JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, 36, 1998

THERMAL STRESSES IN A BIMATERIAL PERIODICALLY LAYERED COMPOSITE DUE TO THE PRESENCE OF INTERFACE CRACKS OR RIGID INCLUSIONS

Andrzej Kaczyński

Institute of Mathematics, Warsaw University of Technology

Stanisław J. Matysiak

Faculty of Geology, University of Warsaw

e-mail: matysiak@albit.geo.uw.edu.pl

The state of thermal stresses for a periodic two-layered elastic space weakened by an interface Griffith crack or a ribbon-like absolutely rigid inclusion is investigated. The analysis is performed within the framework of linear thermoelasticity with microlocal parameters. The resulting thermoelastic problems have been reduced to their mechanical counterparts. The stress singularities as well as the effects of the layering are discussed within the framework of the theory of fracture mechanics.

Key words: layered composite, crack, ribbon inclusion, stress intensity factor

1. Introduction

Extending use of new composite materials in situations involving both mechanical and thermal environments requires the study of their strength and fracture behaviour. Interface cracking is considered to be one of the most commonly encountered types of failure mechanism.

This paper is a continuation of our studies (see Kaczyński and Matysiak, 1997 and references therein) concerning the interface crack and rigid inclusion problems in a bimaterial periodically layered space subjected to mechanical loading. Considering of the problems including thermal effects is the main purpose of the present investigation.

The study is based on the approximate microlocal parameter approach devised by Woźniak (1987). The governing equations of the homogenized model in 2D stationary case of linear thermoelasticity are given in Section 2. We also formulate the general boundary-value problems and present a common method of constructing the solution for an interface Griffith crack and a perfectly rigid ribbon inclusion by using an analogy between the thermal problems and their mechanical counterparts.

The asymptotic analysis of the results for proper assessment of the strength degradation of composites due to the defects under consideration is given in Section 3. Unlike the existing solutions for defects lying on the interface of materials (cf Erdogan, 1972;, Sih and Chen, 1981), the standard (non-oscillating) crack-tip thermal stress singularities are obtained and the stress intensity factors are introduced.

2. Formulation and solution of the problem

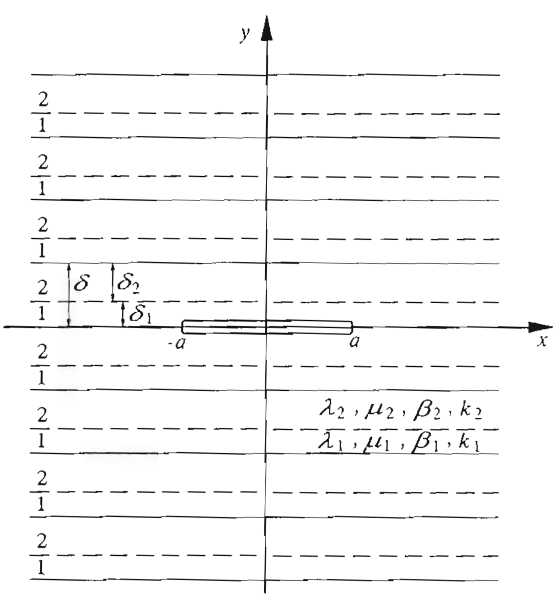

Fig. 1. Middle-cross section of a periodic composite with an interface defect

Let us consider a microperiodic laminated space, the middle cross section of which is shown in Fig.1. A repeated fundamental layer of the thickness δ is composed of two homogeneous isotropic sublayers, denoted by 1 and 2, with thicknesses δ_1 and δ_2. They are characterised by Lamé constants

λ_l and μ_l, thermal conductivities k_l and coefficients of volume expansions $3\beta_l/(3\lambda_l + 2\mu_l)$; here and in the sequel, all the quantities pertaining to the sublayers 1 and 2 will be denoted by the index l or (l) taking the values 1 and 2, respectively.

We deal with the plane steady-state thermoelastic problems related to the xy-plane with the interface defect: a Griffith crack, denoted by C or a perfectly rigid ribbon inclusion, denoted by I, representing a line segment of the length $2a$. This stratified medium is subjected by a certain external loading resulting in the displacement, stress and temperature fields independent of the coordinate z. The perfect mechanical bonding and ideal thermal contact between the layers (excluding the interval $\langle -a, a\rangle$) are assumed. The crack *surfaces are required being free of tractions, and the ribbon faces to be free of* displacements.

