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THERMOELASTIC CONTACT PROBLEM WITH FRICTIONAL HEATING FOR THE PUNCH OF FUNCTIONALLY GRADED MATERIALS

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A contact problem with frictional heating for a body with the functionally graded materials (FGMs) is considered. The FGM body slides on the surface of rigid half-space. The problem is reduced to one integral equation in pressure. The approximate solution was derived basing on the representation of deformed solid surface by a polinomial surface. The influence of the parameters which characterise nonhomogeneity of a medium on a contact region size is investigated.

Key words: thermoelasticity, contact mechanics, frictional heating, functionally graded material

1. Introduction

Static contact problems for a half-space with frictional heating due to sliding were considered by Barber (1976) and by Yevtushenko and Kulchytsky-Zhyhailo (1995) and (1996). Thermoelastic deformation in the contact region leads to interesting physical effects. So, unlike the isothermal case in the absence of frictional heating (the Hertz contact problem), there is assumed that at a fixed sliding speed with the increasing total load there exists the critical value of the contact circle radius. Definitions of the critical values of the contact area size in thermoelasticity contact problems are of great importance because they enable practical sugestions about behaviour of the tribosystem in the friction process (e.g. the thermoelastic instability) to be drawn.

In all the above-mentioned papers it was assumed that the sliders material properties were thermally isotropic and homogeneous. In this paper, using the method put forward by Yevtushenko and Kulchytsky-Zhyhailo (1996), we find the approximate solution of the thermoelastic contact problem with frictional heating in the contact zone for the body with the functionally graded materials

(FGMs). The fracture problems for the FGMs under force loading were studied by Ozturk and Erdogan (1993), by Chen and Erdogan (1996) and by Olesiak and Yevtushenko (1989) and under thermal loading by Noda and Jin (1993), by Jin and Noda (1994) and by Erdogan and Wu (1996).

2. Formulation of the problem

The elastic body and a rigid half-space under the action of the force P so that a circle with the radius a is a contact region. The radius is small in comparison with the curvature radiuses of the body. In addition, the body slides at a constant speed v on the surface of the half-space. Owing to the friction in the contact region thermal energy is generated. The steady heat flow at the interface due to friction induces temperature gradients and thermal stresses and cause the contact pressure distribution and the extent of the contact region changes.

We introduce the dimensionless cylindrical coordinate axes r, φ, z; the z-axis coincides with the normal directed positively into the body and is fixed rigidly to the moving body. In this coordinate system the contact region $0 \leq r \leq 1$, $z = 0$ is motionless.

The following assumptions were made:

- The elastic displacements normal to the body surface due to the tangential trastions, are much smaller than those produced by the normal tractions. Hence, the coupling effect between tangent and normal tractions is negligible and friction influence on the contact area and pressure only through thermal effects.

- The heat generated over the contact area is absorbed by one moving body, and there is no heat transfer across free surfaces of the bodies. The heat input q to the moving body is equal to the rate of frictional heating throughout the contact area, i.e.

$$q = fvp \tag{2.1}$$

  where f is the coefficient of friction, p is the contact pressure.

- The body material is the functionally graded material (FGM) one. The material reveals the following non-homogeneous properties

$$E = E_0 e^{\beta z} \qquad \alpha = \alpha_0 e^{\delta z}$$
$$K = K_0 e^{\gamma z} \qquad \nu = \text{const} \tag{2.2}$$

where $\beta = \beta_0 a$, $\delta = \delta_0 a$, $\gamma = \gamma_0 a$, E is the Young modulus, nu is the Poisson ratio, α and K are the coefficients of linear thermal expansion and heat conductivity, respectively, E_0, α_0, K_0, β_0, ,$S\delta_0$, γ_0, are the material constants.

- The profile of the bodie surface in the region close to the origin is approximately of the form
$$h(r) = \frac{ar^2}{2R} \tag{2.3}$$
where R is the radius of the surface curvature at the origin.

