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FUZZY BOUNDARY ELEMENT METHOD IN THE ANALYSIS OF UNCERTAIN SYSTEMS

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In the paper basic concepts of new methodology of the fuzzy boundary element method are presented. This article deals with fuzzy-set-valued mappings which are solutions of the fuzzy boundary integral equations. Exact fuzzy solutions of the fuzzy boundary integral equations are defined as well as conditional solutions. Computational fuzzy problems and applications are considered in details for boundary potential problems with fuzzy Dirichlet and Neumann type boundary conditions and fuzzy density source functions in a fuzzy domain.

Key words: fuzzy singular integration, fuzzy boundary integral equations, fuzzy boundary element method, fuzzy arithmetical computations

1. Introduction

When a physical problem is transformed into a deterministic boundary problem, we usually cannot be sure that the modelling is perfect. The following three issues can be discussed when considering the nature of this uncertaing: human based uncertainty, system uncertainty and random uncertainty. The prediction of these three types of uncertainty is difficult and the present methods, embodied in the reliability theory, tend to concentrate on the random uncertainty.

The boundary problem may not be known exactly and some functions: i.e., the shape of a structure, material properties, boundary conditions, external or internal excitations, solutions etc. may contain unknown parameters.

Especially, if they are known from some measurements they definitely are subjected to errors. The analysis of the effect of these errors leads to the study of qualitative and quantitative behaviour of the solution uncertainty. There is however, a fundamental difference between the nature of random uncertainty and that of human and system uncertainty. To analyse this type of uncertainty a mathematical approach which is directed at *vagueness* as distinct from randomness is required and this is the potential role of fuzzy sets. Many different interpretations are possible for terminology of uncertain aspects of the Boundary Element Method (BEM). We focus our attention on the fuzzy-set-theoretic description of uncertain phenomena in BEM, and will refer to these approaches as the Fuzzy Boundary Element Method (FBEM). These terms are used here to refer to the boundary element method which accounts for uncertainties in boundary conditions or material properties of a structure as well as the boundary shape. Such uncertainties are usually distributed on a boundary or within a domain of the structure and should be modelled as spatial or spatially-temporal fuzzy fields.

Applications of the FBEM appear to have been initiated in the 1995. The earliest application used the fuzzy boundary integral equation to solve fuzzy boundary value potential problem with uncertain boundary conditions and internal sources (cf Burczyński and Skrzypczyk, 1995, 1996a, 1996b). Then the FBEM was used for elastostatic problems (cf Pilch, 1996; Skrzypczyk and Burczyński, 1997a,b). Modelling uncertainties as fuzzy variables or fuzzy processes suggests the use of fuzzy-set-theoretical methods, which are closely related to a convex modelling of uncertainties. Only linear static problems are studied and applications to non-linear or dynamic problems are left for a future study. Using BEM to solve a boundary value problem in some domain with prescribed boundary conditions on the boundary, one can obtain the Boundary Integral Equation. From now we assume that boundary conditions, material properties, internal prescribed fields and the shape of a boundary are uncertain and we'll model this uncertainty using fuzzy variables. We obtain the Fuzzy Boundary Integral Equation where all operations are in fuzzy sense.

Singular integrals are understood in the sense of fuzzy principal values. New results in singular integration and singular integration over a fuzzy domain are presented by Skrzypczyk (1996), (1997), Skrzypczyk and Burczyński (1998a,b).

Different types of fuzzy solutions are discussed as well as their existence and properties. Linear, quadratic and cubic Fuzzy Boundary Elements are introduced to consider how integral expressions can be discretized to find the system of Fuzzy Algebraic Equations from which the fuzzy boundary values can be found (cf Burczyński and Skrzypczyk, 1997a,b; Skrzypczyk and Burczyński, 1997a,b, 1998a,b).

Illustrative examples from the potential theory are given to comment different aspects of the presented theory. The interval and trapezoid-type fuzzy boundary conditions are considered. To complete the presentation the potential problem in a fuzzy domain is discussed.

Presented methods give the complete methodology how to obtain good approximations of the solutions of uncertain boundary problems with the use of fuzzy analysis.

2. Basic definitions and notation

In the paper we use the following notions.

R^n	–	set of n-dimensional reals
$(R^n, \lvert\cdot\rvert)$	–	n-dimensional Euklidean space with the metric $\lvert\cdot\rvert$
$R(R_+)$	–	set of reals (nonnegative reals respectively)
Γ	–	k-dimensional $(k < n)$ manifold in the Euklidean space R^n.

Let $I(R)$ (similarly $I(R^n)$) denote the set of all closed, bounded intervals $\overline{z} = [z^-, z^+]$ on the real line R (R^n respectively), where z^- and z^+ denote end points of the interval $\overline{z}$. We call other elements of sets $I(R)$ ($I(R^n)$) the interval numbers (interval vectors respectively) (cf Alefeld and Herzberger, 1983; Bauch et al., 1987; Moore, 1966; Neumaier, 1990).

Let $F(R^n)$ be a class of fuzzy sets in R^n, i.e. the set of maps (cf Czogała and Pedrycz, 1985; Dubois and Prade, 1988; Kacprzyk, 1986; Negoita and Ralescu, 1975)

$$F(R^n) := \left\{\mu : \ R^n \to [0,1]\right\}$$

We call a fuzzy number the set $\widetilde{a} \in F(R^n)$ defined by the so called membership function $\mu(x;\widetilde{a}), x \in R^n$ and satisfying some additional conditions (cf Dubois and Prade, 1979, 1988; Felbin, 1992; Kaleva, 1987, 1990) Let additionally

$$\widetilde{a}_\lambda := \left\{x \in R^n : \ \mu(x;\widetilde{a}) \geq \lambda\right\} \qquad 0 < \lambda \leq 1$$

By $F^*(R^n) \subset F(R^n)$ we denote the set of all fuzzy numbers. The interval numbers are naturally particular examples of fuzzy numbers.

