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LONGITUDINAL SHEARING OF ELASTIC-PLASTIC FIBROUS COMPOSITE WITH FRICTIONAL FIBRE-MATRIX INTERFACE[1]

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The problem of macroscopic behaviour of a periodic, elastic-plastic, fibrous composite material with frictional fibre-matrix interface is considered. Anti-plane shear of the material is analysed. The problem is formulated with the help of variational inequalities using the two approaches: displacement-based and stress-based ones. The variational inequalities are solved using the finite element method. In the case of stress-based approach, the Prandtl stress function is utilized. The obtained effective constitutive model of the composite can be used directly in analysis of the problem of torsion of a composite bar with a fine, periodic structure.

Key words: homogenisation, finite element method, duality

1. Introduction

The problem of determination of effective constitutive relations for a periodic, fibrous, elastic-plastic composite material is analysed. It is assumed that there is no adhesion between components of the composite material on the fibre-matrix interface, and the slip is governed by the Coulomb friction law. The problem of anti-plane (longitudinal) shear is studied. It was shown by Suquet (1985) that the mechanical properties of the considered composite can be described by the generalised standard material model.

Following the concepts of homogenisation theory (e.g. Duvaut, 1976; Bensoussan et al., 1978; Sanchez-Palencia, 1980; Suquet, 1985), the cell problem,

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which is to be solved in order to find the effective constitutive relations, is formulated in two dual forms of variational inequalities, which are solved using the displacement-based and stress-based finite element methods.

The problem of anti-plane shear of elastic composite with frictional fibre-matrix interface was studied by Teng and Agah-Tehrani (1992), where the composite cylinder model had been employed to simplify corresponding boundary conditions. The same problem was solved by Więckowski (1995a) using the dual finite element analysis. The problem of plane strain was analysed by Lene and Leguillon (1981), and Lene (1984) but for other models of interface slip, elastic and viscoelastic ones, respectively.

The problem is interesting in analysis of torsion of a periodic, composite bar reinforced with fibres, the diameter of which is very small with respect to dimensions of the bar cross-section. The analysis of such a problem, based on the direct discretisation of the cross-section of the heterogeneous body may be very time-consuming even when using a very fast computer. The computation time is reduced significantly when the problem is solved in two steps: firstly, the effective constitutive relations of the material are determined solving the appropriate boundary value problem for only one repeatable cell of the body, and secondly the standard discretisation is used in order to solve the torsion problem for the homogenised material which, generally, exhibitis anisotropic behaviour.

2. Formulation of the problem

We consider periodic a fibre reinforced composite body, the parts of which are composed of isotropic elastic-plastic materials. The structure of the body, and its typical cell are shown in Fig.1.

The quasi-static evolution problem of anti-plane shear of the composite material is considered. The plane of shear is assumed to be parallel to the fibres, i.e. the macroscopic strain tensor has only two non-zero components, it means $E_{3\alpha} \neq 0$, $E_{\alpha\beta} = E_{33} = 0$, where $\alpha, \beta = 1, 2$

$$E_{ij} = \widetilde{\varepsilon}_{ij} \qquad\qquad \widetilde{(\cdot)} = \frac{1}{\text{meas}(Y)} \int\limits_Y (\cdot)\, dy$$

ε_{ij} denotes the tensor of small deformations, $\text{meas}(Y)$ is the area of the basic repeatable cell. The displacement field has the following components (see e.g.

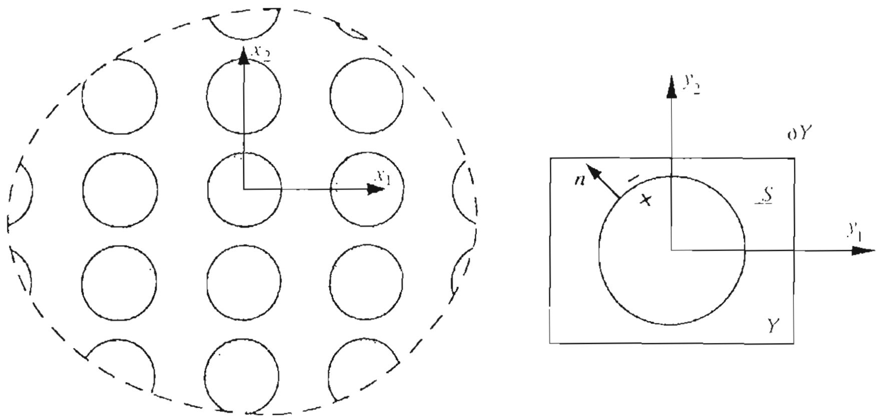

Fig. 1. Composite material structure, and its typical cell

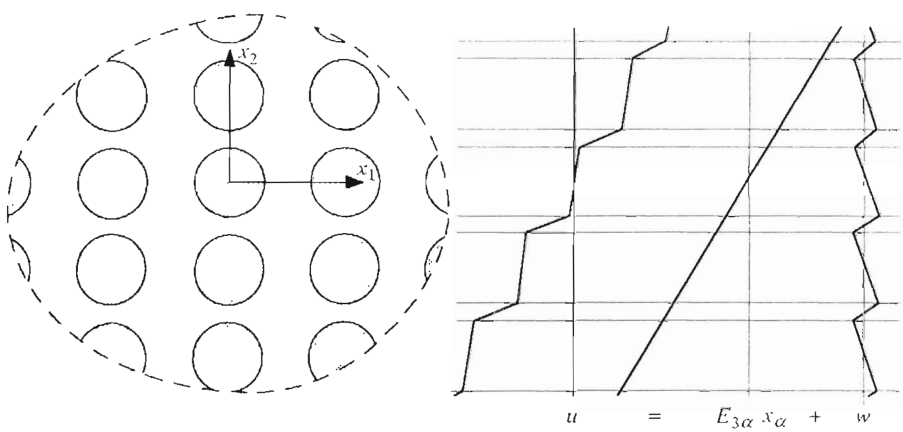

Fig. 2. Displacement decomposition into the homogeneous part $E_{3\alpha}x_\alpha$ and periodic oscillation w

Suquet, 1985; Duvaut, 1976; Bensoussan et al., 1978)

$$u_\alpha = E_{\alpha 3} x_3 \qquad\qquad u_3 \equiv u = E_{3\alpha} x_\alpha + w \tag{2.1}$$

where $w = w(y_\alpha)$ is Y-periodic. The relation $(2.1)_2$ is represented in Fig.2.