Owing to a complicated geometry of the solid and complex boundary conditions, the closed solutions of the considered problems cannot be obtained. The homogenized model of this composite (cf Woźniak, 1987; Kaczyński and Matysiak, 1997) is applied to obtain an approximate solution. Without going into details we present only the final governing equations and constitutive relations of the macro-homogeneous medium (homogenized model) for the stresses $\sigma_{ij}^{(l)}(x,y)$, $(i,j = y \text{ or } x)$, $\sigma_{zz}^{(l)}(x,y)$ and the heat flux vector $[q_x^{(l)}(x,y), q_y^{(l)}(x,y)]$, given in terms of unknown macro-displacements $u(x,y)$, $v(x,y)$ and a macro-temperature $\vartheta(x,y)$ (in the absence of the body forces and heat sources)

$$
\begin{aligned}
&\widetilde{k}\vartheta_{,xx} + K\vartheta_{,yy} = 0\\
&A_2 u_{,xx} + (B + C)v_{,xy} + Cu_{,yy} - K_2\vartheta_{,x} = 0\\
&A_1 v_{,yy} + (B + C)u_{,xy} + Cv_{,xx} - K_1\vartheta_{,y} = 0
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&q_x^{(l)} = -k_l\vartheta_{,x} \qquad\qquad q_y^{(l)} = -K\vartheta_{,y}\\
&\sigma_{yy}^{(l)} = Bu_{,x} + A_1 v_{,y} - K_1\vartheta\\
&\sigma_{xy}^{(l)} = C(u_{,y} + v_{,x})\\
&\sigma_{xx}^{(l)} = D_l v_{,y} + E_l u_{,x} - F_1\vartheta\\
&\sigma_{zz}^{(l)} = \frac{\lambda_l}{2(\lambda_l + \mu_l)}(\sigma_{xx}^{(l)} + \sigma_{yy}^{(l)}) - \frac{\mu_l}{\lambda_l + \mu_l}\beta_l\vartheta
\end{aligned}
\tag{2.2}
$$

The positive constants appearing in the above equations are given in the Appendix. They depend on the material and geometrical characteristics of the composite constituents. It should be emphasised that the condition of perfect bonding between the layers is satisfied ($\sigma_{yy}^{(l)}$, $\sigma_{xy}^{(l)}$, $q_y^{(l)}$ do not depend on l).

The components $\sigma_{xx}^{(l)}$, $\sigma_{zz}^{(l)}$, $q_z^{(l)}$ suffer a discontinuity on the interfaces. Finally, assuming that two sublayers have the same thermo-mechanical properties, we pass directly to the well-known equations of stationary plane thermoelasticity for a homogeneous isotropic body (cf Nowacki, 1986).

Within the framework of the homogenized model we consider the plane boundary-value problem involving an interface defect (crack C or inclusion I) occupying the region $S =< -a.a > \times\{0\}$. Evidently, the associated global conditions on S are

$$\sigma_{yy}^{\pm} = \sigma_{xy}^{\pm} = 0 \quad \text{for} \quad C \qquad\qquad u^{\pm} = v^{\pm} = 0 \quad \text{for} \quad I \tag{2.3}$$

Here and below the quantities assigned with $\pm$ refer to the limiting values as $y \to 0^{\pm}$. Moreover, certain conditions resulting from a given external loading (thermal and mechanical) have to be specified.

Making use of the superposition principle, the problem is separated into two parts and we can write

$$\vartheta = \vartheta^0 + \vartheta^d \qquad (u,v) = (u^0, v^0) + (u^d, v^d) \qquad \sigma = \sigma^0 + \sigma^d \tag{2.4}$$

where the index 0 is associated with the problem of the composite without defect, subjected to a given external load, and the index d refers to the corresponding problem in which some fields are imposed on the defect faces to ensure the conditions (2.3) to be satisfied. It is assumed here that the solution of the first problem on the x-axis is known (see the method of complex potentials devised by Kaczyński and Matysiak (1988) and Kaczyński (1993)). Attention is then focused on the *corrective* solution of the perturbed problem. An efficient approach is based on the *classical* complex representation (similar to that given by Muskhelishvili (1953) in terms of two potentials Φ, Ω and the thermal potential φ_0 (corresponding to the thermal loading), taking on the x-axis the following form (cf Kaczyński and Matysiak, 1989)

$$\begin{aligned} &\sigma_{yy}^{d\pm} - \mathrm{i}t_*\sigma_{xy}^{d\pm} = \Phi^{\pm} + \Omega^{\mp} \\ &2\mu_*[u,_x^{d\pm} + \mathrm{i}t_* v,_x^{d\pm}] = \kappa_*\Phi^{\pm} - \Omega^{\mp} + \beta_*(\varphi_0')^{\pm} - \beta'(\overline{\varphi}_0')^{\mp} \\ &\vartheta^d(x,y) = 2\mathrm{Re}\varphi_0'(\xi) \qquad\qquad \xi = x + \mathrm{i}k_0 y \end{aligned} \tag{2.5}$$

