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FUNDAMENTAL MATRIX OF THE GENERALIZED THERMOELASTIC SYSTEM [1]

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The fundamental matrix of the generalised thermoelastic system is constructed with interaction between temperature and displacement fields as well as finite rate of heat propagation taken into account. Components of the considered matrix are represented in terms of Laplace transforms of the corresponding functions. These functions are represented as integrals over the segments which connect singular points of their Laplace transforms. The characteristic properties of these functions are investigated.

Key words: generalised thermoelasticity, fundamental solutions

1. Introduction

The present contribution contains continuation of our previous papers (1981, 1982, 1985) devoted to investigation of the generalised thermoelastic system of equations proposed by Kaliski (1965), Lord and Shulman (1967) and solved with taking into account the interaction between temperature and displacement fields and the finite rate of heat propagation.

In Mokryk and Pyryev (1981) the Laplace transform of the fundamental matrix of generalised thermoelastic system was constructed. Analytical properties of the matrix components depending on the Laplace transform complex parameter were investigated.

In Mokryk and Pyryev (1982) analytical properties of the characteristic parameters of generalised thermoelastic system were investigated.

In Mokryk and Pyryev (1985) the fundamental solution E_ε of the so called thermoelastic operator $\Lambda(\partial) = \partial_\xi^4 - \partial_\xi^2(a\partial_\tau^2 + b\partial_\tau) + c\partial_\tau^4 + d\partial_\tau^3$ was studied.

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The quantities E_ε, $\Lambda(\partial)$ were defined after Wagner (1994). The investigation of the solution E_s was conducted by Wagner (1994) but without taking into account the finite speed of heat propagation $(c = 0)$. The decomposition theorem for thermoelastic operator $\Lambda(\partial)$ was formulated by Ignaczak (1989).

When studying the transient problems of coupled thermoelastic two groups of methods are usually applied. The first of them is based on time elimination from the corresponding partial differential equations and the second one employs elimination of space coordinates.

The Laplace integral transform method is a member of the first one and allows for considering the problem as an elliptical boundary value one with a parameter. For solving of such a problem the well-known methods can be used; e.g., method of an integral transformation with respect to space coordinate (cf Sherief and Anwar, 1994), method of an expansion into a series (cf Sherief and Anwar, 1994), method of an integral equations (cf Kupradze et al., 1976). operator method. But the main obstacle arises, however, when performing the inverse Laplace transform.

The second group of methods is made up of: method of expansion into a series with respect to the corresponding complete orthogonal system of functions. method of finite integral transformation with respect to a space coordinate (cf Sabodash and Cheban, 1971), method of infinite integral transformation with respect to a space coordinate (cf Lenyuk and Kostenko, 1981). They allow one to consider the studied problem as the Cauchy problem for the corresponding transforms.

Due to substantial mathematical difficulties arising in both the above mentioned groups of methods also the asymptotical methods are applied. For example in Popov (1967) and Sherief and Ezzat (1994) the method of expansion of a solution into a series with respect to a small coupling parameter was used. In Kil'chynska (1971), Norwood and Warren (1969), Wadhawan (1972), Sherief and Dhaliwal (1981), Dhaliwal et al. (1995) the so called method of *short times* Nowacki (1975) is used, i.e. the first terms of near front asymptote were determined. In Popov (1967), Norwood and Warren (1969) the method of *long times* was used.

Kil'chynska (1971) and Lazarov (1980) constructed the *automodelling* solutions to generalised thermoelastic problems.

Recently, numerical methods have been widely applied to solving the dynamical thermoelastic problems; e.g., finite element method (cf Ting and Chen, 1982), method of characteristics (cf Naval and Sabodash, 1976), method of numerical inversion of the Laplace integral transform (cf Sherief and Anwar, 1994).

The uniqueness of general initial boundary-value problem formulation for the generalised linear dynamic thermoelastic was established by Ignaczak (1982) with the us of the associated conservation law involving higher-order time derivatives.