The wanted Y-periodic fields of displacements $w(t)$, and stresses $\sigma(t)$ satisfy the following relations:

(i) the strain-displacement relations

$$\varepsilon_{3\alpha} = \frac{1}{2}(u_{3,\alpha} + u_{\alpha,3}) \qquad \text{on } Y \tag{2.2}$$

(ii) the equilibrium equation

$$\sigma_{3\alpha,\alpha} = 0 \qquad \text{on } Y \tag{2.3}$$

where $\sigma_{3\alpha}$ are the non-zero components of the stress tensor;

(iii) the Prandtl-Reuss constitutive relations, see e.g. Duvaut and Lions (1976), Hill (1950)

$$\left.\begin{aligned} &\dot{\varepsilon}_{3\alpha} = \dot{\varepsilon}^{\rm e}_{3\alpha} + \dot{\varepsilon}^{\rm p}_{3\alpha} \\ &\dot{\varepsilon}^{\rm e}_{3\alpha} = \frac{1}{2G}\dot{\sigma}_{3\alpha} \\ &\dot{\varepsilon}^{\rm p}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha}) \leqslant 0 \qquad \forall \tau \in B \end{aligned}\right\} \qquad \text{on } Y \tag{2.4}$$

where a dot indicates the time derivative, $\varepsilon^{\rm e}$ and $\varepsilon^{\rm p}$ are the elastic and plastic parts of strain, respectively, G denotes the elastic shear modulus, and

$$B = \{\tau \in \mathcal{R}^2 : \quad \tau_{3\alpha}\tau_{3\alpha} \leqslant \tau_0^2\}$$

is the set of plastic admissible stresses, where τ_0 is the plastic limit for shear stresses (the Huber-Mises yield condition is considered);

(iv) the continuity condition for the stress vector on the interface

$$\sigma_T^+ + \sigma_T^- = 0 \qquad \text{on } S$$

where

$$\sigma_T^+ = \sigma_{3\alpha}^+ n_\alpha^+ \qquad\qquad \sigma_T^- = \sigma_{3\alpha}^- n_\alpha^- \qquad\qquad n_\alpha^+ = -n_\alpha^- = n_\alpha$$

$\boldsymbol{n}$ is the unit vector outwardly normal to S (see Fig.1);

(v) the Coulomb law of friction on the interface (e.g. Duvaut and Lions. 1976)

$$
\left.\begin{array}{lll}
|\sigma_T| < g & \rightarrow & [\![\dot{w}]\!] = 0 \\
|\sigma_T| = g & \rightarrow & \exists\lambda \geqslant 0 \quad [\![\dot{w}]\!] = -\lambda\sigma_T
\end{array}\right\} \quad \text{at} \quad x \in S \quad \text{if} \quad g(y) > 0
$$

$$
\sigma_T = 0 \qquad\qquad \text{at} \quad x \in S \quad \text{if} \quad g(y) = 0 \tag{2.5}
$$

where $\sigma_T = \sigma_T^+$, $[\![(\cdot)]\!] = (\cdot)^+ - (\cdot)^-$, g is a non-negative function – the limit for fibre-matrix tractions;

(vi) the initial conditions

$$
w = 0 \qquad \sigma_{3\alpha} = 0 \qquad \text{for} \quad t = 0
$$

where t is the time variable.

In (2.5), $g = g(y) \equiv \mu\sigma_n$, where μ denotes the friction coefficient. and $\sigma_n = \sigma_{\alpha\beta}n_\alpha n_\beta$ is the stress component normal to the interface, which is caused. e.g., by shrinkage of the matrix or external pressure acting in the x_1x_2-plane. Since the stresses $\sigma_{\alpha\beta}$ are not unknown variables in the anti-plane shear problem, the function g does not depend on the solution $\sigma_{3\alpha}$. This means that we have here a simpler case of the Coulomb friction law than in general. This case is known in the literature as Tresca's law of friction.

The displacement field w solving the problem (i)÷(vi) is defined up to an additive constant (see e.g. Bensoussan et al., 1978), which means that the translation of the x_1x_2-cross-section in the x_3-direction does not affect the field of strains and stresses. To avoid the problem of non-uniqueness of the solution for displacement field, the following condition is added

$$
\int\limits_Y w\, dy = 0
$$

3. Finite element solution in displacements

3.1. Variational formulation

Let V and W be the spaces

$$
V = \Big\{ v \in H^1(Y \setminus S): \quad \widetilde{v} = 0, \quad v \ Y\text{-periodic} \Big\}
$$

$$
W = \Big\{ v \in BV(Y \setminus S): \quad \widetilde{v} = 0 \quad v \ Y\text{-periodic} \Big\}
$$

where $H^1(Y \setminus S)$ is the Sobolev space (see Adams, 1975), and $BV(Y \setminus S)$ is the space of functions of bounded variations (see e.g. Matthies et al., 1979). Let P be the set

$$P = \Big\{\tau \in [L^2(Y)]^2 : \quad \tau(x) \in B(x) \quad \text{a.e. on } Y\Big\} \tag{3.1}$$

where $L^2(Y)$ is the space of square integrable functions (Adams, 1975), and the abbreviation a.e. means "almost everywhere". Let $I = [0,T]$ be a time interval, where $T > 0$.

Let us formulate the problem in the variational form. It follows from Eq (2.4) that

$$\Big(\dot{\varepsilon}_{3\alpha} - \frac{1}{2G}\dot{\sigma}_{3\alpha}\Big)(\tau_{3\alpha} - \sigma_{3\alpha}) \leqslant 0 \qquad \forall \tau \in B$$

Integration of the above inequality over $Y \setminus S$, and using Eqs (2.1) and (2.2) and the definition of set P (3.1) lead to the following inequality

$$\int\limits_{Y\setminus S} \Big(\frac{1}{2G}\dot{\sigma}_{3\alpha} - \frac{1}{2}\dot{w}_{,\alpha}\Big)(\tau_{3\alpha} - \sigma_{3\alpha})\; dy \geqslant \int\limits_{S} \dot{E}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha})\; dy \qquad \forall \tau \in P$$

The equilibrium equation (2.3) implies for periodic fields: σ, v, $\dot{w}$

$$\begin{aligned}\int\limits_{Y\setminus S} \sigma_{3\alpha,\alpha}(v - \dot{w})\; dx = \int\limits_{S} \sigma_T([\![v]\!] - [\![\dot{w}]\!])\; ds\; + \\ - \int\limits_{Y\setminus S} \sigma_{3\alpha}(v_{,\alpha} - \dot{w}_{,\alpha})\; dy = 0\end{aligned} \tag{3.2}$$

Because the following inequality is true, see Duvaut and Lions (1976)

$$\sigma_T([\![v]\!] - [\![\dot{w}]\!]) \geqslant g(|[\![v]\!]| - |[\![\dot{w}]\!]|) \qquad \text{on } S$$

Eq (3.2) leads to the inequality

$$\int\limits_{Y\setminus S} \sigma_{3\alpha}(v_{,\alpha} - \dot{w}_{,\alpha})\; dy \geqslant \int\limits_{S} g(|[\![v]\!]| - |[\![\dot{w}]\!]|)\; ds \qquad \forall v \in V$$

Therefore, the considered problem can be written in the variational form as follows:

- Find $(\sigma, \dot{w}) : \ I \to P \times W$ such that a.e. on I

$$
\begin{aligned}
& b(\dot{\sigma}, \tau - \sigma) - (e(\dot{w}), \tau - \sigma) \geqslant (\dot{E}, \tau - \sigma) && \forall \tau \in P \\
& (\sigma, e(v - \dot{w})) \geqslant j(v) - j(\dot{w}) && \forall v \in V \\
& \sigma(0) = 0
\end{aligned}
\tag{3.3}
$$

where the following notation is used

$$
b(\chi, \tau) = \int\limits_Y \frac{1}{2G} \chi_{3\alpha} \tau_{3\alpha} \ dx \qquad\qquad (\chi, \tau) = \int\limits_Y \chi_{3\alpha} \tau_{3\alpha} \ dx
$$

$$
\Big[e(v)\Big]_{3\alpha} = \frac{1}{2} v_{,\alpha}
$$

The variational formulation (3.3) was given separately for the plasticity and friction problems e.g. by Duvaut and Lions (1976), Glowinski et al. (1976), and Glowinski (1984).

