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COMPUTATIONAL ASPECTS OF SATURATED POROUS MEDIA UNDERGOING LARGE DEFORMATIONS

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Large displacements and finite strains of saturated porous media will be considered. A consistent lagrangian description for both solid and fluid phases is presented. The problem of interaction between phases, crucial for computation, is discussed in detail. Variational formulations for the initial boundary value problem and iterative procedures allowing for overcoming nonlinearities are given. These enables one to solve a complex problem of deformation of two-component bodies with deformation-dependent porosity, permeability and separation of particles.

Key words: large deformations, porous media, phase interactions, partial stresses, interface.

1. Introduction

Nonlinear effects in porous media have been investigated by many authors. Among the results most suitable for our analysis, we might mention here the papers of Raats (1968), Szefer (1980), Prevost (1984), Schrefler (1994) and Wilmański (1996). A finite strain formulation in multicomponent continua leads to circumstances which are not trivial from the computational point of view: the lagrangian for solid and eulerian for fluid descriptions contain terms depending on a relative motion between the components (fluid-solid drag force). This fact, fundamental for proper formulation and analysis of the Boundary Value Problems (BVP), must be carefully taken into account. Therefore the proper choice of the form of description is of great importance.

In the present paper a finite deformation of a porous medium, skeleton of which is fulfilled by a fluid will be considered. A consistent material description for both phases has been used. The scope and range of the paper are as follows:

we begin with a short recapitulation of the general theory where the field equations of the system are be given (Section 2). The problem of interactions between phases mentioned above, which is crucial for the deformation process, will be considered in Section 3.

Then, we will pass to a variational formulation of the BVP (Section 4). Section 5 deals with approximations containing both the incremental approch (to overcome nonlinearities) and the iterative process to circumvent the interactions.

2. General theory. Field equations

Consider a two-component body B^α, $\alpha = s, F$ consisting of a porous solid skeleton B^s and an inviscid, incompressible fluid B^F which fulfils the pores. Assuming large deformations of the skeleton and using the lagrangian description one can write the balance equations for each constituent in the form:

— balance of mass

$$\rho^\alpha n^\alpha J^\alpha = \rho_R^\alpha n_R^\alpha \tag{2.1}$$

— balance of momentum

$$\mathrm{Div}\, n_R^\alpha \mathbf{T}_R^\alpha + \boldsymbol{r}^\alpha J^\alpha + \rho_R^\alpha n_R^\alpha \boldsymbol{b}^\alpha = \rho_R^\alpha n_R^\alpha \dot{\boldsymbol{v}}^\alpha \tag{2.2}$$

— balance of angular momentum

$$\mathbf{T}_R^\alpha \boldsymbol{F}^{\top,\alpha} = \boldsymbol{F}^\alpha \mathbf{T}_R^{\top,\alpha} \qquad \alpha = s, F \tag{2.3}$$

Above the following denotations have been used

- $\mathbf{T}_R^\alpha$ – Piola-Kirchhoff partial stress tensor
- $\boldsymbol{b}^\alpha$ – external body forces
- $\boldsymbol{r}^\alpha$ – internal volume forces resulting from interactions between the phases
- $\boldsymbol{v}^\alpha$ – velocity of the constituent particle
- $\boldsymbol{F}^\alpha$ – deformation gradient of the constituent B^α

$$\boldsymbol{F}^\alpha = \frac{d\boldsymbol{x}^\alpha}{d\boldsymbol{X}} = \mathrm{Grad}\,\boldsymbol{x}^\alpha \qquad \boldsymbol{x}^\alpha = \boldsymbol{\Psi}^\alpha(\boldsymbol{X}, t)$$

$$\boldsymbol{\Psi}^\alpha : \ B_R \to B_t \subset E_3 \qquad J^\alpha = \det \boldsymbol{F}^\alpha$$

x^α – position vector of the particle X
n^α – volume fracture, $n^\alpha = dV^\alpha/dV$
n^F – porosity, $n^F = n$
ρ^α – partial mass density

The subscript R stands for the reference configuration B_R which is assumed to be the initial configuration of the skeleton.