The fundamental matrix of the generalised thermoelastic system was studied by Gawinecki (1988), Gawinecki and Hung (1990). The rewiev of papers on the generalised thermoelastic models can be found, e.g., in Chandrasekhavaiah (1986), Podstryhach and Kolyano (1976), Ignaczak (1989).

2. Formulation of the problem

The generalised thermoelastic model proposed by Lord and Shulman (1967), in which the interaction between temperature and displacement fields and finite rate of heat propagation have been taken into account, is considered. This system can be represented in terms of the displacement vector components $\boldsymbol{u} = [u_1, u_2, u_3]$ and the temperature θ

$$\mathbf{J}(\partial_x, \partial_t)\mathbf{U}(\boldsymbol{x}, t) = -\boldsymbol{\Phi}(\boldsymbol{x}, t) \tag{2.1}$$

where $\mathbf{J}(\partial_x, \partial_t)$ – linear matrix differential operator

$$\mathbf{J}(\partial_x, \partial_t) = \Big[J_{nm}(\partial_x, \partial_t)\Big]_{4\times 4} \tag{2.2}$$

with the components

$$J_{nm}(\partial_x, \partial_t) = \left\{ \begin{array}{lll} \mu\delta_{nm}\Pi_2 + (\lambda+\mu)\partial^2_{x_n x_m} & n, m = 1,2,3 & \\ -\gamma\partial_{x_n} & n = 1,2,3 & m = 4 \\ -\eta\ell(\partial_t)\partial^2_{tx_m} & n = 4 & m = 1,2,3 \\ \Pi_3 & n = 4 & m = 4 \end{array} \right.$$

where

$$\Pi_j = \left\{ \begin{array}{ll} \Delta - c_j^{-2}\partial_t^2 & j = 1,2 \\ \Delta - \kappa^{-1}\ell(\partial_t)\partial_t & j = 3 \end{array} \right. \qquad \delta_{nm} = \left\{ \begin{array}{ll} 1 & n = m \\ 0 & n \neq m \end{array} \right.$$

$$\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} \qquad \ell(\partial_t) = 1 + t_r\partial_t$$

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad c_2 = \sqrt{\frac{\mu}{\rho}} \qquad c_q = \sqrt{\frac{\kappa}{t_r}}$$

and

$\mathsf{U}, \boldsymbol{\Phi}$	–	vector functions which are considered as one-column matrices, $\mathsf{U} = [u_1, u_2, u_c, \theta]^\top$, $\boldsymbol{\Phi} = [F_1, F_2, F_3, \ell(\partial_t)Q(\boldsymbol{x},t)/\lambda_\theta]^\top$
$F_n(\boldsymbol{x},t)$	–	components of the body force vector
$Q(\boldsymbol{x},t)$	–	density of an internal heat source
λ, μ	–	Lame coefficients
ρ	–	density
α	–	coefficient of linear thermal expansion
η	–	coupling coefficient (cf Podstryhach et al., 1976), $\eta = \gamma\theta_0/\lambda_\theta$, $\gamma = \alpha(3\lambda + 2\mu)$
θ_0	–	temperature of the body in the undeformed state
λ_θ	–	heat conductivity
κ	–	thermal diffusivity
t_r	–	heat flow relaxation time.

3. Construction of the matrix of fundamental solution

Using the Fourier integral transform with respect to the space coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ and the Laplace transform with respect to time, the solution of Eqs (2.1) can be written in the form (Mokryk and Pyryev, 1981)

$$\mathsf{U}(\boldsymbol{x},t) = \mathsf{L}(\boldsymbol{x},t) * \boldsymbol{\Phi}(\boldsymbol{x},t) \tag{3.1}$$

where $\mathsf{L}(\boldsymbol{x},t) = [L_{nm}(\boldsymbol{x},t)]_{4\times4}$ – matrix of fundamental solution (the delay Green function of free space) of Eqs (2.1), i.e. a square matrix satisfying the equation

$$\mathsf{J}(\partial_x, \partial_t)\mathsf{L}(\boldsymbol{x},t) = -\delta(\boldsymbol{x})\delta(t)\mathsf{I} \tag{3.2}$$