3.2. Finite element approximation

The solution to the problem is found, using an incremental procedure, for finite number of time instants $t_1, \ldots, t_n, \ldots, t_N = T$ $(0 < t_1 < \ldots < t_n < \ldots < t_N)$. The time derivative is approximated by the difference quotient

$$
\dot{\chi}(t_n) \cong \frac{\chi^n - \chi^{n-1}}{t_n - t_{n-1}} = \frac{\Delta \chi^n}{\Delta t_n}
$$

where $\chi^n \equiv \chi(t_n)$.

Let $\mathcal{T}_h$ be a triangulation of Y, $\overline{Y} = \bigcup \overline{K}_i$, $K_i \cap K_j = \emptyset$ for $i \neq j$, where K_i is the triangular subdomain of Y. Let $P_0(K)$ and $P_1(K)$ be the spaces of constant and linear functions, respectively. We define the spaces

$$
\begin{aligned}
H_h &= \Big\{ \tau \in [L^2(Y)]^2 : \quad \tau\Big|_{K_i} \in [P_0(K_i)]^2, \quad K_i \in \mathcal{T}_h \Big\} \\
V_h &= \Big\{ v \in V : \quad v\Big|_{K_i} \in P_1(K_i), \quad K_i \in \mathcal{T}_h \Big\}
\end{aligned}
\tag{3.4}
$$

We can write the discrete problem as follows:

- For $n = 1, 2, \ldots, N$, find $(\sigma_h^n, w_h^n) \in P_h \times V_h$ such that

$$
\sigma_h^n = \sigma_h^{n-1} + \Delta \sigma_h^n \qquad\qquad w_h^n = w_h^{n-1} + \Delta w_h^n
$$

and

$$\begin{aligned}
&b(\Delta\sigma_h^n, \tau - \sigma_h^n) - (e(\Delta w_h^n), \tau - \sigma_h^n) \geqslant (\Delta E^n, \tau - \sigma_h^n) && \forall \tau \in P_h \\
&(\sigma_h^n, e(v - \Delta w_h^n)) \geqslant j(v) - j(\Delta w_h^n) && \forall v \in V_h \\
&\sigma_h^0 = 0
\end{aligned}$$

where $P_h = P \cap H_h$.

Using outcomes of convex analysis (Ekeland and Temam, 1976). we can notice that the increment of displacement $\Delta w_h^n \in V_h$ satisfy the inequality $\forall v \in V_h$

$$\Big(P(\sigma_h^{n-1} + 2G(e(\Delta w_h^n) + \Delta E^n)), e(v - \Delta w_h^n)\Big) + j(v) - j(\Delta w_h^n) \geqslant 0 \tag{3.5}$$

where P is the operator of projection such that

$$\inf_{\tau \in P_h} \|\tau - \sigma\|^2 = \|P(\sigma) - \sigma\|^2 \tag{3.6}$$

$\|\tau\| = \sqrt{b(\tau,\tau)}$. In the considered problem, the operator has a very simple form

$$[P(\sigma)]_\alpha = \frac{\tau_0}{|\sigma|}\sigma_{3\alpha}$$

where $|\sigma| = \sqrt{\sigma_{3\alpha}\sigma_{3\alpha}}$.

Inequality (3.5) is derived in a more detailed way by Więckowski (1995b) for the torsion problem, see also Duvaut and Lions (1976), Johnson (1981).

The stresses a the instant t_n are calculated from the expression, see Johnson (1977), (1981), Samuelsson and Fröier (1979)

$$\sigma_h^n = P\Big(\sigma_h^{n-1} + 2G(e(\Delta w_h^n) + \Delta E^n)\Big)$$

3.3. Iterative solution

Let Λ_h be the space

$$\Lambda_h = \Big\{\lambda \in L^2(S): \quad \lambda\Big|_{\overline{K}_i \cap S} \in P_0(\overline{K}_i \cap S), \quad K_i \in \mathcal{T}_h\Big\}$$

The following procedure is utilized for each instant $t_n \in I$:

(i) initialization of $\Delta w_h^n \in V_h$ and $\lambda \in \Lambda_h$:

$$\Delta w_h^{n(0)} = 0 \qquad\qquad \lambda^{(0)} = 0$$

(ii) calculation of the successive estimation of $\Delta w_h^{n(i)}$ and $\lambda^{(i)}$ from the equations

$$\begin{aligned}
&a(\Delta w_h^{n(i)} - \Delta w_h^{n(i-1)}, v - \Delta w_h^{n(i)}) = \\
&= -\Big(P(\sigma_h^{n-1} + 2G(e(\Delta w_h^{n(i-1)}) + \Delta E^n)), e(v - \Delta w_h^{n(i)})\Big) + \\
&-\frac{1}{2}\int\limits_S \lambda^{(i-1)} g([\![v]\!] - [\![\Delta w_h^{n(i)}]\!])\, ds \qquad \forall v \in V_h \\
&\lambda^{(i)} = \max\Big[-1, \min(1, \lambda^{(i-1)} + \omega g \Delta w_h^{n(i-1)})\Big]
\end{aligned} \tag{3.7}$$

until the required accuracy is achieved, where

$$a(u,v) = \int\limits_{Y\setminus S} \frac{1}{2} G u_{,\alpha} v_{,\alpha}\, dy \qquad \omega > 0$$

4. Finite element solution in stresses

4.1. Variational formulation

Let X be the space of statically admissible fields of stresses

$$X = \Big\{\tau \in [L^2(Y)]^2 : \quad \tau_{3\alpha,\alpha} = 0 \text{ on } Y, \quad \tau_{3\alpha} n_\alpha\Big|_{\partial Y} \ Y\text{-anti-periodic}\Big\}$$

and let C be the following set

$$C = \Big\{\tau \in [L^2(Y)]^2 : \quad |\tau_T| \leqslant g \quad \text{a.e. on } S\Big\}$$

The constitutive relations (2.4) imply the equation

$$\dot{\sigma}_{3\alpha} - 2G(\dot{\varepsilon}_{3\alpha} - \dot{\varepsilon}^{\mathrm{p}}_{3\alpha}) = 0$$

Multiplication of both sides of the above relation by the expression $(\tau_{3\alpha} - \sigma_{3\alpha})$, integration over the domain $Y \setminus S$, and the use of the Green formula lead to the following equation

$$\begin{aligned}
&\int\limits_{Y\setminus S} \frac{1}{2G}\dot{\sigma}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha})\, dy - \int\limits_{Y\setminus S} \dot{E}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha})\, dy + \\
&+\frac{1}{2}\int\limits_{\partial Y} \dot{w}(\tau_T - \sigma_T) ds = \frac{1}{2}\int\limits_S [\![\dot{w}]\!](\tau_T - \sigma_T) ds - \int\limits_{Y\setminus S} \dot{\varepsilon}^{\mathrm{p}}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha}) dy
\end{aligned}$$