Accepting these assumptions, the problem is axi-symmetrical and can be reduced to the solution of the following equations of thermoelasticity for the half-space $z \geq 0$

$$D_1 u + \frac{1-2\nu}{2(1-\nu^2)}\frac{\partial^2 u}{\partial z^2} + \frac{1}{1-\nu}\frac{\partial^2 w}{\partial r \partial z} + \frac{(1-2\nu)\beta}{2(1-\nu^2)}\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\Big) = = \frac{1+\nu}{1-\nu}\alpha_0 \frac{\partial T}{\partial r} e^{\delta z}$$
$$\frac{\partial^2 w}{\partial z^2} + \frac{1-2\nu}{2(1-\nu^2)} D_0 w + \frac{1}{1-\nu}\frac{\partial}{\partial z} D u + \beta\frac{\partial w}{\partial z} + \frac{\nu}{1-\nu}\beta D u = = \frac{1+\nu}{1-\nu}\alpha_0 e^{\delta z}\Big[\frac{\partial T}{\partial z} + (\beta+\delta)T\Big] \tag{2.4}$$

and the heat conduction equation

$$D_0 T + \frac{\partial^2 T}{\partial z^2} + \gamma \frac{\partial T}{\partial z} = 0 \tag{2.5}$$

under the following boundary conditions imposed on the surface $z = 0$

— mechanical

$$\sigma_{zz} = \begin{cases} -p(r) & r \leq 1 \\ 0 & r > 1 \end{cases} \qquad \sigma_{rz} = 0 \qquad r \geq 0$$
$$w = \Delta - \frac{ar^2}{2R} \qquad r \leq 1 \tag{2.6}$$

— thermal

$$K_0\frac{\partial T}{\partial z} = -fvap(r) \qquad r \leq 1 \qquad\qquad \frac{\partial T}{\partial z} = 0 \qquad r > 1$$
$$u, w, t \to 0 \qquad \text{as} \qquad r^2 + z^2 \to \infty \tag{2.7}$$

where

$$D = \frac{\partial}{\partial r} + \frac{1}{r} \qquad\qquad D_i = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{i}{r^2} \qquad i = 0, 1$$

u, w – r and z components of the displacement vector, respectively
T – temperature
σ_{zz}, σ_{rz} – stress components
Δ – displacement of the elastic body as a rigid solid.

The contact pressure must satisfy the equilibrium condition

$$2\pi a^2 \int\limits_0^1 rp(r)\, dr = P \tag{2.8}$$

3. Method of solution

The considered problem was solved using the Hankel integral transform of zero and first orders defined as follows (see Sneddon, 1951)

$$\overline{f}_i(\zeta) = \int\limits_0^\infty yf(y)J_i(\zeta y)\, dy \qquad i = 0, 1$$
$$\overline{f}(r) = \int\limits_0^\infty \zeta\overline{f}_i(\zeta)J_i(\zeta r)\, d\zeta \qquad i = 0, 1 \tag{3.1}$$

The solution of Eq (2.5) under the boundary conditions (2.7) has the form

$$T(r,z) = \frac{afv}{K_0}\int\limits_0^\infty \frac{2\zeta J_0(\zeta r)\mathrm{e}^{-\frac{1}{2}z(\gamma+\sqrt{\gamma^2+4\zeta^2}}}{\gamma+\sqrt{\gamma^2+4\zeta^2}}\, d\zeta \int\limits_0^1 yp(y)J_0(y\zeta)\, dy \tag{3.2}$$

The thermoelastic contact problem can be handed by applying a superposition of the two solutions:

1. For a thermal stresses and deformations induced in the body by a given heat flow (2.1)

2. For the isothermal problem of the body and a rigid half-space contact.

In result the normal displacement of the bodie surface can be find as a sum

$$w(r) = w_p(r) + w_t(r) \tag{3.3}$$

where $w_p(r)$ is the displacement due to distributed contact pressure $p(r)$ and $w_t(r)$ – due to distributed heat flow $q(r)$.