where I is the 4×4 unit matrix. The asterisk $*$ corresponds to the convolution of the corresponding functions with respect to all variables, $\delta(\cdot)$ stands for the Dirac delta function

$$L_{nm}(\boldsymbol{x},t) = \frac{1}{4\pi\mu}\Big\{\frac{\delta_{nm}}{|\boldsymbol{x}|}\delta\Big(t - \frac{|\boldsymbol{x}|}{c_2}\Big) +$$
$$-c_2^2\partial^2_{x_nx_m}\Big[\frac{1}{|\boldsymbol{x}|}\Big(H_2\Big(t - \frac{|\boldsymbol{x}|}{c_2}\Big) + \mathcal{R}^- G_1(|\boldsymbol{x}|,t,1)\Big)\Big]\Big\}$$
$$L_{n4}(\boldsymbol{x},t) = \frac{\gamma}{4\pi\lambda_\theta(\lambda + 2\mu)}\partial_{x_n}\Big[\frac{1}{|\boldsymbol{x}|}G_1(|\boldsymbol{x}|,t,1)\Big] \qquad n,m = 1,2,3$$

$$
\begin{aligned}
&L_{4m}(\boldsymbol{x},t) = \frac{\eta}{4\pi\mu}\frac{c_2^2}{c_1^2}\partial_{x_m}\left\{\frac{1}{|\boldsymbol{x}|}\Big[G_1(|\boldsymbol{x}|,t,0)+t_r G_1(|\boldsymbol{x}|,t,1)\Big]\right\}\\
&L_{44}(\boldsymbol{x},t) = \frac{1}{4\pi\lambda_\theta|\boldsymbol{x}|}\mathcal{R}^+G_1(|\boldsymbol{x}|,t,-1) \qquad\qquad M=\frac{c_1}{c_q}\\
&\mathcal{R}^\pm G_j(|\boldsymbol{x}|,t,k) = \frac{a_2}{c_1^2}G_j(|\boldsymbol{x}|,t,k)+\frac{a_4}{\kappa}G_j(|\boldsymbol{x}|,t,k+1)\pm\\
&\pm\frac{1}{2}G_{j+1}(|\boldsymbol{x}|,t,k) \qquad j=1 \qquad k=\pm 1\\
&a_2 = M^2(1+\varepsilon)-1 \qquad a_4=\frac{1+\varepsilon}{2} \qquad \varepsilon=\frac{\gamma\eta\kappa}{\lambda+2\mu}\\
&H_0(t)=\delta(t) \qquad H_1(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}\\
&H_\alpha(t)=\begin{cases}H_1(t)t^{\alpha-1}/\Gamma(\alpha) & \alpha>0\\ d^N H_{\alpha+N}(t)/dt^N & \alpha\neq 0 \quad \alpha+N>0 \quad N-\text{integer}\end{cases}
\end{aligned}
\tag{3.3}
$$

where the mth column ($m = 1,2,3$) of the fundamental solution matrix represent the displacement and temperature field due to the body force of the components $\delta(\boldsymbol{x})\delta(t)\delta_{nm}$ applied at the point $\boldsymbol{x} = \mathbf{0}$ in the x_m axis direction; the 4th column of the fundamental solution matrix represents the displacement and temperature field in infinite space due to the heat source $\ell(\partial_t)Q(\boldsymbol{x},t)$ in the form $\delta(\boldsymbol{x})\delta(t)$.

4. Functions $G_j(\xi,\tau,n)$

Components of the fundamental solution matrix are represented in terms of the standard functions $G_j(|\boldsymbol{x}|,t,n)$, $j = 1,2,...,14$. Consider the functions $G_j(\xi,\tau,n)$ of dimensionless distance ξ and dimensionless time τ using the following relations

$$
\begin{aligned}
&G_j(\xi,\tau,n) = (\omega_*)^n G_j(|\boldsymbol{x}|,t,n)v_j \qquad \xi=|\boldsymbol{x}|k \qquad \tau=t\omega_*\\
&v_1=v_{12}=\frac{1}{v_{10}}=\left(\frac{k_*}{\omega_*}\right)^2 \qquad v_2=v_9=1 \qquad v_7=v_{13}=\left(\frac{k_*}{\omega_*}\right)^3\\
&v_3=v_5=v_8=v_{14}=\frac{1}{v_4}=\frac{1}{v_6}=\frac{k_*}{\omega_*} \qquad v_{11}=\left(\frac{k_*}{\omega_*}\right)^4 \qquad k_*=\frac{c_1}{\kappa}\\
&\omega_*=\frac{c_1^2}{\kappa}
\end{aligned}
\tag{4.1}
$$