In the above, the constants t_*, μ_*, κ_*2, k_0 are given in the Appendix, and β_*, β' are defined in the cited paper.

The thermal potential φ_0' should be known to reduce is required to reduce the thermoelastic problem to its mechanical counterpart. For a thermally insulated problem it can be easily determined from the Hilbert problem, arising from the boundary conditions

$$q_y^{d\pm} = -q_y^{0\pm} \qquad |x| < a$$
$$q_x^d(\infty) = q_y^d(\infty) = 0 \tag{2.6}$$

Denote now, from the solution of the problem associated with 0, the known values of thermal potential φ_0' and thermal stresses and displacements on the upper and lower faces of the segment $(-a,a)\times\{0\}$ as follows

$$\begin{aligned}\beta_*(\varphi_0')^\pm - \beta'(\overline{\varphi}_0')^\mp &\equiv -[S(x)\pm T(x)]\\ \sigma_{yy}^{0\pm} - \mathrm{i}t_*\sigma_{xy}^{0\pm} &\equiv -[P(x)\pm Q(x)]\\ 2\mu_*[u,_x^{0\pm} + \mathrm{i}t_*v,_x^{0\pm}] &\equiv -[f'(x)\pm g'(x)]\end{aligned} \tag{2.7}$$

In view of Eqs (2.5) and (2.3), (2.4), the fundamental perturbed problem reduces to finding two single-valued, sectionally holomorphic potentials $\varPhi$ and $\varOmega$, satisfying the following boundary conditions

$$\begin{aligned}&\varPhi^\pm + \varOmega^\mp = P(x)\pm Q(x) && |x|<a\\ &\kappa_*\varPhi^\pm - \varOmega^\mp = f'(x)\pm g'(x) + S(x)\pm T(x) && |x|<a\\ &\varPhi(\infty) = \varOmega(\infty)\end{aligned} \tag{2.8}$$

By using the results given by Kaczyński and Matysiak (1997), the solution of the above problem may be written in the common form for the crack C and rigid inclusion I as

$$\varPhi(\widehat{z}) = \frac{F(\widehat{z}) + g_m + g_t}{\sqrt{\widehat{z}^2 - a^2}} + G(\widehat{z})$$
$$\varOmega(\widehat{z}) = -\rho_*\varPhi(\widehat{z}) + 2\rho_*G(\widehat{z}) \tag{2.9}$$

in which the Cauchy integrals F and G of the generalized complex variable $\widehat{z}$ (cf Kaczyński and Matysiak, 1989) and the constants g_m, g_t are defined by means of the known functions $F^*(\rho_*,t)$, $G^*(\rho_*,t)$, $T(t)$

$$F(\widehat{z}) = \frac{1}{4\pi\rho_*}\int\limits_{-a}^{a}\frac{\sqrt{a^2-t^2}F^*(\rho_*,t)}{t-\widehat{z}}\,dt \qquad G(\widehat{z}) = \frac{1}{4\pi\rho_*\mathrm{i}}\int\limits_{-a}^{a}\frac{G^*(\rho_*,t)}{t-\widehat{z}}\,dt \tag{2.10}$$

$$g_m = \frac{\rho+\rho_*}{4\pi(\rho-\rho_*)}\int\limits_{-a}^{a}G^*(\rho_*,t)\,dt \qquad g_t = -\frac{\rho_*}{\pi\mathrm{i}(\rho-\rho_*)}\int\limits_{-a}^{a}T(t)\,dt$$

provided we set in the problem C and I:

— for C

$$\begin{aligned} &\rho_* = -1 && \rho = \kappa_* \\ &F^*(\rho_*,t) = -2P(t) && G^*(\rho_*,t) = -2Q(t) \end{aligned} \tag{2.11}$$

— for I

$$\begin{aligned} &\rho_* = \kappa_* && \rho = -1 \\ &F^*(\rho_*,t) = 2f'(t) + 2S(t) && G^*(\rho_*,t) = 2g'(t) \end{aligned} \tag{2.12}$$

From Eqs(2.5) and (2.9)÷(2.12) it is seen that the proposed model leads to a typical stress singularity at the tips $a^{\pm}$, contrary to the oscillatory singularity appearing in the interface problems posed in conventional formulation.