The following conditions must be satisfied

$$\begin{aligned} &n^s + n^F = 1 && \boldsymbol{r}^s J^s + \boldsymbol{r}^F J^F = \boldsymbol{0} \\ &J^\alpha > 0 && dV^s + dV^F = dV \end{aligned} \tag{2.4}$$

Remark 1. Since the constitutive law for the fluid component is obviuosly expressed by means of the Cauchy stresses $\mathbf{T}^\alpha$, the following formula

$$n_R^\alpha \mathbf{T}_R^\alpha = n^\alpha J^\alpha \mathbf{T}^\alpha \boldsymbol{F}^{-1,\alpha}$$

is valid for porous media. Thus for inviscid fluid it will be

$$n_R \mathbf{T}_R^F = n J^F \mathbf{T}^F \boldsymbol{F}^{-1,F} = n J^F(-p\mathbf{1})\boldsymbol{F}^{-1,F} = -n J^F \boldsymbol{F}^{-1,F} p$$

where p is the fluid pressure.

The above relations result from the equalities

$$\boldsymbol{t}^\alpha dA^\alpha = \boldsymbol{t}_R^\alpha dA_R^\alpha \qquad\qquad dA^\alpha = J^\alpha \sqrt{\boldsymbol{F}^{-1,\alpha}\boldsymbol{N}\boldsymbol{F}^{-1,\alpha}\boldsymbol{N}}\, dA_R^\alpha$$

where $\boldsymbol{t}^\alpha = \mathbf{T}^\alpha \boldsymbol{n}$ $\boldsymbol{t}_R^\alpha = \mathbf{T}_R^\alpha \boldsymbol{N}$ stand for the stress vectors in the current and reference configurations, respectively. Obviously, the vectors $\boldsymbol{n}$ and $\boldsymbol{N}$ stand for the unit outward normals to the considered surfaces (in B_t and B_R, respectively). dA^α, dA_R^α denote the elementary fields of surfaces in B_t and B_R, respectively.

The space deformation gradients are

$$\boldsymbol{F}^{-1,\alpha} = \frac{d\boldsymbol{X}}{d\boldsymbol{x}^\alpha} = \frac{1}{J^\alpha}\mathrm{cof}\boldsymbol{F}^\alpha = \frac{1}{J^\alpha}\frac{\partial J^\alpha}{\partial \boldsymbol{F}^\alpha}$$

where the operator cof denotes the algebraic co-factor.

Remark 2. Very often it is convenient to introduce:

— total stresses

$$\mathbf{T}_R = (1 - n_R)\mathbf{T}^s + n_R \mathbf{T}_R^F$$

— total body forces

$$\rho_R \boldsymbol{b} = \rho_R^s(1-n_R)\boldsymbol{b}^s + \rho_R^F n_R \boldsymbol{b}^F$$

— the mean velocity

$$\rho_R \boldsymbol{v} = \rho_R^s(1-n_R)\boldsymbol{v}^s + \rho_R^F n_R \boldsymbol{v}^F$$

Thus adding Eq (2.2) (for both constituents) and taking into account $(2.4)_2$ one obtains

$$\begin{aligned}
&\operatorname{Div}\mathbf{T}_R + \rho_R \boldsymbol{b} = \rho_R \dot{\boldsymbol{v}} \\
&\operatorname{Div} n_R \mathbf{T}^F + \boldsymbol{r}^F J^F + \rho_R^F n_R \boldsymbol{b}^F = \rho_R^F n_R \dot{\boldsymbol{v}}^F
\end{aligned}$$

Remark 3. In the theory of consolidation (generally considared in soil mechanics) the notion of the so called *water content ratio* θ (firstly introduced by Biot) is very useful. Denoting by $d\theta$ the amount of water impressed from the skeleton one defines

$$\theta = \frac{dQ}{dV_R} = \frac{dV^F - dV_R^F}{dV_R} = \frac{ndV - n_R dV_R}{dV_R} = \frac{nJ^s dV_R - n_R dV_R}{dV_R} = nJ^s - n_R$$

One should distinguish between θ (which depends on the current porosity) and the partial dilatation

$$\Delta^\alpha = \frac{dV^\alpha - dV_R^\alpha}{dV_R^\alpha} = J^\alpha - 1$$

which expresses the volume deformation of the constituent. Hence, it can be written

$$\theta = n(1+\Delta^s) - n_R$$

For small deformations we have immediately $\theta \simeq n - n_R$.

3. Phase interaction forces

In order to describe precisely the interaction densities $\boldsymbol{r}^\alpha$ we apply the decomposition

$$\boldsymbol{r}^\alpha = \boldsymbol{\sigma}^\alpha + \boldsymbol{\tau}^\alpha \tag{3.1}$$

first introduced by Heinrich-Desoyer (1955) and later by Raats (1968) and Szefer (1980). The term $\boldsymbol{\sigma}^\alpha$ corresponds to the deformation contact between phases whereas the second $\boldsymbol{\tau}^\alpha$ stands for the diffusive resistance (Stokes drag).