Characteristic properties of the functions $G_j(|\boldsymbol{x}|, t, n)$ are studied in details by Mokryk and Pyryev (1985). They can be rewritten in terms of the inverse Laplace transform

$$G_j(\xi,\tau,n) = G_j^1(\xi,\tau,n) + G_j^3(\xi,\tau,n) \qquad j = \overline{1,14} \tag{4.2}$$

$$G_j^p(\xi,\tau,n) = \frac{1}{2\pi}\Big(\frac{d}{d\tau} - \chi_2''\Big)^{N+2} \int\limits_{-\infty+\mathrm{i}\chi_2}^{+\infty+\mathrm{i}\chi_2} \widetilde{G}_j^p(\xi,\tau,n)\frac{\exp(-\mathrm{i}\chi\tau)}{(-\mathrm{i}\chi - \chi_2'')^{N+2}}\, d\chi \tag{4.3}$$

$$\chi_2 > \chi_2'' > \max\{0, \chi_0^I\} \qquad c = M^2$$

$$\widetilde{G}_j^p(\xi,\tau,n) = R_j^p(\chi)\frac{\exp[\mathrm{i}\gamma_p(\chi)\xi]}{(-\mathrm{i}\chi)^{n+1}} \qquad p = 1,3$$

$$R_j^{1,3}(\chi) = \sum_{m=1}^{7}\Big\{\delta_{j(2m)} \pm \frac{\chi^2}{[\gamma_1(\chi)]^2 - [\gamma_3(\chi)]^2}\delta_{j(2m-1)}\Big\}B_m^{1,3}(\chi)$$

$$B_1^{1,3}(\chi) = 1 \qquad B_2^{1,3}(\chi) = \frac{1}{B_7^{1,3}(\chi)} = \frac{\gamma_{3,1}(\chi)}{\chi}$$

$$B_3^{1,3}(\chi) = \frac{1}{B_4^{1,3}(\chi)} = \frac{\gamma_{1,3}(\chi)}{\chi} \qquad B_5^{1,3}(\chi) = \frac{1}{B_6^{1,3}(\chi)} = \frac{\gamma_1(\chi)\gamma_3(\chi)}{\chi^2}$$

$$\gamma_{1,3} = \sqrt{\frac{\chi}{2}}\sqrt{a\chi + \mathrm{i}b \pm \Omega(\chi)} \qquad a = 1 + M^2(1+\varepsilon) \qquad b = 1+\varepsilon$$

$$\Omega(\chi) = \sqrt{(a\chi + \mathrm{i}b)^2 - 4\chi(c\chi + \mathrm{i}d)} = \sqrt{p_1(\chi - \chi_+)(\chi - \chi_1)}$$

$$d = 1 \qquad p_1 = a^2 - 4c \qquad p_2 = d(ab - d) - b^2c$$

$$\chi_\pm = \mathrm{i}\chi_0^I \pm \chi_0^R \qquad \chi_0^I = -\frac{ab - 2d}{p_1} \qquad \chi_0^R = \frac{2\sqrt{p_2}}{p_1}$$

