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NONSTANDARD METHOD OF MACRO-MODELLING OF NONPERIODIC MULTILAYERED PLATES

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In this paper a method of macro-modelling of nonperiodic multilayered elastic plates is proposed. The approach is based on certain concepts of the nonstandard analysis, given by Robinson (1966), combined with some a priori postulated physical assumptions devised by Woźniak (1986, 1991) for periodic media. Using this method, the homogenized model of nonperiodic plate will be derived and applied to the evaluation of inhomogeneity effects on a critical force and a free vibration frequency for a simply supported laminated plate.

Key words: nonperiodic plates, macro-model, macro-displacements

1. Introduction

The derivation of effective coefficients for multilayered elastic *periodic* media is a thoroughly studied problem. In this paper a method of macro-modelling of multilayered but *nonperiodic* plates is proposed. The basis of considerations is the microlocal homogenization given by Woźniak (1986) for *micro-periodic media.* In this work the aforementioned approach will be modified and applied to the modelling of elastic multilayered *nonperiodic plates.*

The proposed method of macro-modelling is based on *three heuristic hypothesis* and certain concepts of the *nonstandard analysis.* The nonstandard notions are used only as a mathematical tool and they do not enter the resulting relations. The equations of micromechanics, due to the discontinuous and highly oscillating form of functions describing material properties of the composite body, are a starting point of the considerations. Taking into account the internal constraints for unknown *macro-displacements* (Woźniak, 1986) as well as for micro-displacements, determined in terms of certain extra unknows called *microlocal parameters or correctors* (Matysiak and Nagórko, 1989), and using a method of the nonstandard analysis (Robinson, 1966) we arrive at the homogenized models of nonperiodic multilayered plates. The fundamental relations constitute a system of linear algebraic equations in microlocal parameters and a system of partial differential equations of the second order in the macro-displacements. The obtained macro-models are plausible from the engineering standpoint and may constitute the basis for numerical analysis. In this paper, the derived effective model will be applied to determination of the effect of the micro- heterogeneity on the stability and dynamic behaviour of nonperiodic laminated Reissner-type plates. The problem will be studied within the framework of nonlinear plate theory. The considerations are focused on the multilayered plates made of three homogeneous linear-elastic anisotropic materials, but the method can also be applied to the structures composed of a large number of different materials.

2. Preliminaries

An underformed plate which occupies a region Ω in physical space (parametrized by the Cartesian orthogonal coordinates x_1, x_2, x_3) bounded by the coordinate planes $x_3 = h^+$, $x_3 = h^-$, where $h^+ > 0$, $h^- < 0$ and by the cylindrical surface $\Gamma \equiv \partial\Pi \times (h^-, h^+)$, where Π is a regular region on the plane $0x_1x_2$ is considered. We define $\boldsymbol{x} \equiv (x_1, x_2, x_3) \in \Omega$, $x_\alpha \equiv (x_1, x_2) \in \Pi$, $x_3 \in [h^-, h^+]$, $\tau \in [\tau_0, \tau_f]$ stands for the time coordinate.

The plate is composed of N basic layers bounded by the coordinate planes $x_3 = h^- + \zeta_K$, $K = 0, 1, 2, ..., N$, with $\zeta_0 = 0$, $\zeta_N = h$, where $h = h^+ - h^-$ denotes the thickness of the plate; ζ_{K-1} describes the distance of Kth basic lamina from the boundry plane $x_3 = h^-$. The thickness $\varepsilon_K \equiv \zeta_K - \zeta_{K-1}$, $K = 1, 2, ..., N$, $(\zeta_K > \zeta_{K-1})$, of every basic layer is assumed to be sufficiently small when compared to the thickness h of the plate; i.e. we shall deal with the nonperiodic plates composed of a large number of laminae. Moreover, let every basic layer (ζ_{K-1}, ζ_K), $K = 1, 2, ..., N$, consist of three sublayers $(\zeta_{K-1}, \zeta_{K-1} + \delta_K)$, $(\zeta_{K-1} + \delta_K, \zeta_{K-1} + \delta_K + \widetilde{\delta}_K)$, $(\zeta_{K-1} + \delta_K + \widetilde{\delta}_K, \zeta_K)$, made of three different homogeneous anisotropic linear-elastic materials; by δ_K, $\widetilde{\delta}_K$ we denote the thicknesses of upper and middle sublayers, respectively of the Kth basic unit. Perfect bonding between the layers is assumed. The scheme of the plate and basic notions are shown in Fig.1.

Hereinafter the subscripts i, j run over $1, 2, 3$ and are referred to the Cartesian orthogonal coordinate system $0x_1x_2x_3$; the subscripts $\alpha, \beta, \gamma, \delta$ run over $1, 2$ being related to the coordinate system $0x_1x_2$ on the plane