Let as use the average approach in the form given by Raats (1968)

$$\int\limits_V \boldsymbol{\sigma}^\alpha \, dV = \int\limits_{S_{\alpha\beta}} \boldsymbol{t}_\mu^\alpha \, dS_{\alpha\beta} \tag{3.2}$$

where

$\boldsymbol{t}_\mu^\alpha$ – represents the interface microstress vector, $\boldsymbol{t}_\mu^\alpha = \mathbf{T}_\mu^\alpha \boldsymbol{n}^{\alpha\beta}$

$\mathbf{T}_\mu^\alpha$ – partial microstress tensor

$S_{\alpha\beta}$ – interface between phases (see Fig.1)

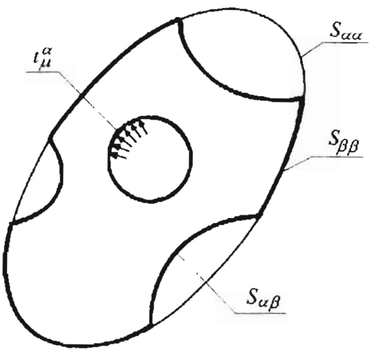

Fig. 1.

Introducing the following denotations

$$S_\alpha = S_{\alpha\beta} \cup S_{\alpha\alpha} \qquad\qquad S = S_{\alpha\alpha} \cup S_{\beta\beta}$$

for the closed surfaces in the domain V, one can write the right-hand side of Eq (3.2) in the form

$$\int\limits_{S_{\alpha\beta}} \boldsymbol{t}_\mu^\alpha \, dS_{\alpha\beta} = \int\limits_{S_\alpha} \boldsymbol{t}_\mu^\alpha \, dS_\alpha - \int\limits_{S_{\alpha\alpha}} \boldsymbol{t}_\mu^\alpha \, dS_{\alpha\alpha} = \int\limits_{S_\alpha} \mathbf{T}_\mu^\alpha \boldsymbol{n} \, dS_\alpha - \int\limits_S \boldsymbol{t}_\mu^\alpha n^\alpha \, dS =$$

$$= \int\limits_{V^\alpha} \operatorname{div} \mathbf{T}_\mu^\alpha \, dV_\alpha - \int\limits_S n^\alpha \mathbf{T}_\mu^\alpha \boldsymbol{n} \, dS = \int\limits_V n^\alpha \operatorname{div} \mathbf{T}_\mu^\alpha \, dV - \int\limits_S n^\alpha \mathbf{T}_\mu^\alpha \boldsymbol{n} \, dS =$$

$$= \int\limits_V n^\alpha \operatorname{div} \mathbf{T}_\mu^\alpha \, dV - \int\limits_V \operatorname{div} n^\alpha \mathbf{T}_\mu^\alpha \, dV = \int\limits_{V_R} (n^\alpha \operatorname{div} \mathbf{T}_\mu^\alpha - \operatorname{div} n^\alpha \mathbf{T}_\mu^\alpha) J^\alpha \, dV_R$$

Assuming that for fluid the same relation between micro-stresses in the current and reference configurations is valid (see Remark 1) and then taking into account the following easy provable formulas

$$\operatorname{div}(J^{-1}\boldsymbol{F}) = 0$$
$$\operatorname{div}(J^{-1}\mathbf{T}_R\boldsymbol{F}) = J^{-1}\operatorname{Grad}\mathbf{T}_R\mathbf{l} = J^{-1}\operatorname{Div}\mathbf{T}_R$$

one obtaines

$$\boldsymbol{\sigma}^F J^F = -n\operatorname{Div}(J^F\boldsymbol{F}^{-F}p) + \operatorname{Div}J^F\boldsymbol{F}^{-F}p = J^F p\boldsymbol{F}^{-1,F}\operatorname{Grad}n \qquad (3.3)$$

that represents the average effects of intrinsic contact between phases in a material description.

The second term in Eq (3.1) which describes the exchange of momentum resulting from different velocity fields of the components (diffusive force) must take the form

$$J^F\boldsymbol{\tau}^F = \mathbf{K}^{-1}(\boldsymbol{v}^F - \boldsymbol{v}^s) \qquad (3.4)$$

where $\mathbf{K}$ is the permeability tensor defined within the framework of constitutive relations.