For full determination of the four-valued function

$$\gamma(\chi) = \sqrt{\chi/2}\sqrt{a\chi + \mathrm{i}b + \Omega(\chi)}$$

it is necessary to introduce four leaves of the Riemann surface of complex variable $\chi = \chi_1 + \mathrm{i}\chi_2$, Fig.1. The leaves are connected along two cuts which join the branch points $\chi_\pm$, χ_3, χ_4. For the definitions of functions $\gamma_1(\chi)$, $\gamma_3(\chi)$ and detailed investigation into their properties the Reader is referred to Mokryk and Pyryev (1982). We can write

$$\gamma_{1,3} = \gamma(\chi) = \frac{\mathrm{i}\chi_2}{c_\pm} \qquad \chi = \mathrm{i}\chi_2 \qquad \chi_2 \to \infty \qquad c_\pm = \sqrt{\frac{2}{a \pm \sqrt{p_1}}}$$

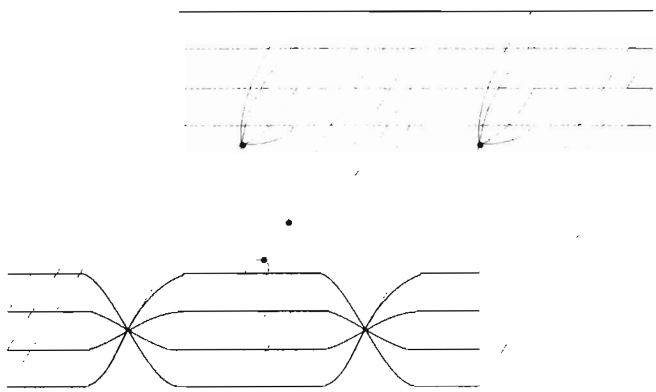

Fig. 1. Four leaves Riemann's surface

In Fig.2 one of the possible closed integration contours for calculation of the integrals (4.3) is presented. The solid line corresponds to the integration contour which is located on the upper sheet of the Riemann surface, while the dashed line corresponds to the integration contour on a lower sheet of the Riemann surface (the third sheet). Different integration contours correspond to five characteristic variants branching points $\chi_{\pm}$, namely, to the points $\chi = \chi_3 = 0$, $\chi = \chi_4 = -\mathrm{i}d/c$.

According to the Cauchy theorem the integral over a closed contour is equal to the sum of residues of the integrand at its singular points. Finally, the integral (4.3) can be write down in the form of a sum of integrals along the edges of cut l_1, l_1', m_1, m_1', along the contours l_2, l_2', m_2, m_2' which connect the branching points $\chi_{\pm}$, along the arcs l_3, l_3', m_3; l_4, l_4', m_4, m_4'; l_8, l_8', m_8, m_8', of circles of sufficiently small radii $\delta_3, \delta_4, \delta_8$ with the centres at the points $\chi = \chi_3 = 0$, $\chi = \chi_4 = -\mathrm{i}d/c$, $\chi = \chi_{\pm}$; along the contour segments l_6, l_6'; l_7, l_7', m_6, m_6' and along the segments l_5, l_5', m_5, m_5' which are located sufficiently far from both the origin and the points $\chi_{\pm}$. Thus, if $N = -2$ we have

$$G_j^{1,3}(\xi,\tau,n) = -H_1\Big(\tau - \frac{\xi}{c_\pm}\Big)\sum_{p=1}^{8}\lim_{\substack{\delta_3,\delta_4,\delta_8\to 0\\ \delta_5\to\infty}} g_{jp}^{1,3}(\xi,\tau,n,\delta_p) \tag{4.4}$$

where

$$g_{jp}^{1,3}(\xi,\tau,n,\delta_p) = \frac{1}{2\pi}\Big(\int\limits_{l_p} + \int\limits_{l_p'} + (1-\delta_{p7})\int\limits_{m_p} + (1-\delta_{p8})\int\limits_{m_p'}\Big)\widetilde{G}_j^{1,3}(\xi,\chi,n)\mathrm{e}^{-\mathrm{i}\chi\tau}\,d\chi$$