If $f : \ R^n \times R^n \to R^n$ is a usual real function, then according to the Zadeh extension principle we can extend f to a fuzzy function $\widetilde{f} : \ F^*(R^n) \times$

$F^*(R^n) \to F^*(R^n)$ in the following way (cf Czogała and Pedrycz, 1985; Dubois and Prade, 1979, 1988; Felbin, 1992; Kacprzyk, 1986; Kaleva, 1987, 1990; Negoita and Ralescu, 1975)

$$\mu\Big(z; \widetilde{f}(\widetilde{u}, \widetilde{v})\Big) = \sup_{z=f(x,y)} \mu(x; \widetilde{u}) \wedge \mu(y; \widetilde{v}) \tag{2.1}$$

where $\wedge$, *vee* denote max and min, respectively. It is a well known result, that

$$\widetilde{f}_\lambda(\widetilde{u}, \widetilde{v}) = \widetilde{f}(\widetilde{u}_\lambda, \widetilde{v}_\lambda) \qquad \forall \widetilde{u}, \widetilde{v} \in F^*(R^n) \qquad 0 \leq \lambda \leq 1 \tag{2.2}$$

if f is a continuous function (Nguyen, 1978). Notice, that $I(R^n) \subset F^*(R^n)$, (cf Guang-Quan, 1991; Kaleva, 1987, 1990).

3. Fuzzy singular integration over a fuzzy domain

Notice, that a fuzzy singular integral in a non-fuzzy domain, in the Fuzzy Principal Value (FPV) of Cauchy type was defined first by Skrzypczyk (1996) and investigated further by Burczyński and Skrzypczyk (1995), (1996a,b), (1997a,b), Skrzypczyk and Burczyński (1997a,b). Fuzzy singular integration over a fuzzy domain was defined first by Skrzypczyk (1997). for further results see Skrzypczyk and Burczyński (1998a).

Let $\mathcal{M}$ denote further a set of manifolds in R^n. By a fuzzy domain $\widetilde{\Gamma}$ we understand a fuzzy subset of manifolds from $\mathcal{M}$, i.e. a map $\mathcal{M}$ into $[0, 1]$. With each map Γ we consider its membership function $\mu(\Gamma; \widetilde{\Gamma})$, $\Gamma \subset \mathcal{M}$. Let $F(\mathcal{M})$ denote a class of fuzzy subsets of $\mathcal{M}$.

One can define a fuzzy domain in another way:

- Classical map Γ_u defined over a set of parameters $U \subset R^p$ with the values in $\mathcal{M}$
- Classical map defined over a fuzzy set $\widetilde{A} \subset F^*(R^p)$ with the values in $\widetilde{\Gamma}$, i.e. such that $\mu(\cdot; \widetilde{A}) \leq \mu(\cdot; \widetilde{\Gamma}) \circ \Gamma$.

We know, that the membership function of a fuzzy singular integral defined over some non-fuzzy domain Γ is as follows

$$\mu_\Gamma\Big(w; \int_\Gamma \widehat{h}(y)\, d\Gamma(y)\Big) \qquad \Gamma \subset \mathcal{M} \qquad w \in R$$

and can be considered as the conditional membership function.

Besides, let $\widetilde{h} : \Gamma \to F^*(R^n)$ be a measurable fuzzy function (cf Aumann, 1965; Kaleva, 1987, 1990; Nanda, 1989), integrable in the FPV-sense over each subset $\Gamma \subset \mathcal{M}$. Then a singular integral (cf Mikhlin, 1986; Piskorek, 1980; Pogorzelski, 1970) of the fuzzy function $\widetilde{h}$ in the FPV-sense over the fuzzy set $\widetilde{\Gamma} \subset F(\mathcal{M})$, will be denoted further as $(FPV)\int_{\widetilde{\Gamma}} \widetilde{h}(y)\, d\Gamma(y)$ and its membership function is defined in the following way $(w \in R)$

$$\mu_\Gamma\Big(w;(FPV)\int_{\widetilde{\Gamma}} \widehat{h}(y)\, d\Gamma(y)\Big) :=$$

$$:= \sup_{\Gamma \subset \mathcal{M}} \Big\{\mu(\Gamma;\widetilde{\Gamma}) \wedge \mu_\Gamma\Big(w;(FPV)\int_{\Gamma} \widetilde{h}(y)\, d\Gamma(y)\Big)\Big\}$$

The integral defined in such a way will be called the generalized fuzzy singular integral of the fuzzy function $\widetilde{h}$ over the fuzzy domain $\widetilde{\Gamma}(x)$, in the Fuzzy Principal Value (FPV) of the Cauchy type (for details see Skrzypczyk, 1996; Skrzypczyk and Burczyński, 1998a,b).

4. Fuzzy boundary integral equations

Using BEM to solve a potential boundary value problem in a domain Ω with the prescribed boundary conditions on the Γ boundary of Ω: Dirichlet (essential) conditions of the type $u(x) = u_0(x)$, for $x \in \Gamma_1$ and Neumann (natural) conditions such as $q(x) = \partial u(x)/\partial n = q_0(x)$, for $x \in \Gamma_2$, $\Gamma = \Gamma_1 \cup \Gamma_2$, one obtains

$$\begin{aligned} c(x)u(x) + \int_\Gamma Q(x,y)u(y)\, d\Gamma(y) + \int_\Omega U(x,y)\xi(y)\, d\Omega(y) &= \\ = \int_\Gamma U(x,y)q(y)\, d\Gamma(y) \qquad\qquad x \in \Gamma & \end{aligned} \tag{4.1}$$

where $\xi(x)$, $x \in \Omega$ is a known source density function and U is a fundamental solution of the Laplace equation $(Q = \partial U/\partial n)$, (see Brebbia and Dominguez, 1989; Brebbia et al., 1984; Burczyński, 1995). We now assume that the values of some of boundary quantities, source density function and contour Γ are uncertain and we shall model this uncertainty using fuzzy variables.