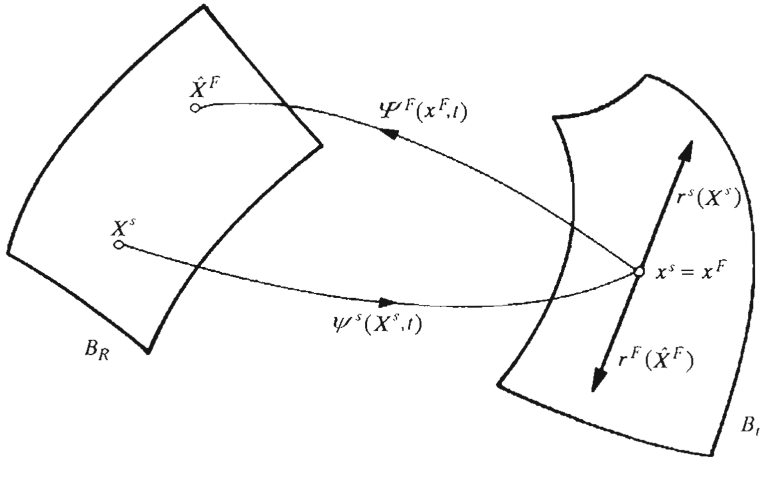

Fig. 2.

Eq (3.4) constitutes the crucial point of computations since the velocities in Eq (3.4) are related to different particles (see Fig.2).

Namely, for $\boldsymbol{x}^s = \boldsymbol{x}^F \Rightarrow \boldsymbol{X}^s \neq \widehat{\boldsymbol{X}}^F$ it is

$$J^s\boldsymbol{r}^s(\boldsymbol{X}^s) + J^F\boldsymbol{r}^F(\widehat{\boldsymbol{X}}^F) = 0 \qquad (3.5)$$

Hence $J^s\boldsymbol{r}^s(\boldsymbol{X}^s) = -J^F\boldsymbol{r}^F(\widehat{\boldsymbol{X}}^F)$.

Then we have

$$
\begin{aligned}
&J^s\boldsymbol{r}^s(\boldsymbol{X}^s) = -(J^F\boldsymbol{\sigma}^F + J^F\boldsymbol{\tau}^F)\Big|_{\widehat{\boldsymbol{X}}^F} = \\
&= -J^F p\boldsymbol{F}^{-1,F}\,\mathrm{Grad}\, n\big|_{\widehat{\boldsymbol{X}}^F} - \mathbf{K}^{-1}(\widehat{\boldsymbol{X}}^F)\Big[\boldsymbol{v}^F(\widehat{\boldsymbol{X}}^F) - \boldsymbol{v}^s(\boldsymbol{X}^s)\Big]
\end{aligned}
\tag{3.6}
$$

Thus on the basis of Eq (2.2) the equations of motion for the constituents obtains the final form

$$
\begin{aligned}
\boldsymbol{X}^s\ :\quad & \mathrm{Div}\,(1-n_R)\mathbf{T}^s_R - J^F p\boldsymbol{F}^{-1,F}\,\mathrm{Grad}\, n(\widehat{\boldsymbol{X}}^F) + \\
& -\mathbf{K}^{-1}(\widehat{\boldsymbol{X}}^F)\Big[\boldsymbol{v}^F(\widehat{\boldsymbol{X}}^F) - \boldsymbol{v}^s(\boldsymbol{X}^s)\Big] + \rho^s_R(1-n_R)\boldsymbol{b}^s = \rho^s_R(1-n_R)\dot{\boldsymbol{v}}^s \\
\widehat{\boldsymbol{X}}^F\ :\quad & -n\,\mathrm{Div}\, J^F p\boldsymbol{F}^{-1,F} + \mathbf{K}^{-1}(\widehat{\boldsymbol{X}}^F)\Big[\boldsymbol{v}^F(\widehat{\boldsymbol{X}}^F) - \boldsymbol{v}^s(\boldsymbol{X}^s)\Big] + \\
& +\rho^F_R n_R\boldsymbol{b}^F = \rho^F_R n_R\dot{\boldsymbol{v}}^F
\end{aligned}
\tag{3.7}
$$

with the substitution

$$
\widehat{\boldsymbol{X}}^F = \boldsymbol{\Psi}^F\Big(\boldsymbol{\psi}(\boldsymbol{X}^s,t),t\Big)
\tag{3.8}
$$

where $\boldsymbol{\Psi}^F$ is the inverse mapping of $\boldsymbol{\psi}^F(\boldsymbol{X},t) = \boldsymbol{x}^F$.