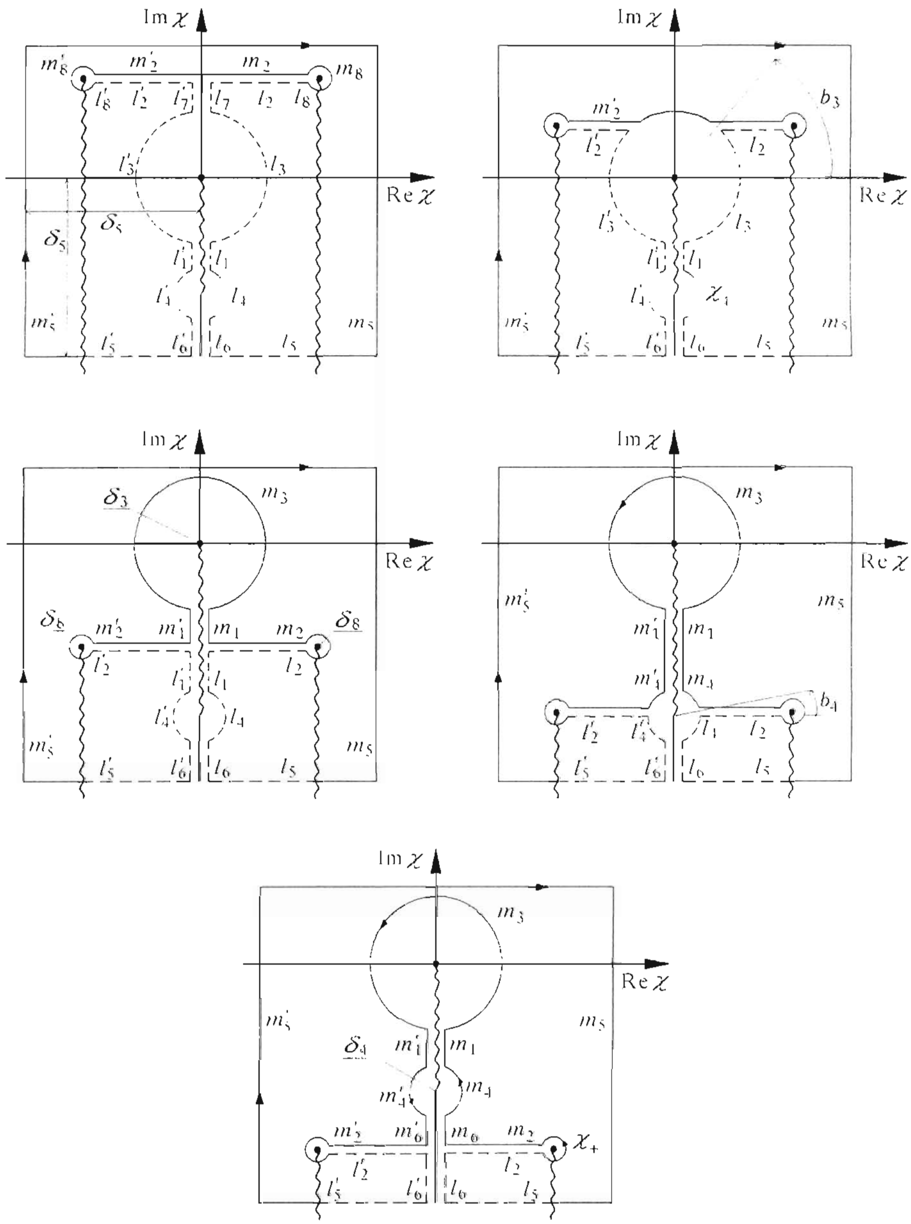

Fig. 2. One of possible contours of integration for integrals calculation

From Fig.2 it can be seen that the integrals along the segments $l_6, l'_6, l_7, l'_7, m_6, m'_6$ disappear because the integrand is a single-valued function. The integrals along the segments l_5, l'_5, m_5, m'_5 tend to zero as $\delta_5 \to \infty$ form the points $\chi = 0$, $\chi = \chi_\pm$ if the following conditions are satisfied

$$\tau - \frac{\xi}{c_\pm} > 0 \qquad\qquad n \geqslant 0$$

The integrals along the circle arcs l_8, l'_8, m_8, m'_8 of the radii δ_8 around the branching points $\chi_\pm$ tend to zero as $\delta_8 \to 0$. The integrals (4.4) for different locations of the points $\chi_\pm$ and $\chi_3 = 0$, $\chi_4 = -\mathrm{i}d/c$ can be represented in terms of the introduced variable integration parameters $\delta_1, \delta_2, b_3, b_4$ which take the corresponding values (see Fig.3) and which $\mathrm{Im}\chi_\pm$ satisfy the following inequalities