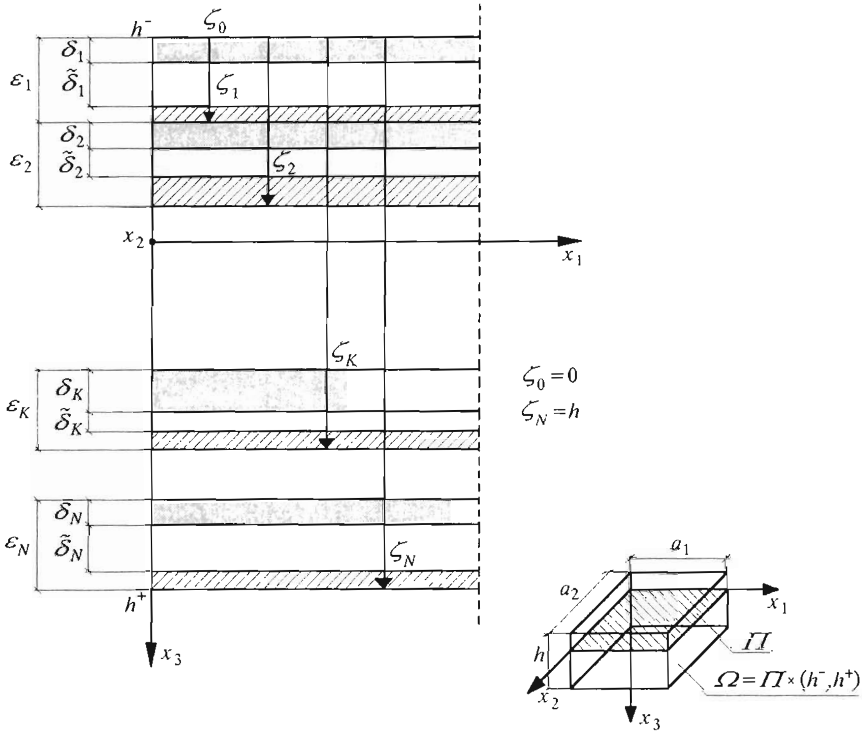

Fig. 1. Scheme of the composite

$x_3 = 0$. Non-tensorial indices a, d take values $1, 2$ and are related to the postulated a priori micro-shape functions. The summation convetion holds with respect to all aforementioned indices.

The composite is loaded on the boundary planes $x_3 = h^+$, $x_3 = h^-$ by means of the known normal surface tractions $p^3_+(x_\alpha, \tau)$, $p^3_-(x_\alpha, \tau)$, respectively, and on the part Γ of the boundary the displacements $\boldsymbol{u}_\Gamma(\boldsymbol{x}, \tau)$, $\boldsymbol{x} \in \Gamma$, are known. By $e_{ij}(\boldsymbol{x}, \tau)$, $t_{ij}(\boldsymbol{x}, \tau)$ we denote strains and stresses, respectively, as the functions defined (almost everywhere) on Ω. Let $u_i(\boldsymbol{x}, \tau)$, $b^3(\boldsymbol{x}, \tau)$ be the displacements and body forces, respectively. The properties of the plate under consideration are determined by a mass density $\rho(x_3)$ and the tensor of elastic modulae $c^{ijkl}(x_3)$. The fields $\rho(x_3)$, $c^{ijkl}(x_3)$ are defined almost everywhere on Ω and assumed to be piecewise constant; they are highly oscillating fields.

We shall define the subsets of $[h^-, h^+]$ by means of

$$L \equiv \bigcup_{K=1}^{N} \left(h^- + \zeta_{K-1}, h^- + \zeta_{K-1} + \eta_K \varepsilon_K\right) \qquad \eta_K = \frac{\delta_K}{\varepsilon_K} \qquad \tilde{\eta}_K = \frac{\tilde{\delta}_K}{\varepsilon_K}$$

$$S \equiv \bigcup_{K=1}^{N} \Big(h^- + \zeta_{K-1} + \eta_K \varepsilon_K, h^- + \zeta_{K-1} + (\eta_K + \widetilde{\eta}_K)\varepsilon_K\Big) \tag{2.1}$$

$$U \equiv \bigcup_{K=1}^{N} \Big(h^- + \zeta_{K-1} + (\eta_K + \widetilde{\eta}_K)\varepsilon_K, h^- + \zeta_K\Big) \qquad K = 1,2,...,N$$

It is assumed that the composite is made of three homogeneous linear-elastic anisotropic materials which in the undeformed state occupy the parts $\Pi \times L$, $\Pi \times S$, $\Pi \times U$ of the region Ω. Hence the mass denisity $\rho(x_3)$ and the tensor of elastic constants $c^{ijkl}(x_3)$ of the nonperiodic plate under consideration will be given by

$$\Big(\rho(x_3), c^{ijkl}(x_3)\Big) = \left\{ \begin{array}{lll} ({}^L\rho, {}^Lc^{ijkl}) & \text{if} & x_3 \in L \\ ({}^S\rho, {}^Sc^{ijkl}) & \text{if} & x_3 \in S \\ ({}^U\rho, {}^Uc^{ijkl}) & \text{if} & x_3 \in U \end{array} \right. \tag{2.2}$$

where ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$ are material constants related to the parts $\Pi \times L$, $\Pi \times S$, $\Pi \times U$, respectively.

We introduce the discrete functions defined at the points $\Big\{h^- + \frac{h}{N}, h^- + \frac{2h}{N}, h^- + \frac{3h}{N}, ..., h^+\Big\}$

$$\overline{\zeta}\Big(h^- + \frac{Kh}{N}\Big) \equiv \zeta_K \qquad \overline{\eta}\Big(h^- + \frac{Kh}{N}\Big) \equiv \eta_K$$
$$\overline{\overline{\eta}}\Big(h^- + \frac{Kh}{N}\Big) \equiv \widetilde{\eta}_K \qquad K = 1,2,...,N \tag{2.3}$$

These discrete functions describe the distribution of layers and sublayers in the composite under consideration.

3. The primary problem – the equations of micromechanics

The governing equations of the plate under consideration will be represented by:

- The strain-displacement relations in which the non-linear terms involving gradients of $u_\alpha(\boldsymbol{x},\tau)$ are neglected

$$e_{\alpha\beta}(\boldsymbol{x},\tau) = u_{(\alpha,\beta)}(\boldsymbol{x},\tau) + \frac{1}{2}u_{3,\alpha}(\boldsymbol{x},\tau)u_{3,\beta}(\boldsymbol{x},\tau) \qquad e_{\alpha 3}(\boldsymbol{x},\tau) = u_{(\alpha,3)}(\boldsymbol{x},\tau)$$
$$e_{33}(\boldsymbol{x},\tau) = u_{3,3}(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \in \Omega \qquad \tau \in [\tau_0, \tau_f] \tag{3.1}$$