The friction law (2.5) implies the following inequality, fulfilled on S (see e.g. Duvaut and Lions, 1976)

$$[\![\dot{w}]\!](\tau_T - \sigma_T) \geqslant 0 \qquad \forall \tau \in C \tag{4.1}$$

From the conditions of periodicity for displacement and stress fields it follows that

$$\int\limits_{\partial Y} \dot{w}(\tau_{3\alpha} - \sigma_{3\alpha}) n_\alpha \, ds = 0 \qquad \forall \dot{w} \in W \qquad \forall \tau, \sigma \in X \tag{4.2}$$

Eqs (4.1) and (4.2), and the inequality in Eq (2.4) lead to the inequality

$$\int\limits_{Y \setminus S} \frac{1}{2G} \dot{\sigma}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha}) \, dy \geqslant \int\limits_{Y \setminus S} \dot{E}_{3\alpha}(\tau_{3\alpha} - \sigma_{3\alpha}) \, dy$$

which is true for each τ and σ belonging to the set

$$K = X \cap P \cap C$$

Thus the problem can be formulated in stresses as follows:

- Find $\sigma \in K$ such that a.e. on $I \equiv [0, T]$ $(T > 0)$

$$\begin{aligned} & b(\dot{\sigma}, \tau - \sigma) \geqslant (\dot{E}, \tau - \sigma) \qquad \forall \tau \in K \\ & \sigma(0) = 0 \end{aligned} \tag{4.3}$$

4.2. Finite element approximation

The same type of time discretisation as that used for the displacement formulation, is considered for the dual one (4.3). The statically admissible fields of stresses are constructed using the Prandtl stress function, ψ, i.e.

$$\tau_{3\alpha} = e_{\alpha\beta} \psi_{,\beta}$$

where the function ψ is approximated using the triangular element with linear shape functions, which means that ψ is an element of the following space

$$Z_h = \Big\{ \psi \in H^1(Y) : \quad \psi\Big|_{K_i} = P_1(K_i), \quad K_i \in \mathcal{T}_h \Big\}$$

It can be proved that fields of stresses belong to the space

$$X_h = X \cap H_h$$

where the space H_h is defined in Eq $(3.4)_1$.

Stress fields generated by the function $\psi \in Z_h$, satisfy the equilibrium equation (2.3) only inside the cell Y. The stress field, to be statically admissible, should also satisfy the periodicity condition implied by the definition of the space X. This condition is fulfilled by periodic location of nodes along the cell boundary ∂Y, and the use of the method of Lagrange multipliers (see Więckowski (1995c) for details).

The discrete problem has the following form:

- For $n = 1, 2, \ldots, N$, find $\sigma_h^n \in K_h$ such that

$$\begin{aligned} &\sigma_h^n = \sigma_h^{n-1} + \Delta\sigma_h^n \\ &b(\Delta\sigma_h^n, \tau - \sigma_h^n) \geqslant (\Delta E^n, \tau - \sigma_h^n) \qquad \forall \tau \in K_h \\ &\sigma_h^0 = 0 \end{aligned}$$

  where

$$K_h = X_h \cap P \cap C$$

4.3. Iterative solution

Again, an adaptation of the iterative algorithm utilized to the problem of torsion of elastic-plastic composite bar with frictional interfaces is applied (see Więckowski, 1995b).

Let us define the following spaces

$$\begin{aligned} M_h &= \Big\{\mu \in L^2(Y): \quad \mu\Big|_{K_i} \in P_0(K_i), \quad K_i \in \mathcal{T}_h\Big\} \\ \Lambda_h &= \Big\{\lambda \in L^2(Y): \quad \lambda\Big|_{\overline{K_i}\cap S} \in P_0(\overline{K_i} \cap S), \quad K_i \in \mathcal{T}_h\Big\} \end{aligned}$$

For each time instant t_n, the following iterative procedure is utilized:

(i) initialization of $\mu \in M_h$ and $\lambda \in \Lambda_h$:

$$\mu^{(0)} = 0 \qquad\qquad \lambda^{(0)} = 0$$

(ii) for each iteration, calculation of the successive estimation of $\Delta\sigma_h^{n(i)}$, $\mu^{(i)}$ and $\lambda^{(i)}$ from the following equations

$$\begin{aligned} &b(\Delta\sigma_h^{n(i)}, \tau - \sigma_h^{n(i)}) = (\Delta E^n, \tau - \sigma_h^{n(i)}) + \\ &-b(\mu^{(i-1)}\sigma_h^{n(i-1)}, \tau - \sigma_h^{n(i)}) - \int\limits_S \lambda^{(i-1)}(\tau_{\rm T} - \sigma_{\rm Th}^{n(i)})\, ds \qquad \forall \tau \in X_h \end{aligned}$$

$$\mu^{(i)} = \max\Big[\mu^{(i-1)} + \Big(1 - \frac{\tau_0}{|\sigma_h^{n(i)}|}\Big), 0\Big]$$

$$\lambda^{(i)} = \begin{cases} 0 & \text{for} \quad \sigma_{\mathrm{T}h}^{n(i)} = 0 \\ \max\Big[\max(\lambda^{(i-1)}, 0) + \varDelta, 0\Big] & \text{for} \quad \sigma_{\mathrm{T}h}^{n(i)} > 0 \\ \min\Big[\min(\lambda^{(i-1)}, 0) - \varDelta, 0\Big] & \text{for} \quad \sigma_{\mathrm{T}h}^{n(i)} < 0 \end{cases}$$

until the required accuracy is achieved, where

$$\varDelta = \omega\Big(1 - \frac{g}{|\sigma_{\mathrm{T}h}^{n(i)}|}\Big) \qquad \omega > 0 \tag{4.4}$$

The multiplier λ can be interpreted as the slip increment defined on the corresponding segment of S.

5. Effective constitutive relations

It was shown by Suquet (1985) that mechanical behaviour of the considered composite body can be described by constitutive model of a generalised standard material, where the total deformation of the body can be split into the elastic and inelastic parts. In the case of anti-plane shear the composite material can be treated as a linearly elastic-plastic body with anisotropic hardening. Let us assume that the directions x_1 and x_2 are parallel to the edges of rectangular repeatable cell of the composite material, which means that there exists the material symmetry with respect to x_3x_1- and x_3x_2-planes. Let B^{eff} denote a set of plastically admissible stresses for a homogenised material equivalent in macro scale to the considered composite

$$B^{\mathrm{eff}} = \Big\{\widetilde{\tau}_{3\alpha} : \quad \Big(\frac{\widetilde{\tau}_{31} - \alpha_{31}}{\tau_{031}}\Big)^2 + \Big(\frac{\widetilde{\tau}_{32} - \alpha_{32}}{\tau_{032}}\Big)^2 \leqslant 1\Big\} \tag{5.1}$$

The effective constitutive relations can be written as follows

$$\dot{E}_{3\alpha} = \dot{E}_{3\alpha}^{\mathrm{e}} + \dot{E}_{3\alpha}^{\mathrm{p}}$$

where the elastic and plastic parts of the strain rate satisfy the following relations

$$\dot{E}_{3\alpha}^{\mathrm{e}} = \frac{1}{2G_{3\alpha}^{\mathrm{eff}}}\dot{\widetilde{\sigma}}_{3\alpha} \qquad \text{(no summation over } \alpha)$$

$$\dot{E}_{3\alpha}^{\mathrm{p}}(\widetilde{\tau}_{3\alpha} - \widetilde{\sigma}_{3\alpha}) \leqslant 0 \qquad \forall \widetilde{\tau}_{3\alpha} \in B^{\mathrm{eff}}$$