The system of equations given above should be discussed together with the initial

$$
\boldsymbol{u}^\alpha(\boldsymbol{X},t_0) = \boldsymbol{u}^\alpha_0(\boldsymbol{X}) \qquad \dot{\boldsymbol{u}}^\alpha(\boldsymbol{X},t_0) = \boldsymbol{v}^\alpha_0(\boldsymbol{X}) \qquad \boldsymbol{X} \in B_R
\tag{3.9}
$$

and boundary conditions

$$
\begin{aligned}
&\mathbf{T}^\alpha_R\boldsymbol{N}(\boldsymbol{X}_0,t) = \boldsymbol{p}^\alpha_R(\boldsymbol{X}_0,t) && \boldsymbol{X}_0 \in S_\sigma \\
&\boldsymbol{u}^\alpha(\boldsymbol{X}_0,t) = \boldsymbol{g}^\alpha(\boldsymbol{X}_0,t) && \boldsymbol{X}_0 \in S_u \\
&(\boldsymbol{v}^F - \boldsymbol{v}^s)\boldsymbol{n} = \begin{cases} w(\boldsymbol{X}_0,t) \\ 0 \end{cases} && \begin{array}{l}\text{for permeable edge } \boldsymbol{X}_0 \in S_w \\ \text{for impermeable edge}\end{array} \\
&S_R = S_\sigma \cup S_u \cup S_w && S_\sigma \cap S_u = \emptyset
\end{aligned}
\tag{3.10}
$$

where

- $\boldsymbol{u}^\alpha$ – denote displacement of the particles
- $\boldsymbol{p}^\alpha_R$ – prescribed surface tractions of the constituents
- w – prescribed outflow velocity of the fluid.

Remark 4. Condition $(3.10)_3$ represents the situation when the outflow w is known a priori. In practice, however, it occurs very often, that the outflow through the permeable edge is induced by the tractions $\boldsymbol{p}_R^{\alpha}$ only. Then obviously one assumes on S_σ : $p = 0$ (or $p = p_a$ – atmospheric pressure) for the fluid constituent.

Thus, the amount of fluid which flows through the boundary results from the continuity of the momentum balance (see $(3.7)_2$)

$$\boldsymbol{v}^F(\widehat{\boldsymbol{X}}^F) - \boldsymbol{v}^s(\widehat{\boldsymbol{X}}^s) = \mathsf{K}\Big[\rho_R^F n_R(\dot{\boldsymbol{v}}^F - \boldsymbol{b}^F) + n\,\mathrm{Div}\,J^F p \boldsymbol{F}^{1,F}\Big] \qquad \text{on } S_w$$

The fluid particles which leave the skeleton constitute a one-phase medium with a moving free boundary (separation of phases outside the skeleton domain occurs). This problem is not discussed here and will be an objective of another paper (our analysis is restricted to the region where no separation of phases takes place, only).

Remark 5. The argument of the inverse function $\boldsymbol{\Psi}^F(\boldsymbol{x}^F, t)$ results from the equality $\boldsymbol{x}^F(\widehat{\boldsymbol{X}}^F, t) = \boldsymbol{x}^s(\boldsymbol{X}^s, t)$. Hence, taking any $\boldsymbol{X}^s$ one must find $\boldsymbol{x}^s = \boldsymbol{\psi}^s(\boldsymbol{X}^s, t)$ and than through-out (3.8) determine the corresponding particle $\widehat{\boldsymbol{X}}^F$. As we see this can be done by considering simultaniously Eqs (3.7), (3.8), only. Overcoming this difficulty is possible only on the basis of an iterative procedure.

Difficulties mentioned above vanish immediately in the case of small deformations (i.e., by assuming small displacements, where the difference between particles $\boldsymbol{X}^s$ and $\widehat{\boldsymbol{X}}^F$ can be neglected).

For the skeleton it is convenient to introduce the symmetric II Piola-Kirchhoff stress tensor

$$\mathsf{S} = \mathsf{T}_R^s \boldsymbol{F}^{-\top,s} \tag{3.11}$$

Thus Eq (2.3) holds automaticaly. It is satisfied also for the inviscid fluid.