Using standart transformations which we omit for the sale of simplicity integral representations of the vertical displacements $w_p(r)$ and $w_t(r)$ are as follows

$$\begin{aligned} w_p(r) &= \frac{2(1-\nu^2)}{E_0} \int\limits_0^\infty \zeta W_p(\zeta) J_0(r\zeta)\, d\zeta \int\limits_0^1 yp(y) J_0(y\zeta)\, dy \\ w_t(r) &= \frac{-\alpha_0(1+\nu)afv}{K_0} \int\limits_0^\infty \frac{2\zeta W_t(\zeta) J_0(\zeta r)\, d\zeta}{\gamma + \sqrt{\gamma^2 + 4\zeta^2}} \int\limits_0^1 yp(y) J_0(y\zeta)\, dy \end{aligned} \tag{3.4}$$

where

$$W_p(\zeta) = \frac{1}{2(1-\nu)} [\mathrm{Im}(m_1)]^{-1} \mathrm{Im}\Big[\frac{(m_2+\beta)(m_1^2 - \zeta^2 - \kappa^2\beta^2)}{m_1(m_1^2-\zeta^2)}\Big]$$

$$W_t(\zeta) = \frac{1}{1-\nu} \Big\{ \frac{\zeta(\zeta^2 - l^2)}{A} [\mathrm{Im}(m_1)]^{-1} \mathrm{Im}\Big[\frac{(m_2 - l)(m_1^2 - \zeta^2 - \kappa^2\beta^2)}{m_1(m_1^2-\zeta^2)}\Big] - \frac{A_w}{A} \Big\}$$

$$A = (l^2-\zeta^2)^2 + 2l\beta(l^2-\zeta^2) + \beta^2(l^2+\kappa^2\zeta^2) \qquad \kappa^2 = \frac{\nu}{1-\nu}$$

$$A_w = (l+2\beta)(l^2-\zeta^2) + l\beta^2 \qquad l = \delta - \frac{\gamma + \sqrt{\gamma^2+4\zeta^2}}{2}$$

m_1, m_2 ($\mathrm{Re}(m_i) < 0$, $i = 1,2$) are the roots the following characteristic equation

$$(m^2-\zeta^2)^2 + 2m\beta(m^2-\zeta^2) + \beta^2(m^2+\kappa^2\zeta^2) = 0$$

and have the form

$$\begin{aligned} m_1 &= -\frac{1}{2}\Big[\beta + \sqrt{\beta^2 + 4(\zeta^2 + \mathrm{i}|\beta|\kappa\zeta)}\Big] \\ m_,21 &= -\frac{1}{2}\Big[\beta + \sqrt{\beta^2 + 4(\zeta^2 - \mathrm{i}|\beta|\kappa\zeta)}\Big] \end{aligned} \tag{3.5}$$

It can be shown straightforward that these functions have the following properties:

1. For $\beta > 0$ there exists a finite limit $W_p(\zeta \to 0)$, $W_t(\zeta \to 0)$
2. For $\beta < 0$ $W_p(\zeta)$ behaves as ζ^{-4}. Since in this case the integral in Eq $(3.4)_1$ is equal to infinity that's why in the following we'll suppose that $\beta \geq 0$
3. For $\zeta \to \infty$ $W_p(\zeta) \to 1/\zeta$, $W_t(\zeta) \to 1/\zeta$.

Let is examine particular cases:

- $\beta \to 0$

  The coefficients m_1 and m_2, Eqs (3.5), at $\beta \to 0$ can be written down in the following asymptotical form

$$m_1 = -\zeta - \frac{\beta}{2}(1 + \mathrm{i}\kappa) \qquad m_2 = -\zeta - \frac{\beta}{2}(1 - \mathrm{i}\kappa)$$

  and, hence

$$A_w = l(l^2 - \zeta^2) \qquad A = (l^2 - \zeta^2)^2 \qquad \mathrm{Im}(m_1) = -\frac{\kappa\beta}{2}$$

$$\mathrm{Im}\Big[\frac{(m_2 + \beta)(m_1^2 - \zeta^2 - \kappa^2\beta^2)}{m_1(m_1^2 - \zeta^2)}\Big] = -\frac{\kappa\beta}{\zeta(1 + \kappa^2)}$$

$$\mathrm{Im}\Big[\frac{(m_2 - l)(m_1^2 - \zeta^2 - \kappa^2\beta^2)}{m_1(m_1^2 - \zeta^2)}\Big] = -\frac{\kappa\beta[2\zeta + l(1 - \kappa^2)]}{2\zeta^2(1 + \kappa^2)}$$

  Finally

$$W_p(\zeta) = \frac{1}{\zeta} \qquad W_t(\zeta) = -\frac{2}{l - \zeta} \tag{3.6}$$

  In addition, when $\gamma = 0$, $\delta = 0$ then $l = -\zeta$ and from Eqs (3.6) it results for the surface vertical displacement in homogeneous half-space (cf Yevtushenko and Kulchytsky-Zhyhailo, 1996a)

$$W_p(\zeta) = W_t(\zeta) = \frac{1}{\zeta}$$

- $\gamma = 0$, $\delta = 0$

  In this case $l = -\zeta$, $A_w = -\zeta\beta^2$, $A = \beta^2\zeta^2/(1 - \nu)$ and in result $W_t(\zeta) = 1/\zeta$.