In Eq (5.1), $\alpha_{3\alpha}$ denotes the kinematic shift which depends on the plastic strain $E^{\rm p}_{3\alpha}$, it means

$$\alpha_{3\alpha} \equiv \alpha_{3\alpha}(E^{\rm p}_{3\alpha}) \qquad \text{(no summation over } \alpha) \tag{5.2}$$

The kinematic shift function $\alpha_{3\alpha}$ can be found by solving the cell problem formulated in Eqs (3.3) or Eqs (4.3). The effective shear modulus $G^{\rm eff}_{\alpha}$ for the x_3x_α-plane can also be found from the solution to the same problems but for sufficiently small strain $E_{3\alpha}$. The effective plastic limit $\widetilde{\tau}_{03\alpha}$ is equal to the value of mean stress, which corresponds to the initiation of plastic zones or slip on fibre-matrix interface, caused by shearing in the x_3x_α-plane.

Eq (5.2), obtained from numerical analysis as a set of pairs $(E^{\rm p}_{3\alpha}, \alpha_{3\alpha})$, can be used directly for defining the hardening function or can be represented at its curvilinear part – with satisfactory accuracy from the engineering point of view – by the following function

$$\overline{\alpha}_{3\alpha} = a_\alpha\sqrt{(b_\alpha E^{\rm p}_{3\alpha} + 1)^2 - 1} \qquad \alpha = 1,2 \qquad \text{(no summation over } \alpha) \tag{5.3}$$

where a_α and b_α are constants of values such chosen that (5.3) is well-fitted to (5.2).

6. Numerical results

Let us consider a composite body with a square basic cell, $l_1 = l_2 = l$ (see Fig.3), for which the ratio of fibre diameter to the cell side length is $R/l = 0.75$.

Calculations have been done for the following values of elastic constants: $E = 3.5\,\text{GPa}$, $\nu = 0.35$ for the matrix (epoxy resin), and $E = 209\,\text{GPa}$, $\nu = 0.3$ for the fibre (steel); and the plastic limits for shear stresses, τ_0: 173.2 and 11.55 MPa for matrix and fibre, respectively. Calculations have been done for two values of limit for the shear tractions on the fibre-matrix interface, g, 8 and 10 MPa. The cases when both the fibre and matrix are assumed to be perfectly elastic, and when there is no slip on the fibre-matrix interface are also considered. All the five variants of data analysed are shown in Table 1. The load has been applied as follows: $E_{31} = 0.005t$, $E_{32} = 0$, $t \in [0,1]$. The load increment has been set as $5 \cdot 10^{-5}$.

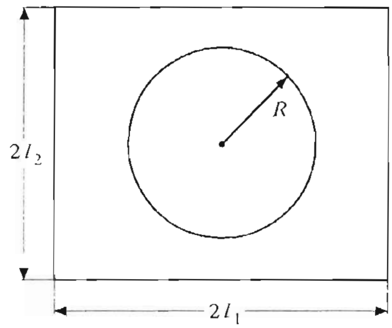

Fig. 3. Dimensions of basic cell

Table 1

variant	σ_0 [MPa] fibre	σ_0 [MPa] matrix	g [MPa]	
1	300	20	∞	(no slip)
2	300	20	8	
3	∞	∞	8	(elastic case)
4	300	20	10	
5	∞	∞	10	(elastic case)

The following convergence criteria have been used to stop the iterative procedures

$$\frac{\|\boldsymbol{q}^{(i)} - \boldsymbol{q}^{(i-1)}\|}{\|\boldsymbol{q}^{(i)}\|} \leqslant \varepsilon = 1 \cdot 10^{-4} \qquad \text{for both the solutions}$$

$$\max\Big(0, \frac{|\sigma_{\rm T}^{(i)}| - g}{g}\Big) \leqslant \varepsilon = 1 \cdot 10^{-4} \qquad \text{additionally for the statically admissible solution}$$

where $\boldsymbol{q}$ denotes the vector of degrees of freedom. Parameters ω involved in Eqs (3.7) and (4.4) have been set as $1 \cdot 10^{-4}$ and $5 \cdot 10^{-5}$, in the displacement and equilibrium models of FEM, respectively.

For a square repeatable cell with a circular fibre, the composite material has the same properties with respect to both the directions x_1 and x_2, i.e. $G_{31}^{\rm eff} = G_{32}^{\rm eff} = G^{\rm eff}$, $\tau_{031} = \tau_{032} = \tau_0$.

The effective shear modulus and plastic shear limit, obtained from both the models of FEM are shown in Table 2.

Table 2. Effective properties of composite

	unit		displacement model	equilibrium model	mean value
G^{eff}	GPa		3.2756	3.2605	3.2680
τ_0	MPa	$g = \infty$	7.0600	6.9075	6.9837
		$g = 8\,\mathrm{MPa}$	4.7986	4.8049	4.8017
		$g = 10\,\mathrm{MPa}$	5.9959	6.0061	6.0010

The relation between $\widetilde{\sigma}_{31}$ and E_{31} obtained by the use of the displacement approach to FEM is shown in Fig.4. The figure shows also the relative difference between the results obtained from both the finite elements models used. The maximum difference between both the results does not exceed 1.3%.

The hardening functions (5.2) are represented in Fig.5 – they have been obtained as mean values taken from both the finite element solutions: kinematically and statically admissible.

Calculations confirm that the shift function $\alpha_{3\alpha}$ can be represented in the form (5.3). Two parameters a and b (in the case of square cell $a_1 = a_2 = a$, $b_1 = b_2 = b$) involved in Eq (5.3) have been found as a result of minimization of the following expression

$$\sum_i \Big(a\sqrt{(bE^{\mathrm{p}}_{3\alpha i} + 1)^2 - 1} - \alpha_{3\alpha i}\Big)^2$$

where $E^{\mathrm{p}}_{3\alpha i}$ denotes the plastic strain obtained for the ith load increment, and $\alpha_{3\alpha i} = \widetilde{\sigma}_{3\alpha} - \widetilde{\tau}_0$. The values of both $E^{\mathrm{p}}_{3\alpha i}$ and $\alpha_{3\alpha i}$ are considered as mean values obtained from both the finite element solutions: kinematically and statically admissible. A gradient minimization method (see Bazaraa and Shetty, 1979; Kręglewski at al., 1984) has been employed to find parameters a and b. For example, in the case of variant No. 3, the following values have been obtained

$$a = 22.572\mathrm{MPa} \qquad\qquad b = 12.758$$

and the relative difference between $\alpha_{3\alpha}$ and $\overline{\alpha}_{3\alpha}$ in the sense of the least square method is

$$\frac{\sqrt{\sum\Big[\overline{\alpha}(E^{\mathrm{p}}_{3\alpha i}) - \alpha_{3\alpha i}\Big]^2}}{\sqrt{\sum \alpha^2_{3\alpha i}}} = 2.02\%$$

which can be considered as a small value from the engineering point of view.