The system (2.1), (3.7) (by introducing Eq (3.11)) need complementation by constitutive relations which we take in the form

$$\begin{aligned} &\mathsf{S} = \mathcal{F}^s(\boldsymbol{C}^s, \theta) \qquad\qquad p = \Pi(\boldsymbol{C}^s, \theta) \\ &\boldsymbol{\tau}^F = \mathsf{K}^{-1}(\boldsymbol{C}^s, \theta, |\boldsymbol{v}^F - \boldsymbol{v}^s|)\Big(\boldsymbol{v}^F - \boldsymbol{v}^s\Big) \end{aligned} \tag{3.12}$$

where $\boldsymbol{C}^s = \boldsymbol{F}^\top \boldsymbol{F}^s$ is the Cauchy-Green deformation tensor and θ is described in Remark 3.

The system (2.1),(3.7) together with the above relations and definitions of θ and $\boldsymbol{C}^s$ enables one to determine the unknown functions n, θ, p, $\boldsymbol{v}^s$, $\boldsymbol{v}^F$, $\mathbf{C}^s$, $\mathbf{S}$ describing the state of the porous, fluid saturated medium.

Remark 6. The form (3.4) of the drag vector $\boldsymbol{r}^F$ (and hence its constitutive equation) results from the representation theorems of the tensor functions. Indeed, any vector function of the kind $\boldsymbol{r}^F(\mathbf{C}^s, \boldsymbol{v}^s, \boldsymbol{v}^F, \theta)$ can be of the form $(3.12)_3$ only.

4. Variational formulation of the BVP

Considering the set of kinematical admissible velocities

$$\widehat{\boldsymbol{v}}^\alpha(\boldsymbol{X}, t, q) = \boldsymbol{v}^\alpha(\boldsymbol{X}, t) + q\boldsymbol{\eta}^\alpha(\boldsymbol{X}, t) \qquad q \in \Re$$

let us define the virtual velocities

$$\delta\boldsymbol{v}^\alpha = \frac{\partial\widehat{\boldsymbol{v}}}{\partial q}dq \qquad\qquad \delta\boldsymbol{v}^\alpha \in V = \left\{\boldsymbol{\eta}^\alpha : \ \boldsymbol{\eta}^\alpha(\boldsymbol{X}_0, t) = 0, \ \ \boldsymbol{X}_0 \in S_u\right\}$$

For the sale of simplicity, let us write the system (3.7) in the compact form

$$\begin{aligned} &\operatorname{Div}(1 - n_R)\mathbf{T}^s_R - J^F\boldsymbol{r}^F + \rho^s_R(1 - n_R)\boldsymbol{b}^s = \rho^s_R(1 - n_R)\dot{\boldsymbol{v}}^s \\ &\operatorname{Div} n_R\mathbf{T}^F_R + J^F\boldsymbol{r}^F + \rho^F_R n_R\boldsymbol{b}^F = \rho^F_R n_R\dot{\boldsymbol{v}}^F \end{aligned} \tag{4.1}$$

Multiplying the first equation by $\delta\boldsymbol{v}^s$, the second one by $\delta\boldsymbol{v}^F$ and than applying the standart procedure for the weak formulation one obtaines the principle of the virtual power

$$\begin{aligned} &\int\limits_{V_R}\left[(1 - n_R)\mathbf{S}\boldsymbol{F}^{\mathsf{T},s} : \nabla\delta\boldsymbol{v}^s + n_R\mathbf{T}^F_R : \nabla\delta\boldsymbol{v}^F\right] dV_R + \\ &+ \int\limits_{V_R} J^F\boldsymbol{r}^F(\delta\boldsymbol{v}^F - \delta\boldsymbol{v}^s)\, dV_R = \int\limits_{V_R}\left[\rho^s_R(1 - n_R)\boldsymbol{b}^s\delta\boldsymbol{v}^s + \rho^F_R n_R\boldsymbol{b}^F\delta\boldsymbol{v}^F\right] dV_R + \\ &+ \int\limits_{S_R}(\boldsymbol{p}^s_R\delta\boldsymbol{v}^s + \boldsymbol{p}^F_R\delta\boldsymbol{v}^F)\, dS_R - \int\limits_{V_R}\left[\rho^s_R(1 - n_R)\dot{\boldsymbol{v}}^s\delta\boldsymbol{v}^s + \rho^F_R n_R\dot{\boldsymbol{v}}^F\delta\boldsymbol{v}^F\right] dV_R \end{aligned} \tag{4.2}$$

where

- (:) – inner product of tensors
- ∇ – gradient operator.