3. Asymptotic analysis

The asymptotic form of solution in a small vicinity of the tips $a^{\pm}$ on the x-axis is found to be (see a principle established by the conditions (2.11) and (2.12))

$$\begin{aligned} &\begin{bmatrix} \sigma^d_{yy}(x,0) \\ \sigma^d_{xy}(x,0) \end{bmatrix} = \frac{\rho_* - 1}{2\rho_*} \begin{bmatrix} k_I^{\pm} \\ k_{II}^{\pm} \end{bmatrix} \frac{1}{\sqrt{2r}} + O(r^0) && x = \pm a \pm r \\ &\sigma^d_{xx}(x,0) = -c^{\pm}\frac{3+\rho_*}{2\rho_*}\frac{k_I^{\pm}}{\sqrt{2r}} + O(r^0) && x = \pm a \pm r \\ &\begin{bmatrix} u^d(x,0) \\ v^d(x,0) \end{bmatrix} = \frac{\rho_* - \kappa_*}{2\mu_*\rho_*} \begin{bmatrix} k_{II}^{\pm} \\ k_I^{\pm} \end{bmatrix} \sqrt{\frac{r}{2}} + O\Big(r^{\frac{3}{2}}\Big) && x = \pm a \mp r \end{aligned} \tag{3.1}$$

where the constants $c^{\pm}$ are given in the Appendix and the parameters $k_I^{\pm}$, $k_{II}^{\pm}$ (superscripts $(\cdot)^-$ and $(\cdot)^+$ refer to the left and right-hand crack (inclusion) tips, respectively), known from the fracture mechanics as the stress intensity factors, are defined by

$$\begin{aligned} k_I^{\pm} - \mathrm{i}t_* k_{II}^{\pm} = \frac{1}{2\pi\sqrt{a}}\Big[\int\limits_{-a}^{a} \sqrt{\frac{a \pm t}{a \mp t}} F^*(\rho_*, t) \pm \\ \pm\mathrm{i}\frac{\rho+\rho_*}{\rho-\rho_*}\int\limits_{-a}^{a} G^*(\rho_*,t)\,dt \mp \mathrm{i}\frac{4\rho_*}{\rho-\rho_*}\int\limits_{-a}^{a} T(t)\,dt\Big] \end{aligned} \tag{3.2}$$

From the solution obtained above within the framework of the homogenized model it is seen that the general character of the asymptotic relations is similar to that in the homogeneous case. Intensification of local thermal stresses is measured by the stress intensity factors. The effects of layering on local crack displacements and inclusion stresses are observed in parameter κ_* (see Appendix).

The general analysis may be used to investigation of the thermal stresses in laminated space containing interface cracks or rigid inclusions and subjected to specific loads, e.g. under concentrated forces and/or heat sources. It becomes possible to compare the results with those obtained in the homogeneous bodies (cf the monographs by Berezhnitsky et al., 1983; Kit and Krivtzun, 1983).

Appendix

Denoting by $\eta = \delta_1/\delta$, $b_l = \lambda_l + 2\mu_l$ $(l = 1, 2)$, $b = (1-\eta)b_1 + 1 + \eta b_2$, the positive coefficients in Eqs (2.1) and (2.2) are given by the following formulae

$$\widetilde{k} = \eta k_1 + (1-\eta)k_2 \qquad K = \frac{k_1 k_2}{(1-\eta)k_1 + \eta k_2} \qquad A_1 = \frac{b_1 b_2}{b}$$

$$A_2 = A_1 + \frac{4}{b}\eta(1-\eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)$$

$$C = \frac{\mu_1 \mu_2}{(1-\eta)\mu_1 + \eta\mu_2} \qquad B = \frac{1}{b}[(1-\eta)\lambda_2 b_1 + \eta\lambda_1 b_2]$$

$$K_1 = \frac{1}{b}[(1-\eta)\beta_2 b_1 + \eta\beta_1 b_2] \qquad D_l = \frac{\lambda_l A_1}{b_l}$$

$$K_2 = \eta\beta_1\lambda_2 + \frac{1}{b}(1-\eta)\beta_2\lambda_1 + \frac{2}{b}[\eta\mu_2 + (1-\eta)\mu_1][\eta\beta_1 + (1-\eta)\beta_2]$$

$$E_l = \frac{1}{b_l}[4\mu_l(\lambda_l + \mu_l) + \lambda_l B] \qquad F_l = \frac{1}{b_l}(2\beta_l\mu_l + \lambda_l K_1)$$

