$ denotes the standard discretized differential operator and $\mathbf{D}$ is the matrix of the elastic constitutive coefficients.

4.2. Operator split technique applied to the semi-discrete problem

The operator split technique is used to approximate the nonlinear semi-discrete problem in terms of a sequence of simpler linear problems. The two ways of operator division were considered, one related to u_h ("equilibrium problem") and the other to β_h ("damage propagation problem"). The proposed scheme can result in two different algorithms depending on the order of the sequence of the operators. These algorithms will be named ALG1 and ALG2, respectively. The ALG1 algorithm first solves the "damage propagation problem", one the displacement field unchanged. At the first stage, the associated ordinary differential equation is solved using the time integration method

$$
\begin{aligned}
&\mathbf{C}[\theta \boldsymbol{\beta}^{n+1} + (1-\theta)\boldsymbol{\beta}^n] + \Delta t[\theta \mathbf{A} \boldsymbol{\beta}^{n+1} + (1-\theta)\mathbf{A}\boldsymbol{\beta}^n] + \\
&+\Delta t[\theta \widetilde{\boldsymbol{F}}^{n+1} + (1-\theta)\boldsymbol{F}^n] = \mathbf{0}
\end{aligned}
\tag{4.4}
$$

where the scalar θ determines the time integration method: i.e., $\theta = 0$ – forward Euler; $\theta = 1$ – backward Euler and $\theta = 1/2$ – trapezoidal rule, respectively. The subscript h was omitted and the superscript n means that the function is approximated at the instant t_n. Besides, $\widetilde{\boldsymbol{F}}^{n+1}$ does not represent the function $\boldsymbol{F}$ evaluated at t_{n+1}, since u_{n+1} is not known a priori. At the first phase $(\dot{\boldsymbol{u}} = \mathbf{0})$ $\widetilde{\boldsymbol{F}}^{n+1}$ is calculated using $\boldsymbol{u}_n$.

The second phase of ALG1 consists in solving the "equilibrium problem"

$$\mathbf{K}(\beta_{n+1})\boldsymbol{u}_{n+1} = \boldsymbol{R}_{n+1} \tag{4.5}$$

where

$$[\boldsymbol{R}_{n+1}]_i = \int\limits_{\Omega} ([\boldsymbol{b}]_k)_{n+1} N_i \; d\Omega - \int\limits_{\Gamma} ([\boldsymbol{g}]_k)_{n+1} N_i \; dA \tag{4.6}$$

The stages of ALG2 algorithm are realised in the inverse order. In numerical implementation of the two algorithms is straigh forward, since both algorithms can be built from a standard finite element code. It can be observed that the "damage propagation problem" is similar to a heat conduction problem, while the "equilibrium problem", is similar to a classical elasticity problem.

5. Analysis of numerical examples

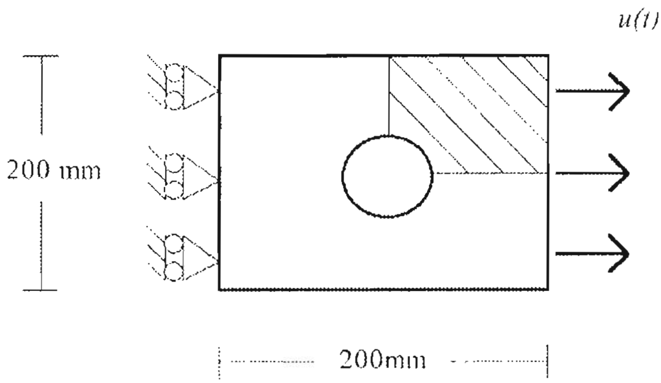

Fig. 1. Plate with a central circular hole

In order to assess the modelling features in a multiaxial stress state, the problem of square plate with a central circular hole is analyzed. The square plate (200 mm × 200 mm × 1 mm) with a central circular hole, which radius is 50 mm, is supported at the left side and subject to a preset displacement $u(t)$ at the opposite side, see Fig.1. Due to symmetry of the problem the analysis is performed for the upper right quarter of the plate only. In the present study a plate made of concrete, which has the following mechanical characteristics: $E = 27.0\,\text{GPa}$, $w = 5.0 \times 10^{-5}\,\text{MPa}$, $C = 1.0 \times 10^{-3}\,\text{MPa s}$ and $k = 0.2\,\text{Mpa mm}^2$ (Frémond and Nedjar, 1996) is considered the preset displacement and the assumed time step are $u(L,t) = \alpha t$ $(\alpha = 5.0 \times 10^{-3}\,\text{mm/s})$ and $\Delta t = 1.0 \times 10^{-4}\,\text{s}$, respectively. The usual bi-linear quadrilateral finite element and the ALG1 algorithm, in which $\theta = 1/2$ is adopted, are used.