$$\begin{array}{lll} \mathrm{Im}\chi_\pm \geqslant \delta_3 & \mathrm{Im}\chi_4 + \delta_4 \leqslant \mathrm{Im}\chi_\pm \leqslant -\delta_3 & \\ |\mathrm{Im}\chi_\pm| \leqslant \delta_3 & |\mathrm{Im}\chi_\pm + \mathrm{Im}\chi_4| \leqslant \delta_4 & \mathrm{Im}\chi_\pm < \mathrm{Im}\chi_4 - \delta_4 \end{array}$$

i.e.

$$g_{j1}^{1,3}(\xi,\tau,n,\delta_1) = \frac{1}{2\pi}\Big[(-1)^{j+1}\int_{\delta_1}^{d/c} S_{j1}^{1,3}(\xi,\tau,x,n)\,dx - \int_{\delta_3}^{\delta_1} S_{j1}^{3,1}(\xi,\tau,x,n)\,dx\Big]$$

$$g_{j2}^{1,3}(\xi,\tau,n,\delta_2) = \frac{1}{2\pi}\mathrm{e}^{\chi_0^l\tau}\int_{\delta_2}^{\chi_0^R} S_{j2}^{1,3}(\xi,\tau,x,n)\,dx \tag{4.5}$$

$$g_{jp}^{1,3}(\xi,\tau,n,b_p) = \frac{1}{2\pi}\Big[(-1)^{j+1}\Big(\int_{-\pi/2}^{b_p} + \int_{\pi-b_p}^{3\pi/2}\Big)S_{jp}^{1,3}(\xi,\tau,\delta_p\mathrm{e}^{\mathrm{i}\varphi},n)\,d\varphi +$$

$$- \int_{b_p}^{\pi-b_p} S_{jp}^{3,1}(\xi,\tau,\delta_p\mathrm{e}^{\mathrm{i}\varphi},n)\,d\varphi\Big] \qquad\qquad p = 3,4$$

The integrands $S_{jp}^{3,1}(\xi,\tau,x,n)$, $m = 1,3$, $p = 1,2$ which appear in Eq(4.5), are combinations of the values of integrands appearing in Eq (4.3)taken on the corresponding segments.

Let us put down the direct representation of the functions $G_1(\xi,\tau,0)$, $G_1(\xi,\tau,1)$, $G_2(\xi,\tau,0)$, which are used in the present paper

$$G_j(\xi,\tau,n) = I_j(\xi,\tau,n,\delta_2)[H_1(\tau_-) - H_1(\tau_+)] + I_j^1(\xi,\tau,n,\delta_1)H_1(\tau_+) + \tag{4.6}$$

$$+(-1)^j I_j^3(\xi,\tau,n,\delta_1)H_1(\tau_-) \qquad j = 1,2 \qquad n = 0,1 \qquad \tau_\pm = \tau - \frac{\xi}{c_\pm}$$

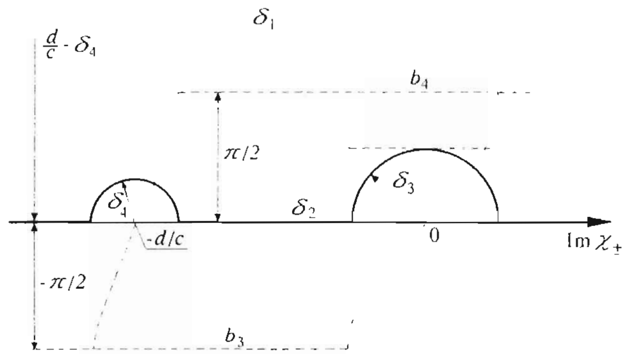

Fig. 3. Variable ranges of integration $\delta_1, \delta_2, b_3, b_4$

where

$$I_j(\xi,\tau,n,\delta_2) = \frac{e^{\chi_0^I\tau}}{2}\int\limits_{\delta_2}^{\chi_0^R}\left(\psi_{jn}^{+}e^{-\alpha_+\xi}+\psi_{jn}^{-}e^{-\alpha_-\xi}\right)dx$$