Let $\widetilde{u}_0$, $\widetilde{q}_0$, $\widetilde{\xi}$ and $\widetilde{\Gamma}$ be fuzzy functions. Define

$$U_\lambda(x\big|\Gamma) := \begin{cases} u: \quad c(x)u(x) + \int\limits_{\Gamma} Q(x,y)u(y)\,d\Gamma(y) + \\ \qquad + \int\limits_{\Omega} U(x,y)\xi(y)\,d\Omega(y) = \int\limits_{\Gamma} U(x,y)q(y)\,d\Gamma(y) \\ u_0(z) \in \overline{u}_{0\lambda}(z)\Big|_{z\in\Gamma_1} \qquad q_0(z) \in \overline{q}_{0\lambda}(z)\Big|_{z\in\Gamma_2} \\ \xi(z) \in \overline{\xi}_\lambda(z) \qquad\qquad z \in \Omega \qquad \Gamma \subset \mathcal{M} \end{cases} \tag{4.2}$$

The conditional exact fuzzy solution $\widetilde{u}_1(x\big|\Gamma)$, $x \in \Gamma \subset \mathcal{M}$ is defined as follows

$$\mu_\Gamma\Big(y; \widetilde{u}_1(x\big|\Gamma)\Big) := \sup\Big\{\lambda: \quad y \in U_\lambda(x\big|\Gamma)\Big\} \qquad x \in \Gamma \subset \mathcal{M} \qquad y \in R^1 \tag{4.3}$$

and the exact fuzzy solution $\widetilde{u}_1(x)$, $x \in \widetilde{\Gamma}$ is defined according to the min-max composition of fuzzy relations

$$\mu\Big(y; \widetilde{u}_1(x)\Big) := \sup_{\Gamma\subset\mathcal{M}} \Big\{\mu(\Gamma;\widetilde{\Gamma}) \wedge \mu_\Gamma\Big(w; \widetilde{u}_1(x\big|\Gamma)\Big)\Big\} \qquad y \in R^1 \tag{4.4}$$

Eq (4.4) describes the membership function function of the first-type fuzzy solution of boundary potential problem defined over a fuzzy domain.

Alternatively let substitute $\widetilde{u}_0$, $\widetilde{q}_0$, $\widetilde{\xi}$ and $\widetilde{\Gamma}$ for u_0, q_0, ξ and Γ, respectively, and let all operations be considered in the fuzzy sense. Thus we consider further the fuzzy analogue of Eq (4.1), as follows

$$\begin{aligned} &\widetilde{c}(x)\widetilde{u}(x) + \int\limits_{\widetilde{\Gamma}} Q(x,y)\widetilde{u}(y)\,d\Gamma(y) + \int\limits_{\widetilde{\Omega}} U(x,y)\widetilde{\xi}(y)\,d\Omega(y) = \\ &= \int\limits_{\widetilde{\Gamma}} U(x,y)\widetilde{q}(y)\,d\Gamma(y) \qquad\qquad x \in \widetilde{\Gamma} \end{aligned} \tag{4.5}$$

Fuzzy integrals are understood in the sense of fuzzy principal value, as defined in Section 3. We want to solve formally Eq (4.5) for $\widetilde{u}$, which will be called a fuzzy solution of the second kind and will be denoted $\widetilde{u}_2$. It is a difficult problem to solve Eq (4.5) in such a way, so we are looking rather for approximate methods (cf Buckley, 1992a,b; Buckley and Qu, 1990a,b, 1991a,b). Now we discuss how to solve Eq (4.5) for the conditional fuzzy function $\widetilde{u}(x)$, $x \in \Gamma \subset \mathcal{M}$. Let

$$\begin{aligned} \widetilde{u}_{0\lambda}(x) &= [u_{0\lambda}^-(x), u_{0\lambda}^+(x)] && x \in \Gamma_1 \\ \widetilde{q}_{0\lambda}(x) &= [q_{0\lambda}^-(x), q_{0\lambda}^+(x)] && x \in \Gamma_2 \\ \widetilde{\xi}_\lambda(x) &= [\xi_\lambda^-(x), \xi_\lambda^+(x)] && x \in \Gamma_1 \end{aligned} \tag{4.6}$$

Assume, that we are looking for the interval-type solution

$$\widetilde{u}_\lambda(x) = [u_\lambda^-(x), u_\lambda^+(x)] \qquad x \in \Gamma \subset \mathcal{M} \tag{4.7}$$

where $0 \leq \lambda \leq 1$. Taking λ-cuts, $\forall 0 \leq \lambda \leq 1$ of the fuzzy Eq (4.5) we obtain formally the infinite set of interval boundary integral equations as follows for $x \in \Gamma \subset \mathcal{M}$

$$\begin{aligned} c(x)[u_\lambda^-(x), u_\lambda^+(x)] &= \int_\Gamma Q(x,y)[u_\lambda^-(y), u_\lambda^+(y)]\, d\Gamma(y) + \\ &+ \int_\Omega U(x,y)[\xi_\lambda^-(y), \xi_\lambda^+(y)]\, d\Omega(y) = \int_\Gamma U(x,y)[q_\lambda^-(y), q_\lambda^+(y)]\, d\Gamma(y) \end{aligned} \tag{4.8}$$