5.1. Damage propagation

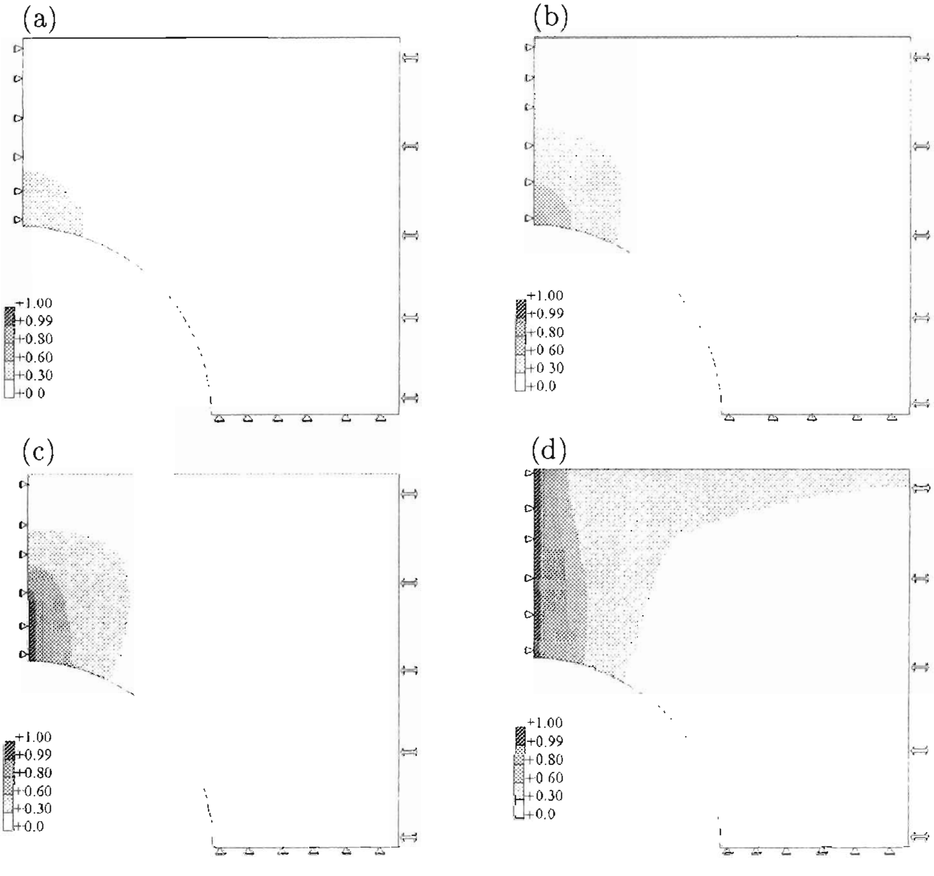

Fig. 2. Damage levels; (a) – $t = 2.5\,\text{s}$, (b) – $t = 2.8\,\text{s}$, (c) – $t = 3.0\,\text{s}$, (d) – $t = 3.35\,\text{s}$

The course of the damage variable $D = 1 - \beta$ on the plate is show in Fig.2. These figures demonstrate that the damage arises at a region near the hole (see Fig.2a,b), what is expected for the body under consideration and subjected to a tensile force. Then the damage propagates towards the free end of the plate, perpendicularly to the load direction, until the plate is completely broken.

5.2. Mesh dependence

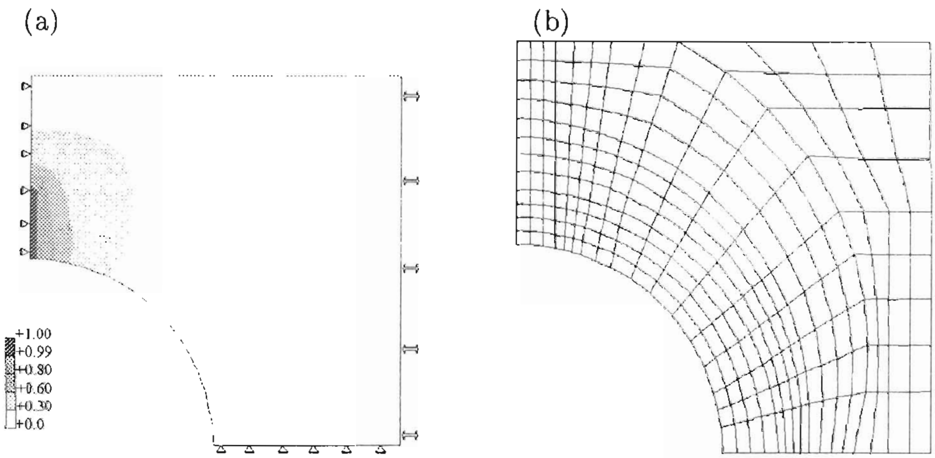

Fig. 3. (a) – Damage levels at $t = 3.0\,\mathrm{s}$, (b) – mesh-1: 274 nodes and 240 elements