Substituting Eqs (3.3) and (3.4) into the boundary condition (2.6) we obtain the following integral equation

$$w_p(r) + w_t(r) = \Delta - \frac{ar^2}{2R} \tag{3.7}$$

4. Numerical solution and analysis

To find an approximate solution Eq (3.7) we use the technique proposed by Yevtushenko and Kulchytsky-Zhyhailo (1996b). We represent the contact pressure in the form

$$\begin{aligned} p(r) &= p_0[d + 1.25(3-2d)r^2]\sqrt{1-r^2} \qquad 0 \le r \le 1 \\ p_0 &= \frac{P}{\pi a^2} \end{aligned} \tag{4.1}$$

where $d = p(0)/p_0$ is an unknown parameter. We note that the contact pressure (4.1) satisfies the equilibrium condition (2.8).

Substituting Eqs (4.1) into the integral representations (3.4) we find

$$\begin{aligned} w_p(r) &= \frac{2(1-\nu^2)P}{E_0 a^2}\Big[\Big(\frac{15}{4} - \frac{3d}{2}\Big)\frac{1-0.5r^2}{4} - \frac{15(3-2d)(1-r^2+0.375r^4)}{64} + \\ &\quad + \Big(\frac{15}{4} - \frac{3d}{2}\Big)w_p^{(1)}(r) - \frac{15(3-2d)}{4}w_p^{(2)}(r)\Big] \\ w_t(r) &= -\frac{\alpha_0(1+\nu)fvP}{K_0 a}\Big[\Big(\frac{15}{4} - \frac{3d}{2}\Big)w_t^{(1)}(r) - \frac{15(3-2d)}{4}w_t^{(2)}(r)\Big] \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} w_p^{(i)}(r) &= \frac{1}{\sqrt{2\pi}}\int\limits_0^\infty [\zeta W_p(\zeta) - 1]\zeta^{-\frac{1}{2}-i}J_{\frac{1}{2}+i}(\zeta)J_0(r\zeta)\;d\zeta \\ w_t^{(i)}(r) &= \frac{1}{\sqrt{2\pi}}\int\limits_0^\infty \frac{2W_t(\zeta)\zeta^{\frac{1}{2}-i}J_{\frac{1}{2}+i}(\zeta)}{\gamma + \sqrt{\gamma^2 + 4\zeta^2}}J_0(\zeta r)\;d\zeta \end{aligned}$$

The displacements $w_j^{(i)}$, $j = p,t$, $i = 1,2$ we represent by a polynomial to the fourth degree of approximation as

$$w_j^{(i)}(r) \approx \widetilde{w}_j^{(i)}(r) = w_{j,0}^{(i)} + w_{j,1}^{(i)}r^2 + w_{j,2}^{(i)}r^4 \qquad \begin{array}{l} j = p,t \\ i = 1,2 \end{array} \tag{4.3}$$

The unknown coefficients $w_{j,k}^{(i)}$, $j = p,t$, $i = 1,2$, $k = 0,1,2$ we determine from physical conditions

$$
\begin{gathered}
w_j^{(i)}(0) = \widetilde{w}_j^{(i)}(0) \qquad\qquad w_j^{(i)}(1) = \widetilde{w}_j^{(i)}(1) \\
\int\limits_0^1 y w_j^{(i)}(y)\,dy = \int\limits_0^1 y \widetilde{w}_j^{(i)}(y)\,dy
\end{gathered}
\tag{4.4}
$$