In Fig.6 ÷ Fig.8, the plastic and slip zones are shown. Diagrams (a) and (b), in these figures, represent the results obtained by the displacement- and

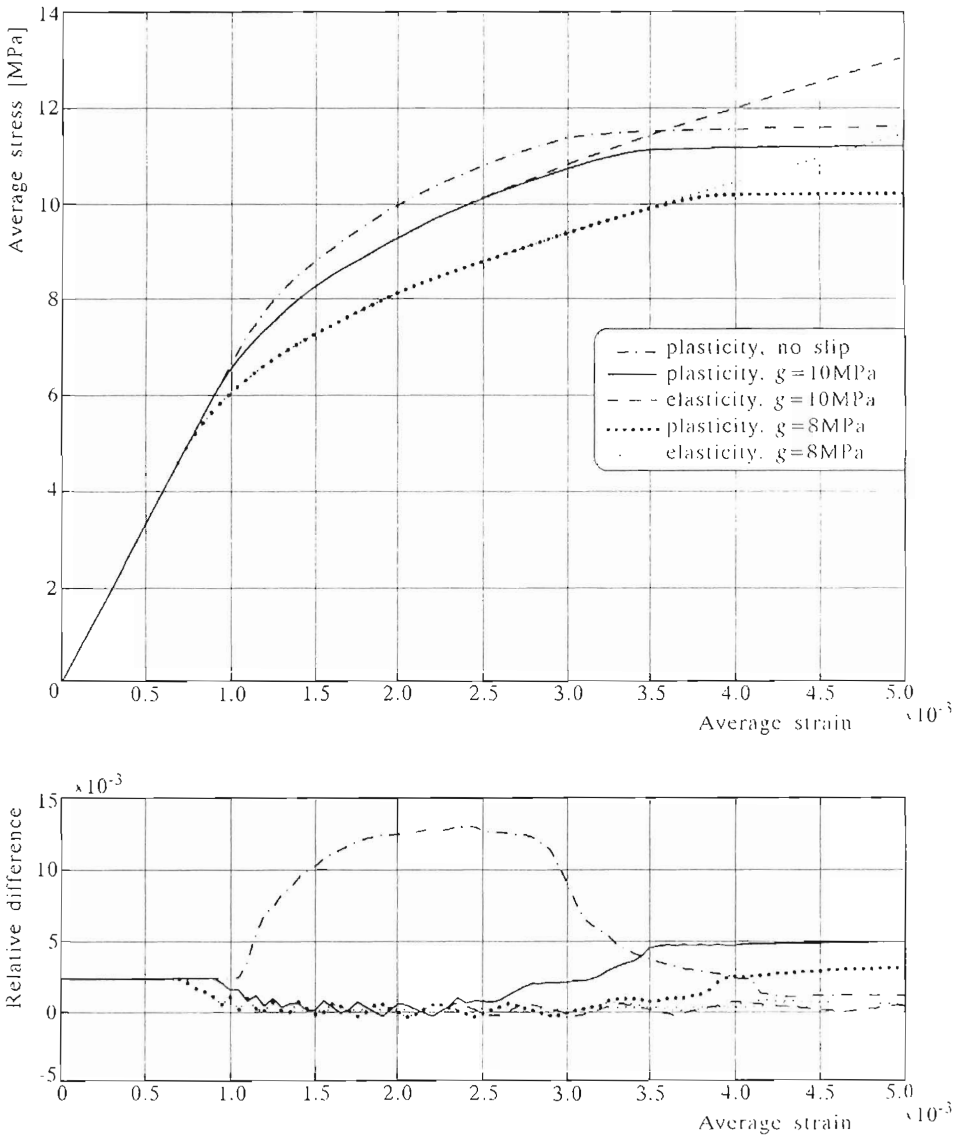

Fig. 4. $\widetilde{\sigma}_{31}$-E_{31} relations

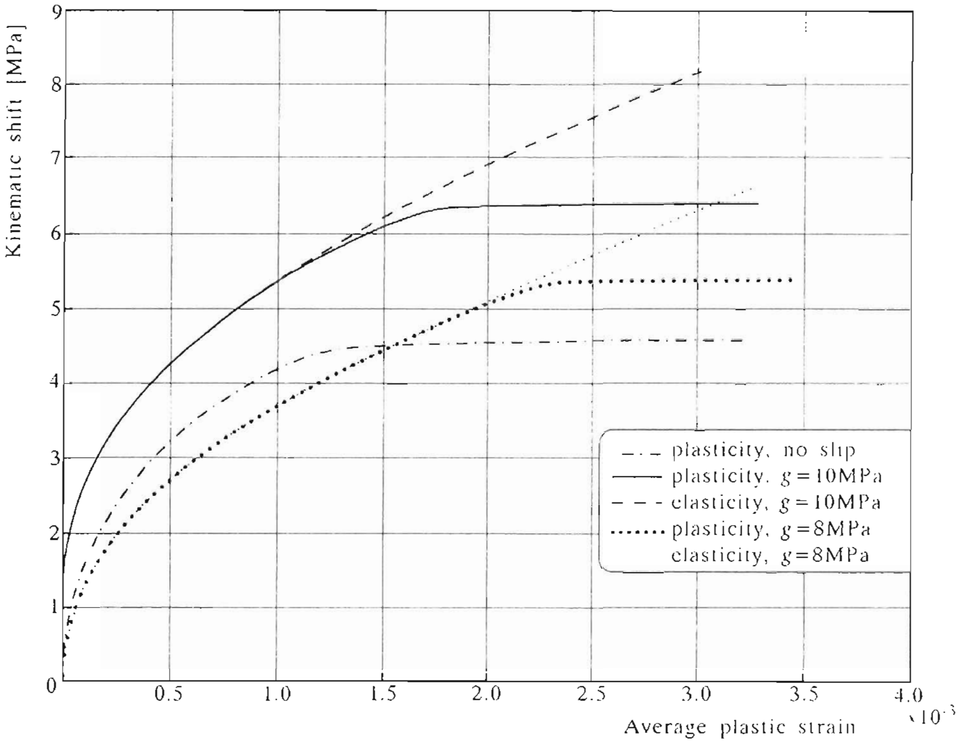

Fig. 5. α_{31}-E^{p}_{31} relations

stress-based methods, respectively. The plastic zones plotted for the variant No. 1 for the two values of strain E_{31}, which correspond to the beginning of non-linear response of the composite and its limit value, is given in Fig.6.

The next two figures present the plastic and slip zones obtained for variants No. 2 and 4, respectively. It can be noticed that the initiation of slip zones corresponds to the point on the $\widetilde{\sigma}$-E path at which the material starts to behave non-linearly. The initiation of plastic zones corresponds to the points at which the $\widetilde{\sigma}$-E paths (Fig.5) for variants $2 \div 4$ start to fork. The instant, at which plastic and slip zones cover all the height of the basic cell, corresponds to the horizontal part of the $\widetilde{\sigma}$-E path. The comparison of Fig.7 and Fig.8 reveals that the plastic zones start to appear heigher (for the right upper quarter of the cell) for a lower value of g.