- The stress-strain relations

$$t^{\alpha\beta}(\boldsymbol{x},\tau) = \overline{c}^{\alpha\beta\gamma\delta}(x_3)e_{\gamma\delta}(\boldsymbol{u})(\boldsymbol{x},\tau) \qquad t^{\alpha 3}(\boldsymbol{x},\tau) = 2c^{\alpha 33\delta}(x_3)e_{3\delta}(\boldsymbol{x},\tau) \tag{3.2}$$

where

$$\overline{c}^{\alpha\beta\gamma\delta}(x_3) = c^{\alpha\beta\gamma\delta}(x_3) - \frac{c^{\alpha\beta 33}(x_3)c^{\gamma\delta 33}(x_3)}{c^{3333}(x_3)}$$

- The virtual work principle

$$\begin{aligned}
&\int\limits_{h^-}^{h^+}\int\limits_{\Omega}[t^{\alpha\beta}\delta e_{\alpha\beta} + 2t^{\alpha 3}\delta e_{\alpha 3} + t^{33}\delta e_{33}]\,d\Pi\,dx_3 = \\
&\int\limits_{\Omega}\Big[p_+^3\delta u_3\Big|_{x_3=h^+} + p_-^3\delta u_3\Big|_{x_3=h^-}\Big]\,d\Pi \\
&\int\limits_{h^-}^{h^+}\int\limits_{\Omega}\rho b^3\delta u_3\,d\Pi\,dx_3 - \int\limits_{h^-}^{h^+}\int\limits_{\Omega}\rho\ddot{u}^i\delta u_i\,d\Pi\,dx_3 \qquad d\Pi \equiv dx_1dx_2
\end{aligned} \tag{3.3}$$

which holds for any admissible virtual displacement field δu_i, such that $\delta u_i(\boldsymbol{x},\tau) = 0$ for $\boldsymbol{x} \in \Gamma$

- The initial and boundary conditions

$$\boldsymbol{u}(\boldsymbol{x},\tau_0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \dot{\boldsymbol{u}}(\boldsymbol{x},\tau_0) = \boldsymbol{v}_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega \tag{3.4}$$

$$\boldsymbol{u}(\boldsymbol{x},\tau) = \boldsymbol{u}_\Gamma(\boldsymbol{x},\tau) \qquad \boldsymbol{x} \in \Gamma \qquad \tau \in [\tau_0,\tau_f] \tag{3.5}$$

The equations of micromechanics (3.1)-(3.5) are the starting point of the analysis.

Now we formulate the following:

Problem P: for known Ω, p_-^3, p_+^3, b^3, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{u}_\Gamma$ and ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$ as well as for the known L, S, U, find the displacements $\boldsymbol{u}(\boldsymbol{x},\tau)$ and stresses $\boldsymbol{t}(\boldsymbol{x},\tau)$, $\boldsymbol{x} \in \Omega$, $\tau \in [\tau_0,\tau_f]$, such that Eqs (3.1) $\div$ (3.5) under conditions (2.2) hold.

The above primary problem does not constitute a mathematical approach which can be successfully applied to numerical calculation of engineering problems. That is why we shall pass from the problem P to a certain effective problem $\widetilde{P}$ and to the computational model of nonperiodic multilayered plates which will be plausible for engineering applications.

In order to formulate the problem $\widetilde{P}$ we shall first formulate a certain auxiliary problem $P^{(1)}$ and then a sequence of problems $P^{(m)}$, $m = 1, 2, , ...$, leading to a nonstandard problem $P^{(\widetilde{\omega})}$, where $\widetilde{\omega}$ is an infinitely large positive integer (Robinson, 1966). The effective problem $\widetilde{P}$ will be obtained using a certain special approximation of the problem $P^{(\widetilde{\omega})}$.

4. Passage to the nonstandard problem – modelling hypotheses

In order to formulate an auxiliary problem $P^{(1)}$ we shall approximate the discrete functions $\overline{\zeta}(x_3)$, $\overline{\eta}(x_3)$, $\overline{\overline{\eta}}(x_3)$ by certain *continuous* and *differentiable* functions $\zeta(x_3)$, $\eta_1(x_3)$, $\eta_2(x_3)$, respectively, defined on the interval $[h^-, h^+]$

$$\zeta : [h^-, h^+] \rightarrow [0, h] \qquad \eta_1 : [h^-, h^+] \rightarrow (0, 1) \qquad \eta_2 : [h^-, h^+] \rightarrow (0, 1) \tag{4.1}$$

The above continuous functions have to satisfy the following conditions

(i) $\zeta(x_3)$ must be a strongly monotone function, such that $\zeta(x_3 = h^-) = 0$, $\zeta(x_3 = h^+) = h$

(ii) $\eta_1(x_3)$, $\eta_2(x_3)$ cannot be highly oscillating functions and

$$\forall x_3 \in [h^-, h^+] \qquad \eta_1(x_3) + \eta_1(x_3) < 1$$

For periodic structure these functions reduce to the constants.

In the problem $P^{(1)}$, the new thickness of Kth basic layer $\varepsilon_K^{(1)}$, $K = 1, 2, ..., N$, the new thicknesses of upper and middle sublayers $\delta_K^{(1)}$, $\widetilde{\delta}_K^{(1)}$ and subsets $L^{(1)}$, $S^{(1)}$, $U^{(1)}$ are determined by the functions (4.1) in the following form

$$\varepsilon_K^{(1)} \equiv \zeta_K^{(1)} - \zeta_{K-1}^{(1)} \qquad \delta_K^{(1)} = \varepsilon_K^{(1)} \eta_K^{(1)} \qquad \widetilde{\delta}_K^{(1)} = \varepsilon_K^{(1)} \widetilde{\eta}_K^{(1)} \tag{4.2}$$