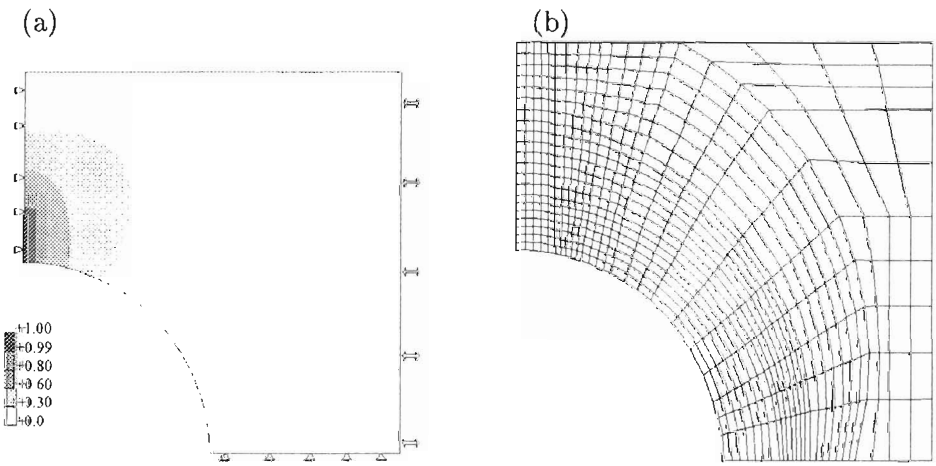

Fig. 4. (a) – Damage levels at $t = 3.0\,\mathrm{s}$, (b) – mesh-2: 594 nodes and 544 elements

To show that the problem solution is not mesh-dependent two different meshes (see Fig.3 and Fig.4) were employed, with different degrees of discretization in the region of the highest of damage level. Besides, the levels of damage at the instant $t = 3.0\,\mathrm{s}$ obtained using these meshes are presented. It can be seen from the figures that the damage distribution are similar.

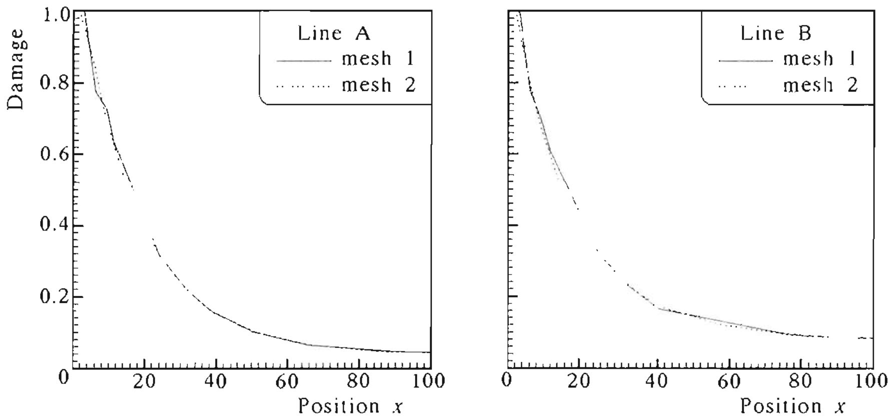

Fig. 5. Damage along the lines A and B for different meshes

Fig.5 allows one to observe the damage propagation along the horizontal lines A ($y = 53.0\,\mathrm{mm}$) and B ($y = 60.0\,\mathrm{mm}$). The shapes of the curves and the damage levels at different points along those lines are almost the same.

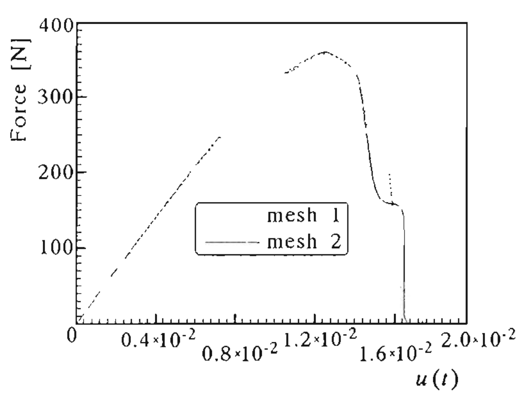

Fig. 6. Force versus displacement curves for different meshes

The curves of the applied force versus displacement obtained using the two meshes are presented in Fig.6 to verify the mesh-dependence. The results represent the behaviour of the global structure. Again, it can be seen, that the shape of the solution is not affected by the spatial discretization. Fig.6

also allows one to observe the softening behaviour. The similar result was also found in several examples (cf Pires-Domingues, 1996). So, although no theoretical result is presented, the presented formulation is believed not to suffer of any numerical pathology due to mesh-dependence.