We solve Eq (4.8) for the interval values $u_\lambda^-(x)$ and $u_\lambda^+(x)$ producing the family of interval functions $\widetilde{u}_{3\lambda}$, $0 \leq \lambda \leq 1$. Following we define the fuzzy solution of the second type $\widetilde{u}$ by the relation

$$\mu\Big(y; \widetilde{u}_3(x\Big|\Gamma)\Big) := \sup\Big\{\lambda : \ y \in \widetilde{u}_{3\lambda}(x\Big|\Gamma)\Big\} \qquad x \in \Gamma \subset \mathcal{M} \qquad y \in R^1 \tag{4.9}$$

Naturally, we are now interested in the relationship between the solutions $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{u}_3$. Following Burczyński and Skrzypczyk (1995), (1996a,b), (1997a,b), Skrzypczyk and Burczyński (1997a,b), and taking into consideration the results of Buckley et al. (1990b), (1991a,a), we have $\widetilde{u}_{1\lambda}(x) = \widetilde{u}_{3\lambda}(x)$, $\forall 0 \leq \lambda \leq 1$, $x \in \Gamma \subset \mathcal{M}$.

5. Fuzzy boundary element method – computational methodology

Now, let us consider how Eq (4.8) can be discretized to find the system of fuzzy algebraic equations from which the boundary values can be found. Assume for simplicity that the body is two-dimensional and its boundary is divided into N elements. Let $\mathcal{M} \supset \Gamma \cong \bigcup_{j=1}^N \Gamma_j$, where Γ_j is the boundary of the jth element. The fuzzy (interval) values of $\widetilde{u}_\lambda$ and $\widetilde{q}_\lambda$ are assumed to be fuzzy constant/linear/cubic etc. over each element.

5.1. Constant fuzzy elements

The points, at which the unknown fuzzy values are considered, are called as usual *nodes* and taken to be in the middle of the element for the so-called *constant fuzzy elements.* Later on we will also discuss the case of *linear fuzzy elements*, i.e. those elements for which the nodes are at the extremes or ends.

For the constant fuzzy elements considered here the boundary is assumed to be divided into N elements, let $\mathcal{M} \supset \Gamma \cong \bigcup_{j=1}^{N} \Gamma_j$, where Γ_j is the boundary of the jth element. The fuzzy (interval) values of $\widetilde{u}_\lambda$ and $\widetilde{q}_\lambda$ are assumed to be fuzzy constant over each element and equal to the fuzzy value at the mid-element node. Eq (4.8) can be discretized for a given point i before applying any fuzzy boundary conditions as follows

$$
\begin{aligned}
\frac{1}{2}\widetilde{u}_\lambda(x_i) &= \sum_{j=1}^{N} \int_{\Gamma_j} Q(x_i,y)\widetilde{u}_\lambda(y)\, d\Gamma(y) + \int_{\Omega} U(x_i,y)\widetilde{\xi}_\lambda(y)\, d\Omega(y) = \\
&= \sum_{j=1}^{N} \int_{\Gamma_j} U(x_i,y)\widetilde{q}_\lambda(y)\, d\Gamma(y) \qquad\qquad x_i \in \bigcup_{j=1}^{N} \Gamma_j
\end{aligned}
\tag{5.1}
$$

or with respect to the interval ends of λ-cuts

$$
\begin{aligned}
\frac{1}{2}[\widetilde{u}_\lambda^-(x_i),\widetilde{u}_\lambda^+(x_i)] &= \sum_{j=1}^{N} \int_{\Gamma_j} Q(x_i,y)[\widetilde{u}_\lambda^-(y),\widetilde{u}_\lambda^+(y)]\, d\Gamma(y) + \\
&+ \int_{\Omega} U(x_i,y)[\widetilde{\xi}_\lambda^-(y),\widetilde{\xi}_\lambda^+(y)]\, d\Omega(y) = \\
&= \sum_{j=1}^{N} \int_{\Gamma_j} U(x_i,y)[\widetilde{q}_\lambda^-(y),\widetilde{q}_\lambda^+(y)]\, d\Gamma(y) \qquad\qquad x_i \in \bigcup_{j=1}^{N} \Gamma_j
\end{aligned}
\tag{5.2}
$$

The point ith is one of the boundary nodes. Note that for this type of fuzzy element (i.e. fuzzy constant) the boundary must be always *smooth* as the node is at the centre of the element, hence the multiplier of $\widetilde{u}_\lambda(x_i)$ is 0.5.

5.2. Linear/quadratic/higher order fuzzy elements

Up to now we have only considered the case of fuzzy constant elements, i.e. those with the values of fuzzy variables assumed to be the same all over the element. Now, let us consider a linear variation of fuzzy $\tilde{u}$ and $\tilde{q}$. In this case the nodes are considered to be at the ends of the element, see Fig.1.

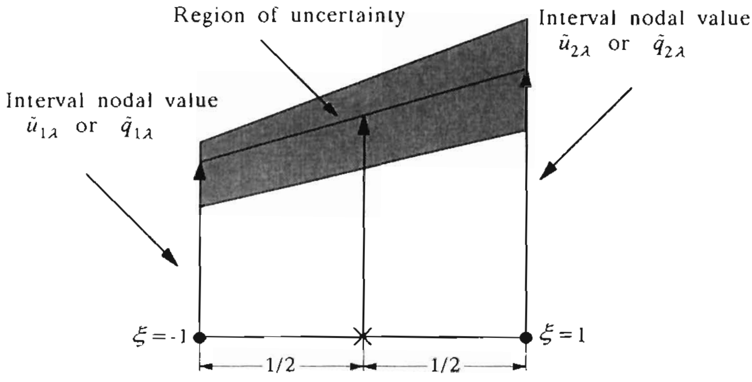

Fig. 1. Fuzzy boundary linear element

The governing fuzzy integral statement can now be written as Eq (4.8). After discretizing the boundary into a series of N elements Eq (4.8) can be rewritten as

$$c_i\tilde{u}_\lambda(x_i) = \sum_{j=1}^{N}\int_{\Gamma_j} Q(x_i,y)\tilde{u}_\lambda(y)\,d\Gamma(y) + \int_{\Omega} U(x_i,y)\tilde{\xi}_\lambda(y)\,d\Omega(y) = \tag{5.3}$$

$$= \sum_{j=1}^{N}\int_{\Gamma_j} U(x_i,y)\tilde{q}_\lambda(y)\,d\Gamma(y) \qquad\qquad x_i \in \bigcup_{j=1}^{N}\Gamma_j$$