$$\begin{aligned}
L^{(1)} &\equiv \bigcup_{K=1}^{N} \Big(h^- + \zeta_{K-1}^{(1)}, h^- + \zeta_{K-1}^{(1)} + \eta_K^{(1)} \varepsilon_K^{(1)} \Big) \\
S^{(1)} &\equiv \bigcup_{K=1}^{N} \Big(h^- + \zeta_{K-1}^{(1)} + \eta_K^{(1)} \varepsilon_K^{(1)}, h^- + \zeta_{K-1}^{(1)} + (\eta_K^{(1)} + \widetilde{\eta}_K^{(1)}) \varepsilon_K^{(1)} \Big) \\
U^{(1)} &\equiv \bigcup_{K=1}^{N} \Big(h^- + \zeta_{K-1}^{(1)} + (\eta_K^{(1)} + \widetilde{\eta}_K^{(1)}) \varepsilon_K^{(1)}, h^- + \zeta_K^{(1)} \Big)
\end{aligned} \tag{4.3}$$

where

$$\zeta_K^{(1)} \equiv \zeta_K\Big(h^- + \frac{Kh}{N}\Big) \qquad \eta_K^{(1)} \equiv \eta_1\Big(h^- + \frac{Kh}{N}\Big)$$
$$\widetilde{\eta}_K^{(1)} \equiv \eta_2\Big(h^- + \frac{Kh}{N}\Big) \qquad K = 1, 2, ..., N \tag{4.4}$$

In the problem $P^{(1)}$, the tensor of elastic constants $c^{ijkl}(x_3)$ and the mass density $\rho(x_3)$ will be given by (4.5)

$$\Big(\rho(x_3), c^{ijkl}(x_3)\Big) = \begin{cases} ({}^L\rho, {}^Lc^{ijkl}) & \text{if} \quad x_3 \in L^{(1)} \\ ({}^S\rho, {}^Sc^{ijkl}) & \text{if} \quad x_3 \in S^{(1)} \\ ({}^U\rho, {}^Uc^{ijkl}) & \text{if} \quad x_3 \in U^{(1)} \end{cases} \tag{4.5}$$

Now we formulate the following

Problem $P^{(1)}$**:** for known Ω, p^3_-, p^3_+, b^3, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{u}_\Gamma$ and ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$, as in the problem P, and for known $L^{(1)}$, $S^{(1)}$, $U^{(1)}$, find the displacements $\boldsymbol{u}(\boldsymbol{x},\tau)$ and stresses $\boldsymbol{t}(\boldsymbol{x},\tau)$, $\boldsymbol{x} \in \Omega$, $\tau \in [\tau_0, \tau_f]$, such that Eqs (3.1) ÷ (3.5) under conditions (4.5) hold.

The method of macro-modelling of nonperiodic multilayered plates proposed in the paper is based on three hypotheses. The first hypothesis is reffered to as

The layer distribution hypothesis. If the discrete functions (2.3) ÷ (2.5) can be approximated by the continuous and differentiable functions (4.1) satisfying the aforementioned conditions (i), (ii), then the solution to the boundary-initial value problem P can be approximated by the solution to the problem $P^{(1)}$.

The above hypothesis cannot be accepted for the periodically-laminated media.

Now we formulate a sequence of problems $P^{(m)}$, where m is an arbitrary positive integer. The problem $P^{(m)}$ describes a certain composite which is made of mN number of thin basic layers.

In the problem $P^{(m)}$, the new thickness of Rth basic layer $\varepsilon_R^{(m)}$, $R = 1, 2, ..., mN$, and new thicknesses of upper and middle sublayers $\delta_R^{(m)}$, $\widetilde{\delta}_R^{(m)}$, respectively are defined as follows

$$\varepsilon_R^{(m)} \equiv \zeta_R^{(m)} - \zeta_{R-1}^{(m)} \qquad \delta_R^{(m)} = \varepsilon_R^{(m)}\eta_R^{(m)} \qquad \widetilde{\delta}_R^{(m)} = \varepsilon_R^{(m)}\widetilde{\eta}_R^{(m)} \tag{4.6}$$

where

$$\zeta_R^{(m)} \equiv \zeta_R\Big(h^- + \frac{Rh}{mN}\Big) \qquad \eta_R^{(m)} \equiv \eta_1\Big(h^- + \frac{Rh}{mN}\Big) \qquad \tilde{\eta}_R^{(m)} \equiv \eta_2\Big(h^- + \frac{Rh}{mN}\Big) \tag{4.7}$$

In the problem $P^{(m)}$, the subsets $L^{(m)}$, $S^{(m)}$, $U^{(m)}$ and functions $c^{ijkl}(x_3)$, $\rho(x_3)$ are given by Eqs (4.3), (4.5), respectively, in which the superscript (1) and the subscript $K = 1,2,...,N$ must be replaced by (m) and $R = 1,2,...,mN$, respectively.

Then an arbitrary auxiliary problem $P^{(m)}$, $m = 1,2,...$, will be stated as follows

Problem $P^{(m)}$**:** for known Ω, p_-^3, p_+^3, b^3, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{u}_\Gamma$ and ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$, as in the problem P, and for known $L^{(m)}$, $S^{(m)}$, $U^{(m)}$, find the displacements $\boldsymbol{u}(\boldsymbol{x},\tau)$ and stresses $\boldsymbol{t}(\boldsymbol{x},\tau)$, $\boldsymbol{x} \in \Omega$, $\tau \in [\tau_0,\tau_f]$, such that Eqs (3.1) ÷ (3.5) hold.

Define $\varepsilon^{(1)} \equiv \max \varepsilon_K^{(1)}$, $K = 1,2,...,N$, as the maximum thickness of a layer in a plate described by the problem $P^{(1)}$. The second hypothesis, given by Woźniak (1986), will be referred to as

The homogenization hypothesis. If $\varepsilon^{(1)} \equiv \max \varepsilon_K^{(1)} \ll h$, $K = 1,2,...,N$, then the problem $P^{(1)}$ can be approximated by the problem $P^{(m)}$ for every positive integer m.