Substituting $w_p(r)$ and $w_t(r)$, Eqs (4.2), into the contact condition (3.7), taking into account Eq (4.3) and comparing the coefficients at r^2 and r^4, we obtain

$$
\begin{aligned}
&\frac{9d}{32} - \frac{15}{64} - \Big(\frac{15}{4} - \frac{3d}{2}\Big)(w_{p,1}^{(1)} - g a_H a_0 w_{t,1}^{(1)}) + \\
&+\frac{15(3-2d)}{4}(w_{p,1}^{(2)} - g a_H a_0 w_{t,1}^{(2)}) = \frac{3a_0^3}{16} \\
&\frac{45(3-2d}{512} - \Big(\frac{15}{4} - \frac{3d}{2}\Big)(w_{p,2}^{(1)} - g a_H a_0 w_{t,2}^{(1)}) + \\
&+\frac{15(3-2d)}{4}(w_{p,2}^{(2)} - g a_H a_0 w_{t,2}^{(2)}) = 0
\end{aligned}
\tag{4.5}
$$

where

$$
a_0 = \frac{a}{a_H} \qquad\qquad a_H^3 = \frac{3PR(1-\nu^2)}{4E_0} \qquad\qquad g = \frac{f v \alpha_0 E_0}{2K_0(1-\nu)}
$$

a is the radius of the contact circle in the isothermal Hertz problem under fixed load P (cf Johnson, 1985).

Thus, the initial problem has been reduced to the solution of the system of two non-linear algebraical equations for determination of the dimensionless parameters a_0 and d. The input parameters of the problem are the dimensionless values: $\beta_0 a^*$, $\gamma_0 a^*$, $\delta_0 a^*$, $g a_H$ and ν, where a^* – some characteristic linear size.

At first, we investigate the solution of the problem in the isothermal case ($g = 0$). The dependence of the dimensionless radius of the contact region a_0 on the dimensionless parameter $\beta_0 a_H$ is given in Fig.1. If the value of $\beta_0 a_H$ increases, then the radius a_0 decreases. According to the numerical analysis carried out the dependence of the solution on the parameter ν is negligible for $\nu = 0.25 - 0.45$. In further calculations we use $\nu = 0.25$.

Now we analyse the influence of parameters β_0, δ_0, γ_0 on the critical radius of the contact circle. Note that the critical radius is the limiting value

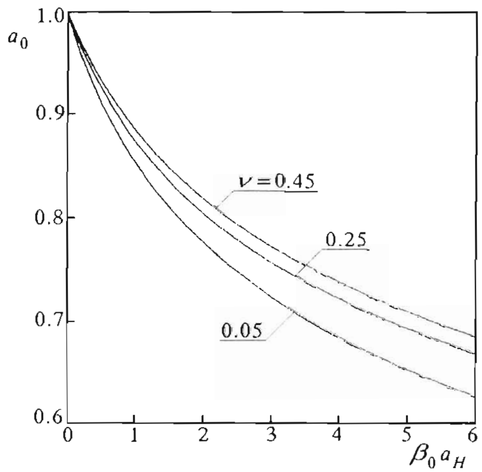

Fig. 1. Dimensionless radius a_0 versus the parameter $\beta_0 a_H$ $(1 - \nu = 0.05,$ $2 - \nu = 0.25,$ $3 - \nu = 0.45)$

of the contact circle under an infinite rise of the force P. From the solution of the contact problem with frictional heating for the isotropic, homogeneous body the critical contact radius can be determinated as follows $a_{cr} = 2/g$ (cf Yevtushenko and Kulchytsky-Zhyhailo, 1995).

Dependence of the dimensionless value $\overline{a}_{cr}/a_{cr}$ ($\overline{a}_{cr}$ is the critical value of the contact area in the contact problem for the FGM body) on the dimensionless parameters $\beta_0 a_{cr}$, $\delta_0 a_{cr}$, $\gamma_0 a_{cr}$ is presented in Fig.2. Basing on these figures the following conclusions can be drawn:

- Except for some domain Ω of the coordinate angle II ($\delta_0 a_{cr} < 0$, $\gamma_0 a_{cr} > 0$) levels can be approximated by straight lines which form with the line $\delta_0 = -\gamma_0$ an angle of nearly $90°$

- In the domain Ω the maximum values $\overline{a}_{cr}/a_{cr}$ are lie on the line, which makes with the bisector of the coordinate angle II an angle not more than $15°$

- The value of $\overline{a}_{cr}/a_{cr}$ decreases as β_0/a_{cr} increases. This effect takes place in the domain Ω.