$c_i = \theta/2\pi$, where θ is the non-fuzzy internal angle of the corner in radians.

The values of $\tilde{u}_\lambda$ and $\tilde{q}_\lambda$ at any point on the element can be defined in terms of their nodal fuzzy values and two linear interpolating functions ϕ_1 and ϕ_2, which are given in terms of the homogeneous coordinate ξ, i.e.

$$\tilde{u}_\lambda(\xi) = \phi_1\tilde{u}_{1\lambda} + \phi_2\tilde{u}_{2\lambda}$$
$$\tilde{q}_\lambda(\xi) = \phi_1\tilde{q}_{1\lambda} + \phi_2\tilde{q}_{2\lambda} \tag{5.4}$$

where ξ is the dimensionless coordinate varying from -1 to $+1$ and the two interpolating functions are

$$\phi_1 = \frac{1}{2}(1-\xi) \qquad\qquad \phi_2 = \frac{1}{2}(1+\xi) \tag{5.5}$$

The integrals in Eq (5.3) are more difficult to calculate than those for the fuzzy constant element as the $\widetilde{u}_\lambda$ and $\widetilde{q}_\lambda$ vary fuzzy linearly over each Γ_j and hence it is not possible to take them out of the integrals.

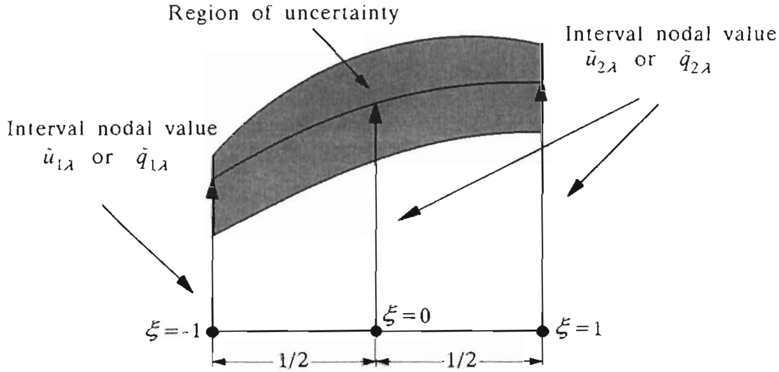

Fig. 2. Fuzzy boundary quadratic element

Approximations based on higher order fuzzy elements i.e. quadratic, cubic etc. can be calculated in a similar way, see Fig.2.

5.3. Methodology of fuzzy arithmetical computations

If we now assume that the position of ith point can vary from 1 to N one obtains a system of N fuzzy algebraic equations resulting from Eqs (5.2) or (5.3). This set of fuzzy equations can be expressed in a matrix form as

$$\mathbf{H}_\lambda \widetilde{\boldsymbol{U}}_\lambda = \mathbf{G}_\lambda \widetilde{\boldsymbol{Q}}_\lambda + \widetilde{\boldsymbol{V}}_\lambda \tag{5.6}$$

where $\mathbf{H}_\lambda$ and $\mathbf{G}_\lambda$ are two $N \times N$ non-fuzzy matrices and $\widetilde{\boldsymbol{U}}_\lambda$, $\widetilde{\boldsymbol{Q}}_\lambda$, $\widetilde{\boldsymbol{V}}_\lambda$ are the fuzzy vectors of length N, $\forall\lambda \in [0,1]$. Notice that N_1 fuzzy values of $\widetilde{u}_\lambda$ and N_2 fuzzy values of $\widetilde{q}_\lambda$ are known on Γ_1 and Γ_2, respectively, hence there are only N fuzzy unknowns in Eqs (5.6). One has to rearrange the system to obtain the standard system of fuzzy algebraic equations

$$\widetilde{\mathbf{A}}_\lambda \widetilde{\boldsymbol{X}}_\lambda = \widetilde{\boldsymbol{F}}_\lambda \qquad \forall 0 \leq \lambda \leq 1 \tag{5.7}$$

where $\widetilde{\boldsymbol{X}}_\lambda$ is a fuzzy (interval) vector of unknown λ-cuts $\widetilde{u}_\lambda$ and $\widetilde{q}_\lambda$ fuzzy boundary values. Eq (5.7) can now be solved and all the boundary values are then known. Let $\forall 0 \leq \lambda \leq 1$

$$X_\lambda := \Big\{ \boldsymbol{X} : \ \mathbf{A}_\lambda \boldsymbol{X} = \boldsymbol{F}_\lambda, \ \mathbf{A}_\lambda = [a_{\lambda ij}], \ \boldsymbol{F}_\lambda = [f_{\lambda i}], \quad a_{\lambda ij} \in \tilde{a}_{\lambda ij}, \ f_{\lambda i} \in \tilde{f}_{\lambda i}, \ i, j = 1, 2, ..., N \Big\} \tag{5.8}$$

Define $\tilde{\boldsymbol{X}}_1(\Gamma)$, $\Gamma \subset \mathcal{M}$, a fuzzy subset of R^N, by its membership function

$$\mu\Big(\boldsymbol{x}; \tilde{\boldsymbol{X}}_1(\Gamma)\Big) := \sup\Big\{\lambda : \ \boldsymbol{x} \in X_\lambda\Big\} \qquad \boldsymbol{x} \in R^N \qquad \Gamma \subset \mathcal{M} \tag{5.9}$$

We call $\tilde{\boldsymbol{X}}_1(\Gamma)$ an exact fuzzy conditional solution of FBEM for arbitrary $\Gamma \subset \mathcal{M}$. In further considerations the parameter Γ is fixed and we can, therefore, omit it.