5.3. Accuracy of the proposed algorithm

The performance of the proposed algorithm is examined solving the problem of rectangular clamped plate (see Fig.7), with a preset displacement $u_x(x = L, y) = \alpha t$ and $u_y(x = L, y) = 0$. The plane strain state and the following boundary conditions: $\beta = \partial\beta/\partial x = 0$ in $x = 0$ and $x = L$ are assumed.

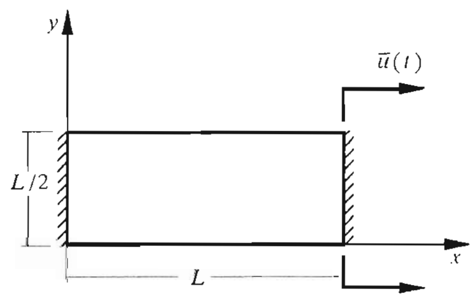

Fig. 7. Rectangular plate with the preset displacement

The following values were taken: $\alpha = 5.0 \times 10^{-3}$ mm/s, $L = 20.0$ mm, $E = 50.0$ GPa, $w = 0.025$ MPa, $C = 1$ MPa s and $k = 0.1$ Mpa mm^2. A mesh of 289 nodal points and 256 elements was used to solve the problem (see Fig.8). The dashed line shows the longitudinal line at the central region of the plane, where the analyzed nodal points (137 up to 153) are localized.

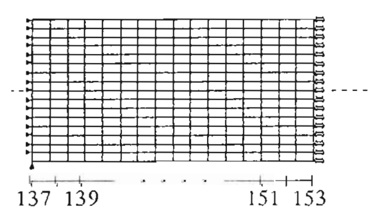

Fig. 8. Finite element mesh

In order to assess the accuracy of the proposed numerical method, the results obtained using the ALG1 (with thrre values taken for θ) and with a coupled solution scheme are compared. The coupled damage problem defined by the Eqs (4.2) is nonlinlear and thus was solved with the aid of a Newton technique associated to the backward Euler integration method. Fig.9 presents the results obtained using different numerical methods at the instant $t = 5.5$ s.

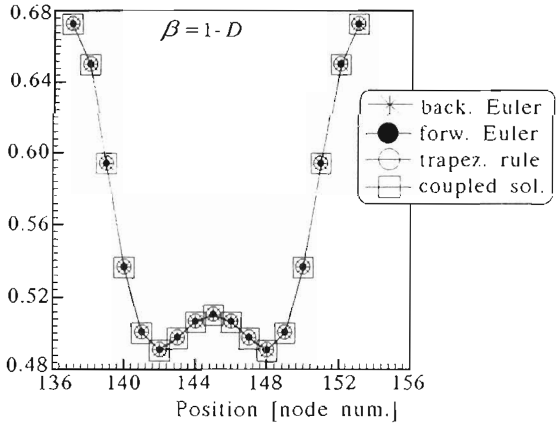

Fig. 9. Different numerical methods, $t = 5.5$ s

It can be observed that the results are quite similar using different algorithms.The same kind of behavior was obtained for the displacements. Comparisons were also made using the ALG2, arriving at similar conclusions. Therefore, the results presented in this section prove that the operator split method was able to give the results at the same precision level that other methods capable of solving the coupled problem. It has to be emphasized that the mentioned above is expensive.

6. Final remarks

A simple numerical method was used to approximate the solution to the nonlinear damage propagation problem without the necessity for radical modification of an ordinary finite element code. This simple numerical method is constructed combining the FEM and an operator split technique that transforms the global nonlinear problem into a sequence of linear problems. The proposed numerical method has succeeded, as it was documented by the presented examples.

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Nieliniowe sprężyste kruche zniszczenie – rozwiązanie numeryczne przy użyciu metod operatorowych

Streszczenie

Celem poprawności sformułowania problemu w pewnych teoriach zniszczenia do modelu konstytutywnego materiałów sprężystych wprowadza się wyższe gradienty zmiennych opisujących zniszczenie. Chociaż teorie takie umożliwiają matematycznie poprawne modelowanie zjawisk lokalizacji odkształceń, to z punktu widzenia numeryki stosowanie ich uważa się zwykle za bardzo skomplikowane. W niniejszej pracy przedstawiono zastosowanie numeryczne teorii zniszczenia z wyższymi gradientami do materiałów sprężystych. Do przybliżonego rozwiązania otrzymanych nieliniowych zadań matematycznych zaproponowano prostą metodę numeryczną opartą na metodzie elementów skończonych. Użycie metody operatorowej pozwoliło uniknąć sprężenia między zmiennymi opisującymi zniszczenie i odkształcanie.

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