The known transfer principle (Robinson, 1966) implies that the sequence of problems $P^{(m)}$ is leading to a nonstandard problem $P^{(\tilde{\omega})}$, where $\tilde{\omega}$ is an infinitely large positive integer. The problem $P^{(\tilde{\omega})}$ describes a certain composite which is made of an infinite number of infinitely thin layers.

The formulation of the problem $P^{(\tilde{\omega})}$ is similar to that of the problem $P^{(m)}$. We replace the entities Ω, Π, p_-^3, p_+^3, b^3, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{u}_\Gamma$ and ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$, $\zeta(x_3)$, $\eta_1(x_3)$, $\eta_2(x_3)$ with the standard entities, which will be denoted by ${}^*\Omega$, ${}^*\Pi$, ..., ${}^*\eta_1(x_3)$, ${}^*\eta_2(x_3)$, respectively. Instead of the functions $\rho(x_3)$, $c^{ijkl}(x_3)$, $\boldsymbol{u}(\boldsymbol{x},\tau)$, $\boldsymbol{t}(\boldsymbol{x},\tau)$ we introduce the nonstandard functions ${}^{(\tilde{\omega})}\rho(x_3)$, ${}^{(\tilde{\omega})}c^{ijkl}(x_3)$, ${}^{(\tilde{\omega})}\boldsymbol{u}(\boldsymbol{x},\tau)$, ${}^{(\tilde{\omega})}\boldsymbol{t}(\boldsymbol{x},\tau)$, respectively. Now, Eqs (4.6), (4.7) have the form

$$\varepsilon_R^{(\tilde{\omega})} \equiv \zeta_M^{(\tilde{\omega})} - \zeta_{M-1}^{(\tilde{\omega})} \qquad \delta_M^{(\tilde{\omega})} = \varepsilon_M^{(\tilde{\omega})}\eta_M^{(\tilde{\omega})} \qquad \tilde{\delta}_M^{(\tilde{\omega})} = \varepsilon_M^{(\tilde{\omega})}\tilde{\eta}_M^{(\tilde{\omega})} \tag{4.8}$$

where

$$\zeta_M^{(\widetilde{\omega})} \equiv {}^*\zeta\Big(h^- + \frac{Mh}{\widetilde{\omega}N}\Big) \qquad \eta_M^{(\widetilde{\omega})} \equiv {}^*\eta_1\Big(h^- + \frac{Mh}{\widetilde{\omega}N}\Big)$$
$$\widetilde{\eta}_M^{(\widetilde{\omega})} \equiv {}^*\eta_2\Big(h^- + \frac{Mh}{\widetilde{\omega}N}\Big) \qquad M = 1,2,...,\widetilde{\omega}N \tag{4.9}$$

The subsets $L^{(\widetilde{\omega})}$, $S^{(\widetilde{\omega})}$, $U^{(\widetilde{\omega})}$ and functions ${}^{(\widetilde{\omega})}c^{ijkl}(x_3)$, ${}^{(\widetilde{\omega})}\rho(x_3)$ are determined by Eqs (4.3), (4.5) in which the superscript (1) and the subscript $K = 1,2,...,N$ must be replaced by $(\widetilde{\omega})$ and $M = 1,2,...,\widetilde{\omega}N$, respectively.

Now, we introduce the third hypothesis (Woźniak, 1991).

The micro-macro localization hypothesis. The approximate solution to the nonstandard problem $P^{(\widetilde{\omega})}$ can be expected in the class of functions given by

$$u_\alpha^{(\widetilde{\omega})}(\boldsymbol{x},\tau) = {}^*W_\alpha(x_\alpha,\tau) + x_3{}^*D_\alpha(x_\alpha,\tau) + h_a(x_3){}^*Q_\alpha^a(x_\alpha,\tau)$$
$$u_3^{(\widetilde{\omega})}(\boldsymbol{x},\tau) = {}^*W_3(x_\alpha,\tau) \qquad \boldsymbol{x} \in {}^*\Omega \qquad \tau \in {}^*[\tau_0,\tau_f] \qquad \alpha, a = 1,2 \tag{4.10}$$

where

– ${}^*W_i$, ${}^*D_\alpha$, ${}^*Q_\alpha^a$ are (sufficiently regular) arbitrary and independent unknown standard functions, the fields W_i, D_α are called macro-displacements (Woźniak, 1991), the functions Q_α^a are called microlocal (or correction) parameters (Matysiak and Nagórko, 1989)

– $h_a(x_3)$ are postulated a priori, linear independent, nonstandard micro-shape functions (Woźniak, 1991); they take only infinitesimal values but their derivatives take standard values.

The micro-shape functions introduced in this paper are represented graphically in Fig.2.

It can be observed that Eqs (4.10) represents a certain generalization of the known kinematic hypothesis for the Reissner-type plate theory, which takes into account the effect of plate inhomogeneity on the distribution of displacements.

In the nonstandard structure M^*, using the micro-macro localization hypothesis (4.10) as well as the strain-displacement relations (3.1) and the stress-strain relations (3.2), after neglecting the terms involving micro-shape functions (but not their derivatives!) we obtain the strain tensor and the