Assume now that no $\mathbf{A} \subset \tilde{\mathbf{A}}_\lambda$ is singular $\forall 0 \leq \lambda \leq 1$. We wish to find the set of solutions $\tilde{\boldsymbol{X}}_1$ and its relation to Eq (5.7), where the interval multiplication and addition are used to evaluate its left-hand side. We now try to solve Eq (5.7) for the $x_{1\lambda}^-$ and $x_{1\lambda}^+$, $i = 1, 2, ..., N$, $0 \leq \lambda \leq 1$, and hope they are the λ-cuts of fuzzy numbers $\tilde{x}_i$, $i = 1, 2, ..., N$. In any case, assume that this method does produce fuzzy numbers $\tilde{x}_i$, $i = 1, 2, ..., N$.

Define $\tilde{\boldsymbol{X}}_2(\Gamma)$, $\Gamma \subset \mathcal{M}$, a fuzzy subset of R^N, by its membership function

$$\mu\Big(\boldsymbol{x}; \tilde{\boldsymbol{X}}_2(\Gamma)\Big) := \min_{1 \leq i \leq N} \Big\{\mu(x_i; \tilde{x}_i)\Big\} \qquad \boldsymbol{x} = [x_i] \in R^N \qquad \Gamma \subset \mathcal{M} \tag{5.10}$$

We can prove that $\tilde{\boldsymbol{X}}_{1\lambda} \subseteq [\tilde{\boldsymbol{X}}_{2\lambda}^-, \tilde{\boldsymbol{X}}_{2\lambda}^+]$, $\forall 0 \leq \lambda \leq 1$, $\Gamma \subset \mathcal{M}$.

Many authors (cf Moore, 1965; Neumaier, 1990; Skrzypczyk and Pownuk, 1997) discussed methods for computing an interval vector $\tilde{\boldsymbol{X}}_{2\lambda}$ containing $\tilde{\boldsymbol{X}}_{1\lambda}$. The exact calculation of $\tilde{\boldsymbol{X}}_{1\lambda}$ is very difficult for multidimensional problems. The interval vector $\tilde{\boldsymbol{X}}_{1\lambda} = [\tilde{\boldsymbol{X}}_{2\lambda}^-, \tilde{\boldsymbol{X}}_{2\lambda}^+]$, $\forall 0 \leq \lambda \leq 1$ defines a region in an N-dimensional space bounded by the planes $x_i = x_{i\lambda}^-$ and $x_i = x_{i\lambda}^+$, $i = 1, 2, ..., N$. Since $\tilde{\boldsymbol{X}}_{1\lambda}$ will usually not be a rectangle in R^N, we would expect $\tilde{\boldsymbol{X}}_{1\lambda}$ to be a proper subset of $\tilde{\boldsymbol{X}}_{2\lambda}$. Naturally, the smallest $\tilde{\boldsymbol{X}}_{2\lambda}$ is of interest. We shall use $\tilde{\boldsymbol{X}}_2$ as the approximate fuzzy solution of Eq (5.7).

6. Numerical results

6.1. Interval boundary conditions

The following example shows how the presented methods work for fuzzy boundary conditions. Analyse a simple potential problem. Consider the case of a square close domain of the type shown in Fig.3, where the boundary has been discretized into 12 fuzzy constant elements with 5 internal points (cf Brebbia and Dominguez, 1989; Burczyński and Skrzypczyk, 1995, 1996a,b, 1997a,b; Skrzypczyk and Burczyński, 1998a).

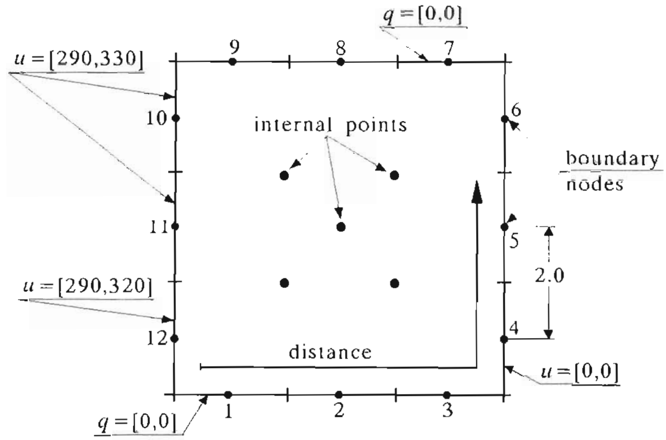

Fig. 3. Interval potential problem

It is assumed, that the boundary conditions u_0 and q_0 in the considered potential problem are the following interval functions

$$\tilde{u}_0(x) = [u_0^-(x), u_0^+(x)] \qquad x \in \Gamma_1$$
$$\tilde{q}_0(x) = [q_0^-(x), q_0^+(x)] \qquad x \in \Gamma_2 \tag{6.1}$$

and $\tilde{\xi}(x) \equiv 0$, $\forall x \in \Omega$. Numerical values are given in Fig.3.