Remark 7. All dynamical and kinematical quantities in Eq (4.2); like, stresses $\mathbf{S}$, $\mathbf{T}_R^F$ (or p), deformation measures $\boldsymbol{F}^\alpha$, J^F, $\mathbf{C}^s$ and topological characteristics n, θ should be expressed in terms of displacements $\boldsymbol{u}^s$, $\boldsymbol{u}^F$ as basic unknowns of the deformation process.

Remark 8. Principle (4.2) (with Remark 7) represents the pure kinematical approach of the analysis which differs from the mixed velocity $\boldsymbol{v}^\alpha$ – pressure p formulation, very often used in fluid mechanics. We prefer here the kinematical approach since by virtue of the constitutive equation $(3.12)_2$ the pressure p can be eliminated. Moreover, our exprience shows that this formulation has also some computational advantages (facility to construct the kinematical admissible fields, loss of numerical instabilities).

Then, introducing for simplicity the denotation

- $B(\boldsymbol{u}^\alpha, \delta\boldsymbol{v}^\alpha)$ – first integral in Eq (4.2)
- $R(\boldsymbol{v}^\alpha, \delta\boldsymbol{v}^\alpha)$ – second term of the left-hand side which describe the virtual power of interactions
- $L(\boldsymbol{b}^\alpha, \boldsymbol{p}^\alpha, \delta\boldsymbol{v}^\alpha)$ – the virtual power of the external forces (two first integrals on the right-hand side)
- $L_b(\dot{\boldsymbol{v}}^\alpha, \delta\boldsymbol{v}^\alpha)$ – last term in Eq (4.2)

one can rewrite Eq (4.2) in the abstract form for $\delta\boldsymbol{v}^s, \delta\boldsymbol{v}^F \in V$

$$B(\boldsymbol{u}^\alpha, \delta\boldsymbol{v}^\alpha) + R(\boldsymbol{v}^\alpha, \delta\boldsymbol{v}^\alpha) = L(\delta\boldsymbol{v}^\alpha) - L_b(\dot{\boldsymbol{v}}^\alpha, \delta\boldsymbol{v}^\alpha) \tag{4.3}$$

convenient for further considerations.

Eq (4.3) must be solved together with Eq (3.8).

5. Incremental formulation. Discretization

To solve any initial-boundary-value problem of the type (4.3) with Eq (3.9) the incremental approach will be obviously applied. As it is known, it consist in the mapping of the prescribed quantities $(\boldsymbol{b}^\alpha, \boldsymbol{p}^\alpha) : [0,1] \to f(\lambda) = (\lambda\boldsymbol{b}^\alpha, \lambda\boldsymbol{p}^\alpha)$

and then in partition of the interval $[0,1]$: $0 = \lambda_0 < \lambda_1 < ... < \lambda_N = 1$. Then it follows

$$\begin{aligned} &\boldsymbol{u}^\alpha : \ [0,1] \rightarrow \boldsymbol{u}^\alpha(\lambda) \\ &\boldsymbol{u}^\alpha_{i+1} = \boldsymbol{u}^\alpha(\lambda_{i+1}) = \boldsymbol{u}^\alpha(\lambda_i + \Delta\lambda) = \boldsymbol{u}^\alpha_i + \Delta\boldsymbol{u}^\alpha \end{aligned} \tag{5.1}$$

Similarly, all remaining quantities $\boldsymbol{F}^\alpha$, $\mathbf{T}^\alpha_R$, n etc. have the same (like Eq (5.1)) representation.

Assuming that all functionals in Eq (4.3) are at least one-sided Gateaux directional differentiable, we obtain the incremental formulation of the virtual power principle