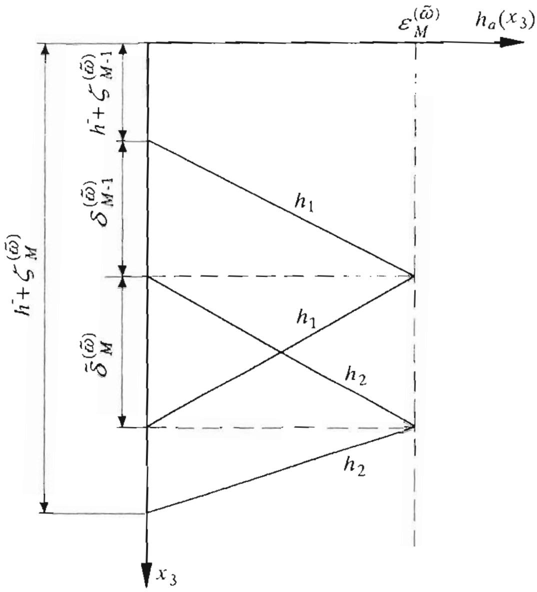

Fig. 2. The graph of the micro-shape functions $h_1(x_3)$, $h_2(x_3)$

stresses ${}^{*L}t^{\alpha j}$, ${}^{*S}t^{\alpha j}$, ${}^{*U}t^{\alpha j}$ in the parts ${}^*\Pi \times L^{(\widetilde{\omega})}$, ${}^*\Pi \times S^{(\widetilde{\omega})}$, ${}^*\Pi \times U^{(\widetilde{\omega})}$ of the region ${}^*\Omega$, respectively. Then, taking into account the known theorems of nonstandard integral calculus (Robinson, 1966; Woźniak, 1991) we arrive at the virtual work principle, which includies only the standard entities; so we can pass from the nonstandard structure ${}^*M$ to the classical structure M. The approximate solution to the nonstandard problem can be found as the solution to a certain problem for the macro-displacements and the correction parameters. This problem does not involve any nonstandard entity; it will be called effective or microlocal and denote by $\widetilde{P}$.

5. The effective problem – the equations of macromechanics

Let us define

$$T^{\alpha j}(\boldsymbol{x},\tau) \equiv {}^{L}t^{\alpha j}(\boldsymbol{x},\tau)\eta_1(x_3) + {}^{S}t^{\alpha j}(\boldsymbol{x},\tau)\eta_2(x_3) + \\ + {}^{U}t^{\alpha j}(\boldsymbol{x},\tau)[1-\eta_1(x_3)-\eta_2(x_3)] \tag{5.1}$$

$$\widetilde{\rho}(x_3) \equiv {}^{L}\rho\eta_1(x_3) + {}^{S}\rho\eta_2(x_3) + {}^{U}\rho[1-\eta_1(x_3)-\eta_2(x_3)] \tag{5.2}$$

The tensor $T^{\alpha j}(\boldsymbol{x},\tau)$ will be called the mean stress tensor. The scalar $\widetilde{\rho}(x_3)$ will be called the mean mass density. Since the functions W_i, D_α, Q^a_α are arbitrary and independent, then after introducing denotations

$$
\begin{aligned}
N^{\alpha\beta}(x_\alpha,\tau) &= \int\limits_{h^-}^{h^+} T^{\alpha\beta}\,dx_3 & M^{\alpha\beta}(x_\alpha,\tau) &= \int\limits_{h^-}^{h^+} x_3T^{\alpha\beta}\,dx_3 \\
\widehat{Q}^{\alpha}(x_\alpha,\tau) &= \int\limits_{h^-}^{h^+} T^{\alpha 3}\,dx_3 & p^3(x_\alpha,\tau) &\equiv p^3_+ + p^3_- + \int\limits_{h^-}^{h^+} \widetilde{\rho}(x_3)b^3\,dx_3 \qquad (5.3)\\
\widetilde{f} &\equiv \int\limits_{h^-}^{h^+} \widetilde{\rho}(x_3)\,dx_3 & \widehat{f} &\equiv \int\limits_{h^-}^{h^+} \widetilde{\rho}(x_3)x_3^2\,dx_3
\end{aligned}
$$

and using the divergence theorem as well as the du Bois-Reymonde lemma, we obtain from the virtual work principle the following equations of homogenized model

— the plate equations of motion

$$
\begin{aligned}
&N^{\alpha\beta}{}_{,\beta}(x_\alpha,\tau) = \widetilde{f}\ddot{W}^{\alpha}(x_\alpha,\tau)\\
&M^{\alpha\beta}{}_{,\beta}(x_\alpha,\tau) - \widehat{Q}^{\alpha}(x_\alpha,\tau) = \widehat{f}\ddot{D}^{\alpha}(x_\alpha,\tau) \qquad (5.4)\\
&\widehat{Q}^{\alpha}_{,\alpha}(x_\alpha,\tau) + \Big(N^{\alpha\beta}(x_\alpha,\tau)W_{3,\alpha}(x_\alpha,\tau)\Big)_{,\beta} + p^3(x_\alpha,\tau) = \widetilde{f}\ddot{W}^{3}(x_\alpha,\tau)
\end{aligned}
$$

— the system of linear algebraic formulas for correctors

$$
P^{\alpha 33\delta}_{ab}Q^b_\delta(x_\alpha,\tau) = -h[\![c^{\alpha 33\delta}_a]\!]\Big(W_{3,\delta}(x_\alpha,\tau) + D_\delta(x_\alpha,\tau)\Big) \qquad (5.5)
$$

where

$$
P^{\alpha 33\delta}_{ab} \equiv \begin{cases}
\int\limits_{h^-}^{h^+}\Big(\dfrac{{}^{L}c^{\alpha 33\delta}}{\eta_1(x_3)} + \dfrac{{}^{S}c^{\alpha 33\delta}}{\eta_2(x_3)}\Big)\,dx_3 & \text{if} \quad a=b=1\\
\int\limits_{h^-}^{h^+}\Big(\dfrac{{}^{S}c^{\alpha 33\delta}}{\eta_2(x_3)} + \dfrac{{}^{U}c^{\alpha 33\delta}}{1-\eta_1(x_3)-\eta_2(x_3)}\Big)\,dx_3 & \text{if} \quad a=b=2\\
-\int\limits_{h^-}^{h^+}\dfrac{{}^{S}c^{\alpha 33\delta}}{\eta_2(x_3)}\,dx_3 & \text{if} \quad \begin{cases} a=1 \ \wedge \ b=2\\ \text{or}\\ a=2 \ \wedge \ b=1\end{cases}
\end{cases} \qquad (5.6)
$$

$$
[\![c^{\alpha 33\delta}_1]\!] \equiv {}^{L}c^{\alpha 33\delta} - {}^{S}c^{\alpha 33\delta} \qquad [\![c^{\alpha 33\delta}_2]\!] \equiv {}^{S}c^{\alpha 33\delta} - {}^{U}c^{\alpha 33\delta} \qquad (5.7)
$$