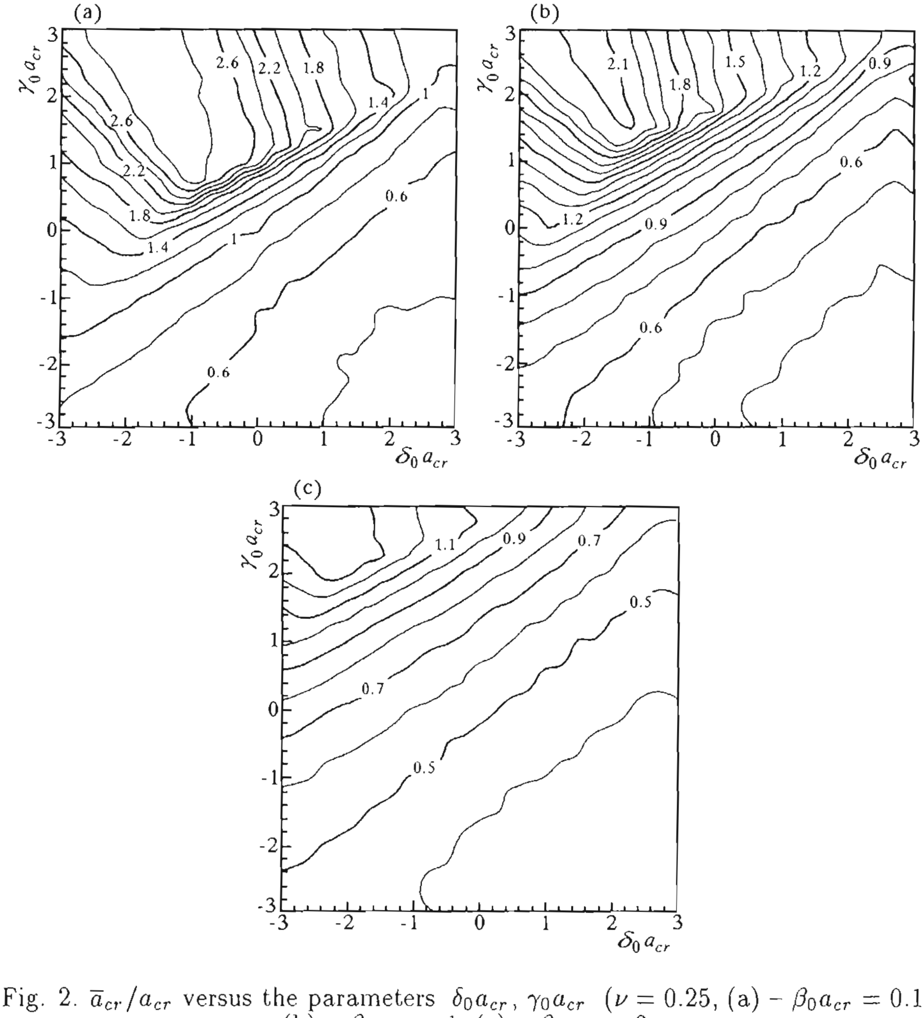

Fig. 2. $\overline{a}_{cr}/a_{cr}$ versus the parameters $\delta_0 a_{cr}$, $\gamma_0 a_{cr}$ ($\nu = 0.25$, (a) – $\beta_0 a_{cr} = 0.1$, (b) – $\beta_0 a_{cr} = 1$, (c) – $\beta_0 a_{cr} = 3$

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Termosprężyste zagadnienie kontaktowe z generacją ciepła spowodowanego tarciem dla stempla o ciągłej niejednorodności

Streszczenie

Rozpatrzono kontaktowe zagadnienie z uwzględnieniem generacji ciepla dla stempla z materiału o ciąglej niejednorodności. Ten stempel ślizga się po powierzchni sztywnej półprzestrzeni. Zagadnienie zostało zredukowane do jednego równania całkowego na nieznane ciśnienie. Przybliżone rozwiązanie zostalo znalezione stosując reprezentację odkształconej powierzchni ciała przez wielomianową powierzchnię. Zbadano wplyw parametrów charakteryzujących niejednorodność stempla na rozmiar obszaru kontaktu.

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