Moreover, the non-monotonic strain evolution for the component E_{31} has been considered according to the upper diagram presented in Fig.9. The cor-

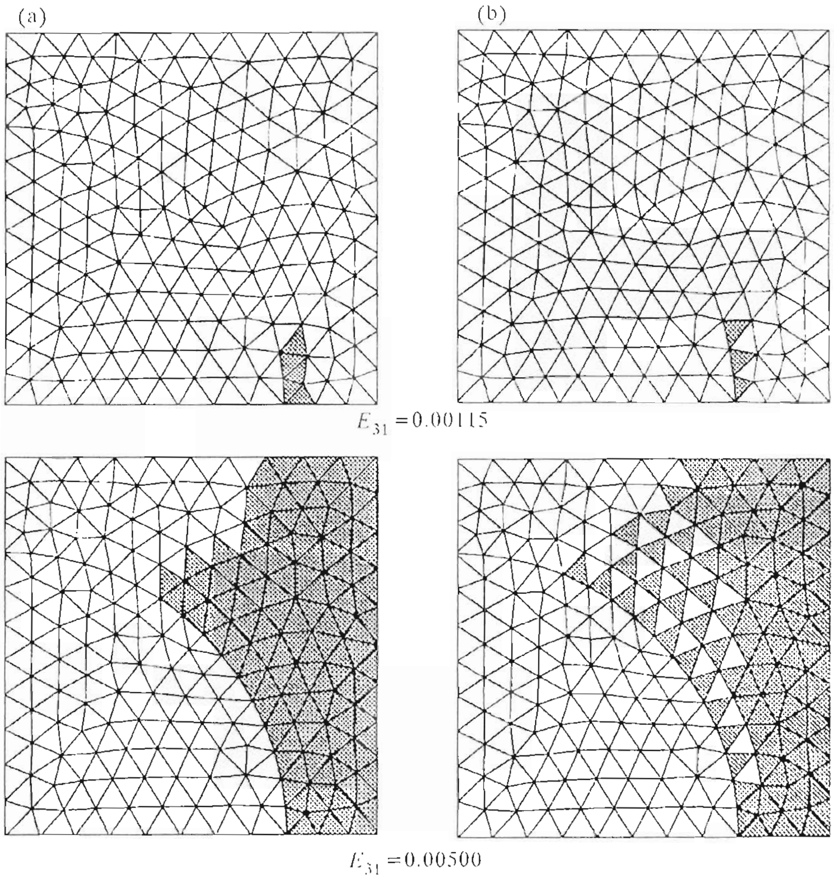

Fig. 6. Plastic zones, no interface slip

responding $\widetilde{\sigma}_{31}$-E_{31} paths are shown in the lower diagram in the same figure for variants of data No. 4 and 5 – the Bauschinger effect can be observed.

The results show a very good agreement between the two applied approaches although the number of elements used to discretisation of the quarter of the cell is relatively small – the region is divided into 336 triangles.