A solution to Eq (5.5) can be written in the form

$$
Q^b_\delta(x_\alpha,\tau) = -hK^{bd}_{\delta 33\gamma}[\![c^{\gamma 33\beta}_d]\!]\Big(W_{3,\beta}(x_\alpha,\tau) + D_\beta(x_\alpha,\tau)\Big) \qquad (5.8)
$$

where $K^{bd}_{\delta 33\gamma}$ are defined by

$$P^{\alpha 33\gamma}_{ab} K^{bd}_{\delta 33\gamma} = \delta^d_a \delta^\alpha_\delta \tag{5.9}$$

Let us observe that $[\![c^{\alpha 33\delta}_a]\!] = 0$ implies that $Q^a_\alpha(x_\alpha,\tau) = 0$ and the micro-local effects in this case disappear. After eliminating the correctors from the homogenized model, by means of Eq (5.9), we arrive at the equations of macromechanics expressed only in terms of macro-displacements:
— The plate constitutive equations

$$\begin{aligned}
N^{\alpha\beta}(x_\alpha,\tau) &= \overline{B}^{\alpha\beta\gamma\delta}\Big[W_{\gamma,\delta}(x_\alpha,\tau) + \frac{1}{2}W_{3,\gamma}(x_\alpha,\tau)W_{3,\delta}(x_\alpha,\tau)\Big] + \\
&+ \overline{F}^{\alpha\beta\gamma\delta} D_{\gamma,\delta}(x_\alpha,\tau) \\
M^{\alpha\beta}(x_\alpha,\tau) &= \overline{F}^{\alpha\beta\gamma\delta}\Big[W_{\gamma,\delta}(x_\alpha,\tau) + \frac{1}{2}W_{3,\gamma}(x_\alpha,\tau)W_{3,\delta}(x_\alpha,\tau)\Big] + \\
&+ \overline{G}^{\alpha\beta\gamma\delta} D_{\gamma,\delta}(x_\alpha,\tau) \\
\widehat{Q}^{\alpha}(x_\alpha,\tau) &= (B^{\alpha 33\beta} - H^{\alpha 33\beta})[W_{3,\beta}(x_\alpha,\tau) + D_\beta(x_\alpha,\tau)]
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
&H^{\alpha 33\delta} \equiv h^2 [\![c^{\alpha 33\delta}_a]\!] K^{ad}_{\delta 33\gamma} [\![c^{\gamma 33\beta}_d]\!] \\
&\overline{B}^{\alpha\beta\gamma\delta} \equiv \int\limits_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)\, dx_3 \qquad\qquad B^{\alpha 33\delta} \equiv \int\limits_{h^-}^{h^+} C^{\alpha 33\delta}(x_3)\, dx_3 \\
&\overline{F}^{\alpha\beta\gamma\delta} \equiv \int\limits_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)x_3\, dx_3 \qquad\qquad \overline{G}^{\alpha\beta\gamma\delta} \equiv \int\limits_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)x_3^2\, dx_3 \\
&\overline{C}^{\alpha\beta\gamma\delta} \equiv {}^{L}\overline{C}^{\alpha\beta\gamma\delta}\eta_1(x_3) + {}^{S}\overline{C}^{\alpha\beta\gamma\delta}\eta_2(x_3) + {}^{U}\overline{C}^{\alpha\beta\gamma\delta}[1 - \eta_1(x_3) - \eta_2(x_3)]
\end{aligned} \tag{5.11}$$

It can be proven that tensors $\overline{B}^{\alpha\beta\gamma\delta}$, $\overline{F}^{\alpha\beta\gamma\delta}$, $\overline{G}^{\alpha\beta\gamma\delta}$, $(B^{\alpha 33\beta} - H^{\alpha 33\beta})$ are positive definite.
— Combining Eqs (5.10) and (5.4) we arrive at the system of five nonlinear differential equations of motion in five basic unknows: W_i, D_α. However, in stability and vibration problems the system of governing equations will take the form

$$
\begin{aligned}
&(B^{\alpha 33\beta} - H^{\alpha 33\beta})[W_{3,\alpha\beta} + D_{(\alpha,\beta)}] + N^{\alpha\beta} W_{3,\alpha\beta} + p^3 + \\
&+\widetilde{f}\ddot{W}^{\alpha} W_{3,\alpha} - \widetilde{f}\ddot{W}_3 = 0 \\
&\underline{\overline{F}^{\alpha\beta\gamma\delta}[W_{\gamma,\delta\beta} + W_{3,\gamma} + W_{3,\delta\beta}]} + \overline{G}^{\alpha\beta\gamma\delta} D_{\gamma,\delta\beta} + \\
&-(B^{\alpha 33\beta} - H^{\alpha 33\beta})[W_{3,\beta} + D_{\beta}] - \widehat{f}\ddot{D}^{\alpha} = 0
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
&N^{\alpha\beta}{}_{,\beta} - \widetilde{f}\ddot{W}^{\alpha} = 0 \\
&N^{\alpha\beta} = \overline{B}^{\alpha\beta\gamma\delta}\Big[W_{\gamma,\delta} + \frac{1}{2} W_{3,\gamma} W_{3,\delta}\Big] + \underline{\overline{F}^{\alpha\beta\gamma\delta} D_{(\gamma,\delta)}}
\end{aligned}
\tag{5.13}
$$

The underlined terms in Eqs (5.12), (5.13) depend on $\overline{F}^{\alpha\beta\gamma\delta}$ and represent the coupling between $N_{\alpha\beta}$ and $M_{\alpha\beta}$ in the plate constitutive relations (5.10). The equations of motion (5.12), (5.13) and the constitutive relations (5.10) have to be considered together with the appropriate initial and boundary conditions which have a form analogous to that used in the well known Reissner-type plate theories. The effect of the heterogeneity of a laminated plate is described in Eqs (5.10) and (5.12) by the tensor with components $H^{\alpha 33\beta}$, defined by Eq $(5.11)_1$. It can be shown that for homogeneous plates (in this case $Q^a_\alpha(x_\alpha,\tau) = 0$ and $\overline{F}^{\alpha\beta\gamma\delta} = 0$), after neglecting the inertia terms $\widetilde{f}\ddot{W}^{\alpha}$ and non-linear terms, the resulting equations (5.12), (5.13) reduce to the well known equations of the Reissner-type plate theory.