$$\begin{aligned} &B'_{u^s}(\boldsymbol{u}^\alpha_i;\delta\boldsymbol{v}^\alpha)\Delta\boldsymbol{u}^s + B'_{u^F}(\boldsymbol{u}^\alpha_i;\delta\boldsymbol{v}^\alpha)\Delta\boldsymbol{u}^F + R'_{u^s}(\boldsymbol{v}^\alpha_i;\delta\boldsymbol{v}^\alpha)\Delta\boldsymbol{u}^s + \\ &+R'_{u^F}(\boldsymbol{v}^\alpha_i;\delta\boldsymbol{v}^\alpha)\Delta\boldsymbol{u}^F + L_b(\Delta\dot{\boldsymbol{v}}^\alpha,\delta\boldsymbol{v}^\alpha) = L(\Delta\boldsymbol{b}^\alpha,\Delta\boldsymbol{p}^\alpha,\delta\boldsymbol{v}^\alpha) + \\ &-B(\boldsymbol{u}^\alpha_i;\delta\boldsymbol{v}^\alpha) - R(\boldsymbol{v}^\alpha_i;\delta\boldsymbol{v}^\alpha) \qquad\qquad i = 0,1,2,...,N \qquad \alpha = s,F \end{aligned} \tag{5.2}$$

Here the terms with the superscript $(\cdot)'$ denote suitable Gateaux derivatives of the functionals.

The above equation being linear with respect to the increments $\Delta\boldsymbol{u}^s$, $\Delta\boldsymbol{u}^F$ gives a recursive "step by step" procedure for finding a solution to the boundary-value problem. This procedure, combined with the finite element technique in space and with the finite difference discretization in the time domain, leads to a matrix equation of the type

$$\mathbf{M}\Delta\ddot{\widehat{\boldsymbol{u}}} + \mathbf{C}\Delta\dot{\widehat{\boldsymbol{u}}} + \mathbf{K}\Delta\widehat{\boldsymbol{u}} = \Delta\boldsymbol{F} + \boldsymbol{Q}_t \tag{5.3}$$

where

$\Delta\widehat{\boldsymbol{u}}$	–	incremental nodal displacement vector
$\mathbf{M},\mathbf{C},\mathbf{K}$	–	matrices obtained from Eq (5.2) by inspection
$\Delta\boldsymbol{F}$	–	external force-increment vector
$\boldsymbol{Q}_t$	–	residual vector resulting from linearization of the nonlinear terms.

Difficulties in solving the large displacement and finite strain boundary value problem arise not only due to strong nonlinearities of the system (4.1), but (as mentioned previously) also from the fact that the function (3.8) must be taken into account. This can be done iteratively as follows:

one starts with the given particles $\boldsymbol{X}^s$ and $\boldsymbol{X}^F$ at the time t_0 (related to B_R). We are looking for the localizations

$$\boldsymbol{x}^s_{N+1} = \boldsymbol{X}^s + \sum_{i=0}^{N} \boldsymbol{\Phi}^s(\boldsymbol{X}^s)\Delta\widehat{\boldsymbol{u}}^s_i$$
$$\boldsymbol{x}^F_{N+1} = \boldsymbol{X}^F + \sum_{i} \boldsymbol{\Phi}^F(\boldsymbol{X}^F)\Delta\widehat{\boldsymbol{u}}^F_i$$

where $\boldsymbol{\Phi}^\alpha$ – matrix of the shape functions.

Then taking the equality

$$\boldsymbol{x}^s_i = \boldsymbol{x}^F_i$$

we solve the equation

$$\boldsymbol{x}^s_i = \boldsymbol{X}^F + \sum_{i=0}^{N} \boldsymbol{\Phi}^F(\boldsymbol{X}^F)\Delta\widehat{\boldsymbol{u}}^F_i \Rightarrow \boldsymbol{X}^F_{i+1}$$

Now we pass to the new finite element coordinates and find new $\boldsymbol{x}^s_{i+1}$ and $\boldsymbol{x}^F_{i+1}$. This procedure shuld be repeated until

$$|\boldsymbol{x}^s_{i+1} - \boldsymbol{x}^F_{i+1}|^{N+1} \leqslant \varepsilon \qquad \text{sufficiently small}$$

A numerical example of a multilayered porous subsoil in the uni-axial state of deformation will be given separately.

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Obliczeniowe aspekty nasyconych ośrodków porowatych przy dużych deformacjach

Streszczenie

W pracy rozważano duże przemieszczenia i skończone odkształcenia nasyconych ośrodków porowatych. Przedstawiono konsekwentny opis Lagrange'a dla obu faz: ciekłej i stałej. Szczegółowo omówiono zasadniczy dla obliczeń problem interakcji faz. Podano wariacyjne sformułowanie problemu początkowo-brzegowego oraz iteracyjne procedury dla pokonania nieliniowości. To pozwala rozwiązywać złożone problemy deformacji ośrodków dwu-składnikowych ze zmienną, zależną od deformacji porowatością, przepuszczalnością i separacją cząstek.

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