Let the interval solution be denoted as $\tilde{u}(x) = [u^-(x), u^+(x)]$, $x \in \Gamma$. Since all boundary values are interval functions it is enough to solve the potential problem in the interval formulation only.

To compare the fuzzy results with the deterministic ones the potential problem under consideration was solved for mid-point boundary values, i.e.

$$u_0(x) = \frac{1}{2}\Big(u_0^-(x) + u_0^+(x)\Big) \qquad x \in \Gamma_1$$
$$q_0(x) = \frac{1}{2}\Big(q_0^-(x) + q_0^+(x)\Big) \qquad x \in \Gamma_2 \tag{6.2}$$

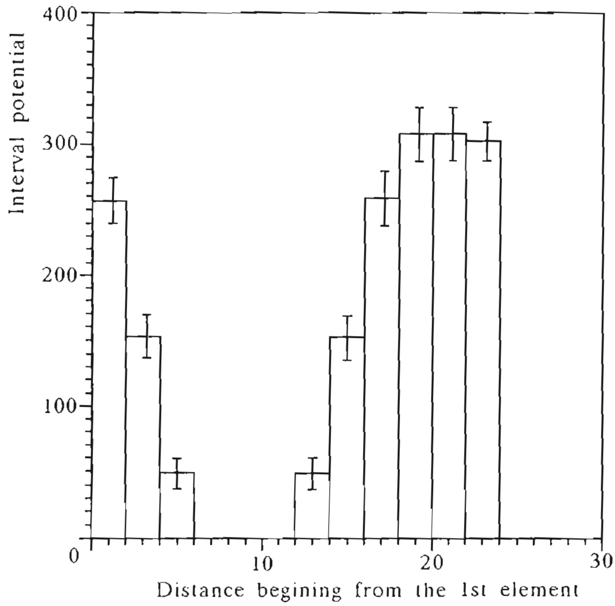

Fig. 4. Interval and mid-point potential solution

To illustrate a specific character of the interval calculations fuzzy results for internal points are presented in details in Table 1. To demonstrate qualitative character of fuzziness of results all values of the potential are presented graphically as interval functions of domain circumference, see Fig.4.

Table 1. Fuzzy results of internal values

Internal points			
x	y	Potential	
		u^-	u^+
2.00	2.00	154.8040	256.0090
2.00	4.00	155.5196	257.2826
3.00	3.00	101.6876	206.7500
4.00	2.00	47.9807	156.7115
4.00	4.00	48.2566	157.0701

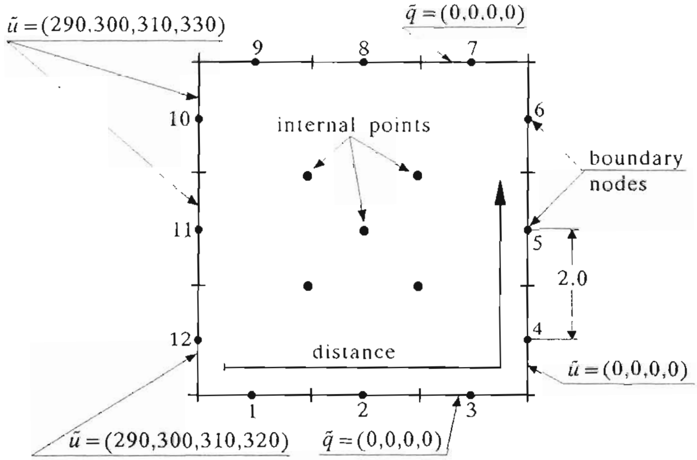

Fig. 5. Fuzzy potential problem with fuzzy trapezoidal boundary conditions

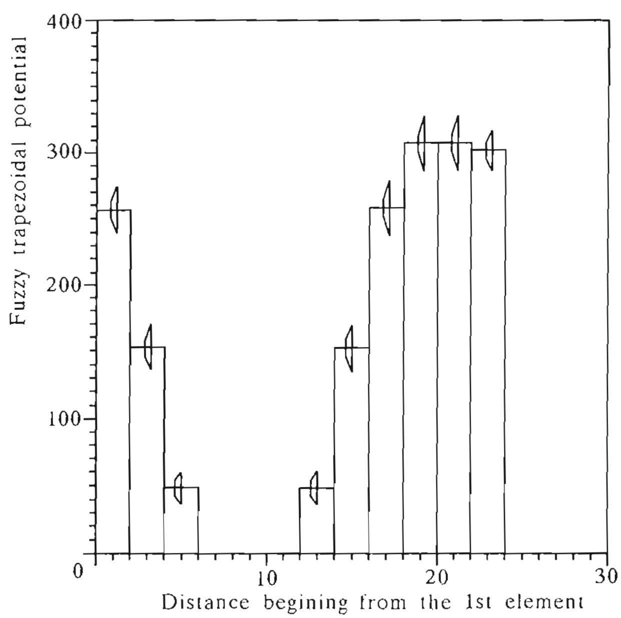

Fig. 6. Trapezoidal membership functions of boundary solutions

6.2. Trapezoidal boundary conditions

Consider now the same potential problem as in subsection 6.1 but with fuzzy boundary conditions of the trapezoidal membership functions. Such membership functions can be characterised as the ordered quadruple equivalent to the points of the trapezoid. Numerical values of the considered boundary values are given in Fig.5.

Since only boundary conditions are of the fuzzy character, the exact fuzzy solution has membership functions also of the trapezoidal shape. The results are presented in Fig.6. The values of membership functions of the solution at the internal points have trapezoidal character too. The numerical values are omitted for simplicity (cf Skrzypczyk and Burczyński, 1997a,b).