The effective problem can be stated as follows:

Problem $\widetilde{P}$: for known Ω, p^3_+, p^3_-, b^3, initial and boundary conditions for macro-displacements and ${}^L\rho$, ${}^L c^{ijkl}$, ${}^S\rho$, ${}^S c^{ijkl}$, ${}^U\rho$, ${}^U c^{ijkl}$ as well as for known $\eta_1(x_3)$, $\eta_2(x_3)$, find the macro-displacements $W_i(x_\alpha,\tau)$, $D_\alpha(x_\alpha,\tau)$, satisfying the equations of homogenized model.

6. Example

In order to illustrate the general results obtained in the paper we shall apply Eqs (5.12), (5.13) to the analysis of the stability and free vibrations of a rectangular plate which is simply suported on the edges $x_1 = 0$, $x_1 = a_1$. We assume that $h^+ = -h^- = h/2$ and then $\overline{F}^{\alpha\beta\gamma\delta} = 0$. We shall consider this problem as one-dimensional, setting $x_\alpha \equiv x_1$. For simplicity we shall neglect the inertia terms $\widetilde{f}\ddot{W}^\alpha$ and the body forces. We also assume that $p^3 = 0$. Let

$$W_3(x_1,\tau) = \sum_{m=1}^{\infty} A_m \sin \frac{m\pi}{a_1} x_1 e^{-i\omega_m\tau}$$
$$D_1(x_1,\tau) = \sum_{m=1}^{\infty} B_m \cos \frac{m\pi}{a_1} x_1 e^{-i\omega_m\tau} \tag{6.1}$$

Using the aforementioned assumptions and subsituting (6.1) into (5.12) we obtain for $A_m \neq 0$, $B_m \neq 0$

$$\begin{vmatrix} \widetilde{f}\omega_m^2 - \mathcal{A}\lambda_m^2 + \overline{N}^{11}\lambda_m^2 & -\mathcal{A}\lambda_m \\ -\mathcal{A}\lambda_m & \widehat{f}\omega_m^2 - \overline{G}^{1111}\lambda_m^2 - \mathcal{A} \end{vmatrix} = 0 \tag{6.2}$$

where

$$\mathcal{A} = B^{1331} - H^{1331} \qquad \overline{N}^{11}(\tau) = -N^{11}(\tau) \qquad \lambda_m = \frac{m\pi}{a_1}$$

Let us introduce the parameters

$$\xi \equiv \frac{H^{1331}}{B^{1331}} \qquad s^2 = \frac{B^{1331}(a_1)^2}{\overline{G}^{1111}\pi^2}$$

where ξ $(0 \leq \xi < 1)$ characterizes the relative heterogenity of laminated plate structure (for $\zeta = 0$ we are dealing with a homogeneous plate) and s is the plate slenderness parameter.

We shall restrict ourselves to the analysis of the following two cases:

- if $\omega_m^2 = 0$, then for a critical force we obtain the condition

$$\overline{N}^{11}_{kr} = \frac{B^{1331}(1-\xi)}{1+(1-\xi)s^2} \tag{6.3}$$

  which describes the effect of nonperiodically laminated plate structure heterogeneity on the plate stability.

- if $\overline{N}^{11} = 0$ then, after neglecting the terms $\widehat{f}\omega_m^2$ we obtain the formula

$$\omega_1^2 = \frac{\pi^2}{\widetilde{f}a_1}\frac{B^{1331}(1-\xi)}{1+(1-\xi)s^2} \tag{6.4}$$

  which has a form similar to Eq (6.3) and characterizes the effect of laminated plate structure on the plate free vibration frequency.

7. Conclusions

The main aim of this paper was to derive an effective model of certain nonperiodically-laminated plates which takes into account the micro-local effects and which is plausible from the engineering standpoint and may constitute the basis for numerical analysis.

From Eqs (6.3),(6.4), it follows that the effects of heterogenity of the plate under consideration on a critical force and free vibration frequency are negligibly small. However, if ξ is close to 1 then the heterogenity of laminated plate structure leads to a sudden decrease of the critical force and free vibration frequency. These conclusions are similar to those obtained by Konieczny et al. (1995), which where related to the theory of nonperiodic laminated plates with interlaminar imperfections.

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Niestandardowa metoda makro-modelowania nieperiodycznie uwarstwionych płyt

Streszczenie

W pracy zaprezentowana jest metoda efektywnego makromodelowania sprężystych płyt o nieperiodycznie warstwowej strukturze. Model zhomogenizowany wyprowadzony został w oparciu o metodę mikrolokalnego modelowania kompozytów mikroperiodycznych opracowaną przez Woźniaka (1986); metoda ta została przez autorów

pracy zmodyfikowana i rozszerzona do modelowania płyt warstwowych o strukturze nieperiodycznej. W proponowanej procedurze makro-modelowania wykorzystane są pojęcia i twierdzenia z analizy niestandardowej (Robinson, 1966). Wyprowadzony model zhomogenizowany nieperiodycznie uwarstwionej płyty typu Reissnera zastosowany będzie do zbadania wpływu niejednorodności struktury kompozytu na wartość siły krytycznej i częstość drgań własnych płyty swobodnie podpartej na brzegach.

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