The constants appearing in the complex representation (2.5) are given as follows

$$t_* = \sqrt[4]{\frac{A_1}{A_2}} \qquad \mu_* = \frac{A_+ A_-}{2(A_* - A_-)} \qquad \kappa_* = \frac{A_* + A_-}{A_* - A_-}$$

provided

$$A_* = \sqrt[4]{A_1 A_2}\sqrt{\frac{(A_+ + 2C)A_-}{C}} \qquad A_\pm = \sqrt{A_1 A_2} \pm B \qquad k_0 = \sqrt{\frac{\widetilde{k}}{K}}$$

The constants $c^{\pm}$ in Eqs (3.1) are defined as

$$c^{+} = c^{(1)} \qquad c^{-} = c^{(2)}$$

provided

$$c^{(l)} = 1 + \frac{2\mu_l(2\lambda_l + 2\mu_l - A_+)}{(\lambda_l + 2\mu_l)A_+}$$

References

1. BEREZHNITSKY L.T., PANASYUK V.V., STASCHUK N.G., 1983, *Interaction of Rigid Linear Inclusions and Cracks*, Izd. Naukova Dumka, Kiev (in Russian

2. ERDOGAN F., 1972, Fracture Problems of Composite Materials, *Engng. Fracture Mech.*, **4**, 811-840

3. KACZYŃSKI A., 1993, Stress Intensity Factors for an Interface Crack in a Periodic Two-Layered Composite Under the Action of Heat Sources, *Int. J. Fract.*, **62**, 183-202

4. KACZYŃSKI A., MATYSIAK S.J., 1988, On the Complex Potentials of the Linear Thermoelasticity with Microlocal Parameters, *Acta Mechanica*, **72**, 245-259

5. KACZYŃSKI A., MATYSIAK S.J, 1989, Thermal Stresses in a Laminate Composite with a Row of Interface Cracks, *Int. J. Engng. Sci.*, **27**, 2, 131-147

6. KACZYŃSKI A., MATYSIAK S.J, 1997, Some 2D Interface Crack and Rigid Inclusion Problems in Microperiodically Layered Elastic Composites, *J. Theor Appl. Mech.*, **35**, 4, 751-762

7. KIT G.S., KRIVTZUN M.G., 1983, *Thermoelastic Plane Problems for Bodies with Cracks*, Izd. Naukova Dumka, Kiev (in Russian)

8. MATYSIAK S.J, WOŹNIAK C., 1988, On the Microlocal Modelling of Thermoelastic Composites, *J. Tech. Phys.*, **29**, 1, 85-97

9. MUSKHELISHVILI N.I., 1966, *Some Basic Problems of the Mathematical Theory of Elasticity*, Izd. Nauka, Moskva (ed.5, in Russian)

10. NOWACKI W., 1986, *Thermoelasticity*, Polish Scientific Publishers and Pergamon Press, Warszawa and Oxford

11. SIH G.C., CHEN E.P. (EDIT.), 1981, *Cracks in Composite Materials*, Mechanics of Fracture 6, Martinus Nijhoff, The Netherlands

12. WOŹNIAK C., 1987, A Nonstandard Method of Modelling of Thermoelastic Periodic Composites, *Int. J. Engng. Sci.*, **25**, 483-499

Naprężenia cieplne w periodycznym dwuwarstwowym kompozycie spowodowane obecnością szczelin i sztywnych inkluzji międzywarstwowych

Streszczenie

Zbadano stan naprężeń cieplnych dla periodycznej dwuwarstwowej przestrzeni sprężystej, osłabionej szczeliną Griffitha bądź doskonale sztywną, lamelkową inkluzją. Analizę przeprowadzono w ramach liniowej termosprężystości z parametrami mikrolokalnymi. Wynikające zagadnienia termosprężyste zostały sprowadzone do ich odpowiedników mechanicznych. Przedyskutowano z punktu widzenia teorii pękania osobliwości naprężeń i wpływ uwarstwienia.

Manuscript received October 15, 1997; accepted for print November 25, 1997