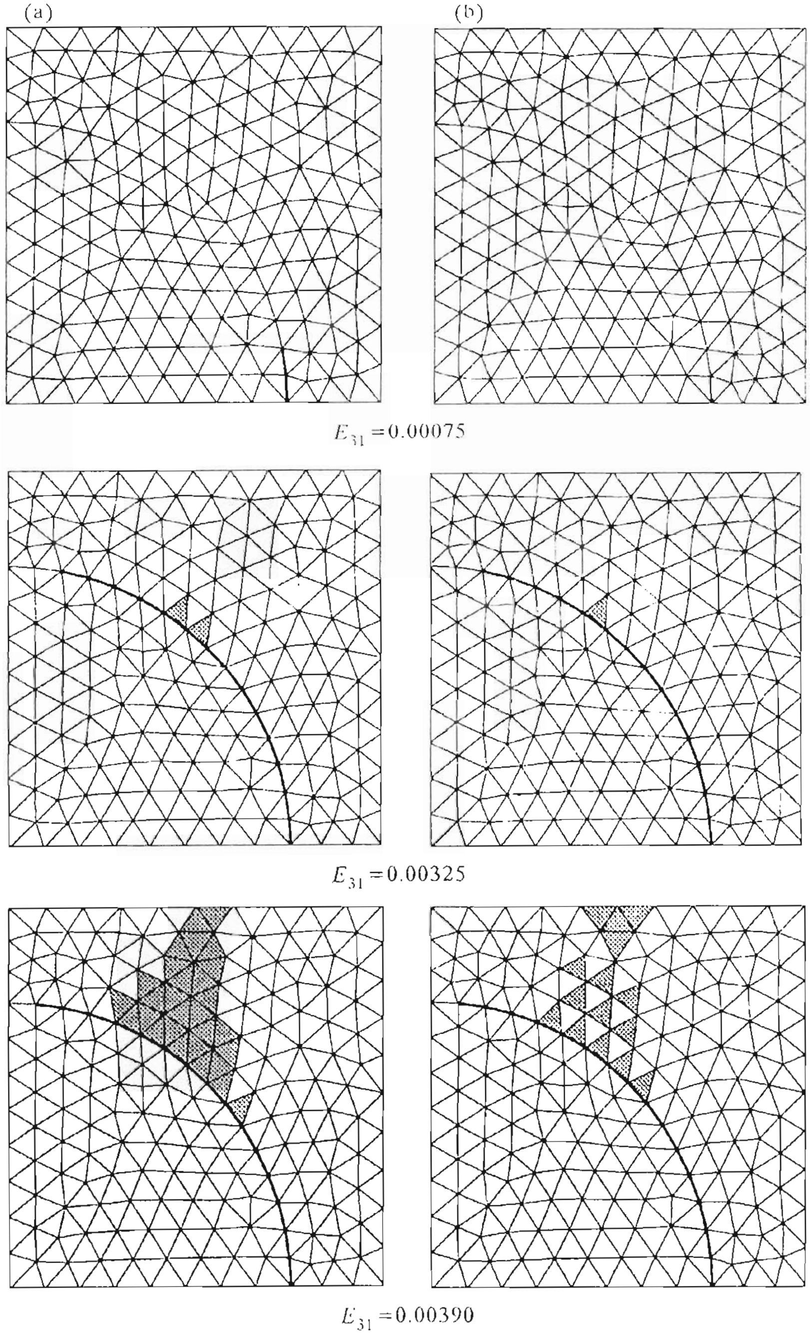

Fig. 7. Plastic and slip zones, $g = 8\,\mathrm{MPa}$

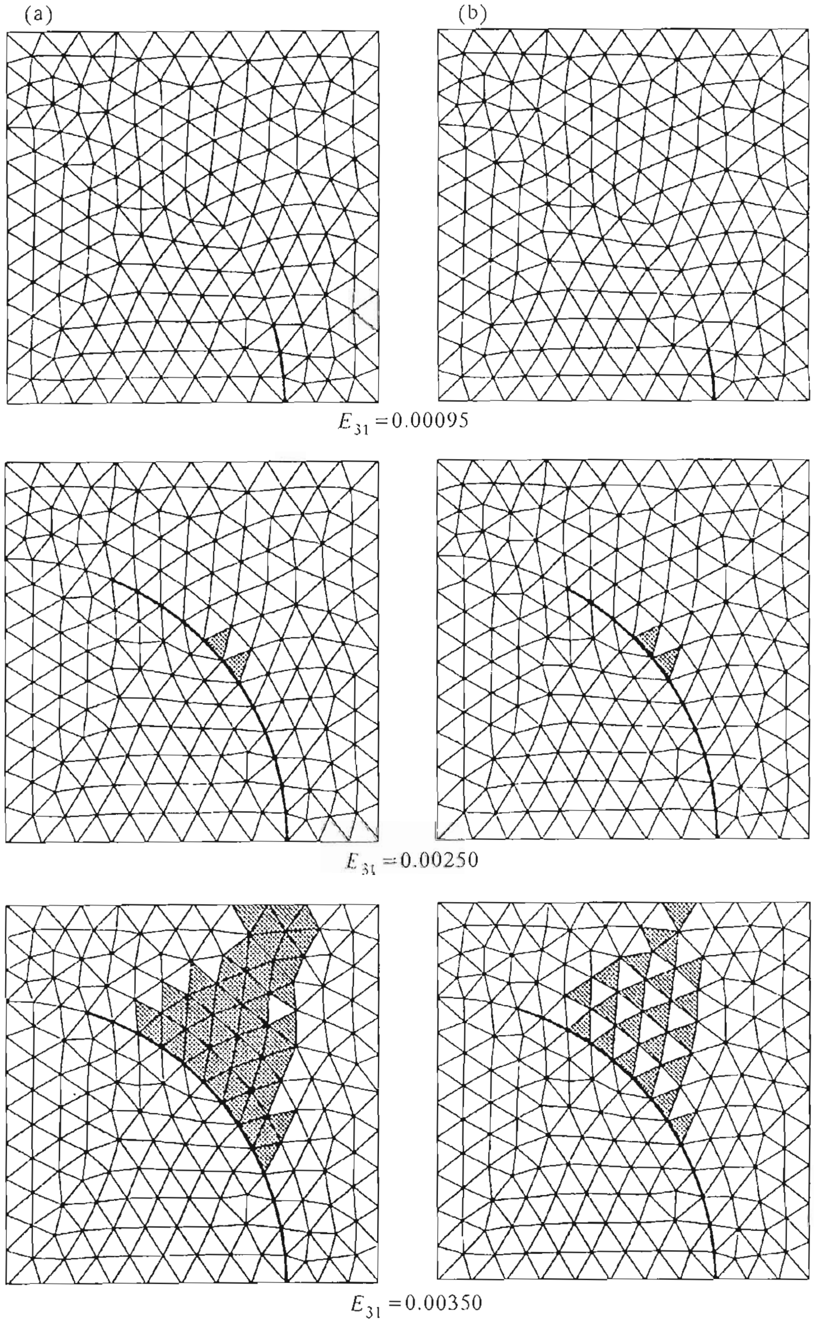

Fig. 8. Plastic and slip zones, $g = 10\,\mathrm{MPa}$

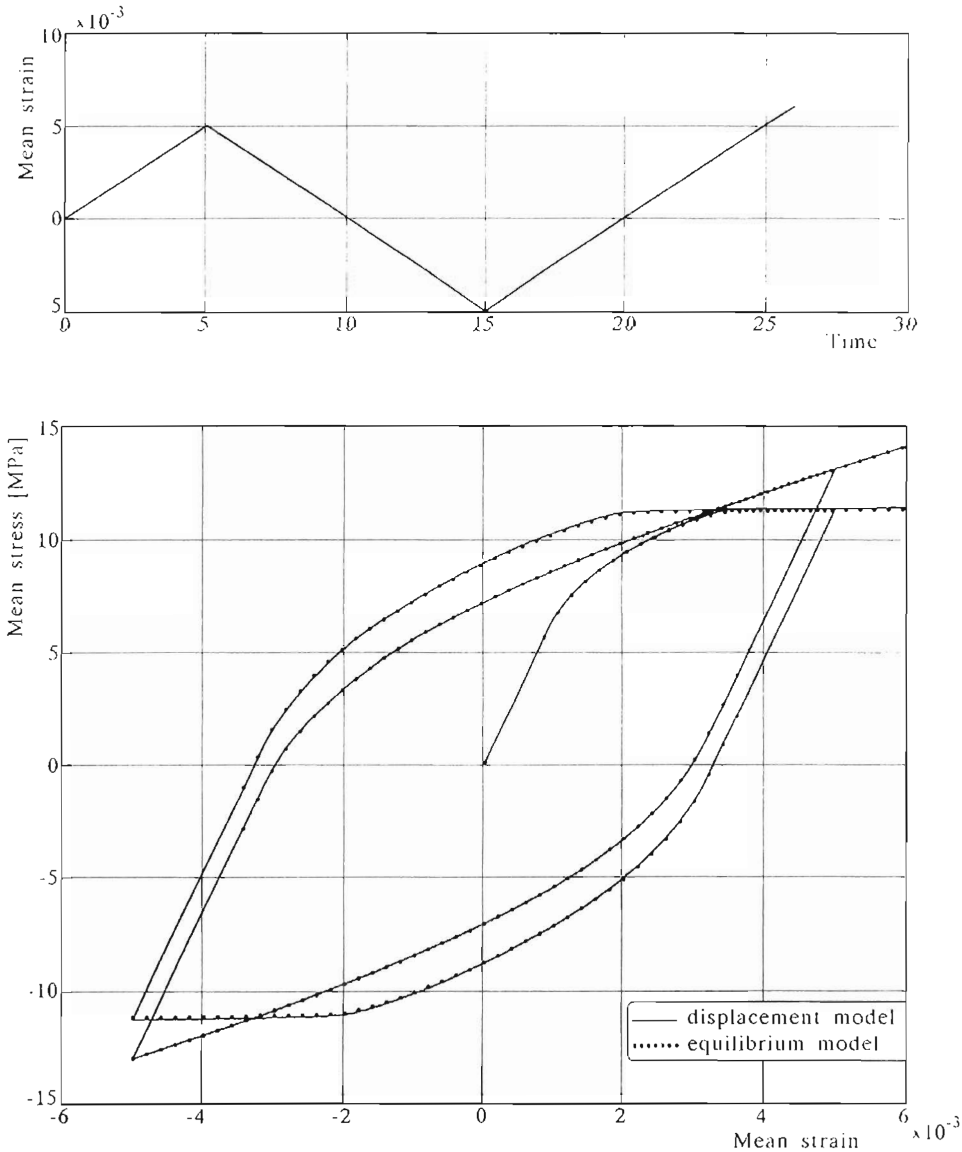

Fig. 9. $\widetilde{\sigma}_{31}$-E_{31} paths for non-monotonic deformation

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Podłużne ścinanie sprężysto-plastycznego kompozytu włóknistego z uwzględnieniem tarcia na powierzchniach włókien

Streszczenie

W pracy rozważono zagadnienie efektywnych własności periodycznego, sprężysto-plastycznego kompozytu włóknistego, poddanego stanowi anty-płaskiego ścinania, uwzględniając zjawisko tarcia na powierzchni łączącej włókno i matrycę. Zagadnienie zostało sformułowane w przemieszczeniach i naprężeniach w postaci nierówności wariacyjnych, które rozwiązano za pomocą metody elementów skończonych. W przypadku sformułowania naprężeniowego zastosowano funkcję naprężeń Prandtla. Otrzymane efektywne związki konstytutywne dla rozważanego kompozytu mogą być bezpośrednio wykorzystane w analizie zagadnienia swobodnego skręcania pręta kompozytowego o periodycznej strukturze włóknistej.

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