6.3. Potential problem in the fuzzy domain

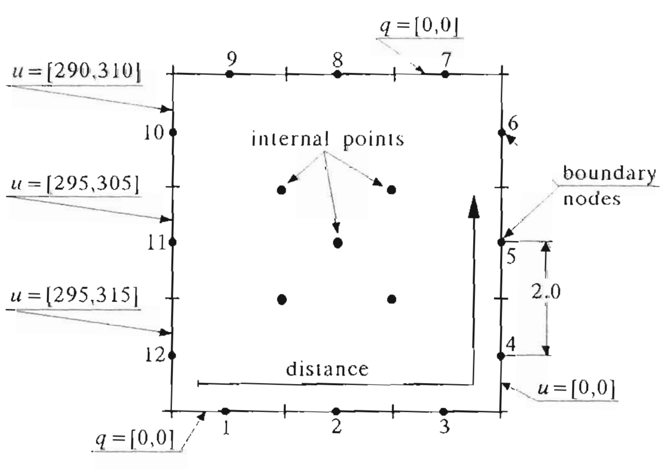

Fig. 7. Fuzzy boundary conditions

Now, consider the same potential problem. Assume, that boundary functions are of the interval character. This assumption is made for simplicity of calculations only. The methodology of calculations allows one to consider any shapes of membership functions. The boundary conditions u_0 and q_0 are independent with respect to the boundary fluctuations and $\widehat{\xi}(x) \equiv 0, \forall x \in \widetilde{\Omega}$.

The numerical values for boundary conditions are presented in Fig.7. Additionally assume, that the considered domain is fuzzy – its boundary is a tetragon with apexes which can take uncertain positions. Let the coordinates $(x_i, y_i, i = 1,2,3,4)$ be known with accuracy $(\pm\tau_i, \pm\sigma_i, i = 1,2,3,4)$, respectively, see Fig.8. In such a way, the considered domain is the fuzzy function of 8 fuzzy parameters, see Section 3.

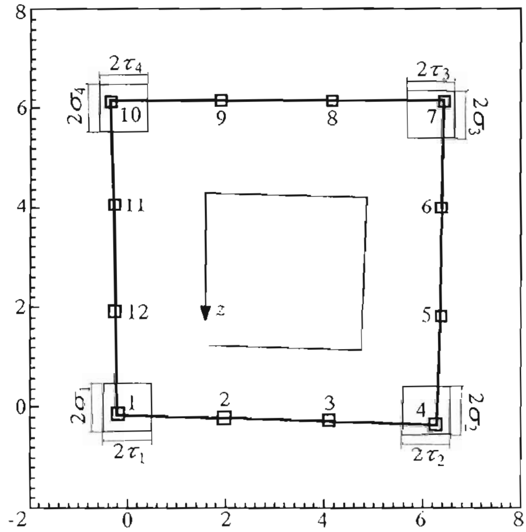

Fig. 8. Fuzzy domain of the boundary problem

At first we analyse the appropriate conditional solutions with respect to the fuzzy boundary. We know, that each conditional solution is the interval function, see Section 4. Denote this solution as $\tilde{u}(x|\Gamma) = [u^-(x), u^+(x)]$, $x \in \Gamma \subset \mathcal{M}$. If all conditional solutions are known, we use the max-min formula to obtain the global membership functions for interesting solutions – boundary or internal.

In Fig.9 the global boundary solution for the potential function is given. For comparison one conditional solution is presented for the particular case of the domain signed in Fig.8. It is also possible to analyse the potential problem in fully complicated form, with all fuzzy elements – internal sources, boundary conditions and the shape of boundary.

7. Conclusions

This paper is a continuation of earlier works summarising our knowledge

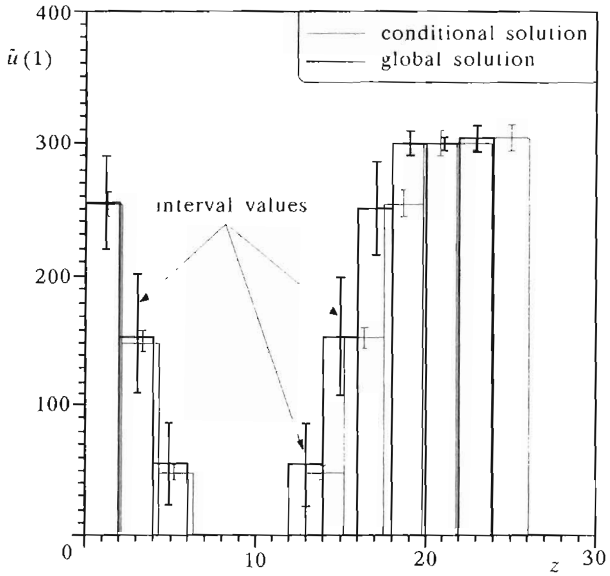

Fig. 9. Global fuzzy solution and the conditional one

about applications of the boundary element method to fuzzy analysis. It shows the new theoretical and computational methodology of the fuzzy analysis in potential theory. Sample applications are presented to potential problems with boundary conditions not sharply given by characterised by fuzzy functions of the interval type and trapezoidal-type membership functions.

A main conclusion is that fuzzy sets can be effectively used to estimate system uncertainty in boundary problems and random uncertainty can be calculated with a new technique called the FBEM approach.

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Metoda rozmytych elementów brzegowych w analizie systemów niepewnych

Streszczenie

W pracy przedstawiono elementy nowej koncepcji metody rozmytych elementów brzegowych. Artykuł omawia odwzorowania o wartościach rozmytych, które są rozwiązaniami rozmytych brzegowych równań całkowych. Analiza przeprowadzona jest na przykładzie problemu brzegowego klasycznej teorii potencjału z uwzględnieniem rozmytych warunków brzegowych typu Dirichleta i Neumanna, rozmytej funkcji gęstości źródeł oraz rozmytego kształtu obszaru.

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