$$\psi_{10}^{\pm} = -\frac{1}{l_0}\cos\varphi_\pm \qquad \psi_{11}^{\pm} = \frac{1}{l_0k_0^2}(x\sin\varphi_\pm - \chi_0^I\cos\varphi_\pm)$$

$$\psi_{20}^{\pm} = \pm l_0\psi_{11}^{\pm} \qquad l_0 = \sqrt{(\chi_0^R)^2 - x^2} \qquad k_0 = \sqrt{(\chi_0^I)^2 - x^2}$$

$$\varphi_\pm = \xi\gamma_\pm - \tau x \qquad \gamma_\pm = \frac{1}{2\sqrt{2}}\left(\sqrt{Z^0 + X_\pm^0}K_+ + \sqrt{Z^0 - X_\pm^0}K_-\right)$$

$$K_\pm = \sqrt{k^0 \pm \chi_0^I} \qquad \alpha_\pm = \frac{1}{2\sqrt{2}}\left(\sqrt{Z^0 + X_\pm^0}K_- - \sqrt{Z^0 - X_\pm^0}K_+\right)$$

$$Z^0 = \sqrt{(X_\pm^0)^2 + (Y^0)^2} \qquad X_\pm^0 = \chi_0^I a + b \pm l_0\sqrt{p_1} \qquad Y^0 = xa$$

$$I_j^m(\xi,\tau,n,\delta_1) = \frac{1}{\pi}\left((-1)^j\delta_{m3}\int\limits_{\delta_1}^{d/c} + \delta_{m1}\int\limits_{\delta_3}^{\delta_1}\right)\cdot$$

$$\cdot\frac{e^{-\tau x}}{(-x)^{n+1}}\theta_{jn}\sin(\xi\sqrt{Ax})\,dx - (-1)^j\,d_{jn} \qquad m = 1,3$$

$$k^0 = \sqrt{(\chi_0^I + x)^2 + (\chi_0^R)^2} \qquad \theta_{11} = \theta_{10} = -\frac{x}{k^0} \qquad \theta_{20} = 1$$

$$A = \frac{1}{2}(k^0\sqrt{p_1} + b - xa) \qquad d_{10} = 0 \qquad d_{20} = -1$$

$$d_{11} = \frac{\sqrt{p_1}}{b} \begin{cases} \operatorname{sgn}\chi_0^I & |\chi_0^I| > \delta_3 \\ 2\pi^{-1}\arcsin(\chi_0^I/\delta_3) & |\chi_0^I| < \delta_3 \end{cases}$$

$$\delta_1 = \begin{cases} \delta_3 & -\chi_0^I \leqslant \delta_3 \\ -\chi_0^I & \delta_3 \leqslant \chi_0^I \leqslant d/c \\ d/c & d/c \leqslant \chi_0^I \end{cases} \qquad \delta_2 = \begin{cases} 0 & |\chi_0^I| \leqslant \delta_3 \\ \sqrt{(\delta_3)^2 - (\chi_0^I)^2} & |\chi_0^I| \geqslant \delta_3 \end{cases}$$

δ_1, δ_2 – variable ranges of integration
δ_3 – arbitrary small value.

Using the results of analysis of the functions $\gamma_{1,3}(\chi)$ carried out by Mokryk and Pyryev (1982) and expanding the functions $\tilde{G}_j^{1,3}(\xi,\chi,n)$, $j = \overline{1,14}$ into a series in the neighbourhood of the characteristic points we obtain some characteristic asymptotical representations of the functions $G_j^{1,3}(\xi,\tau,n)$, $G_j(\xi,\tau,n)$, $j = \overline{1,14}$.

Basing on the expansion of the functions $\tilde{G}_j^{1,3}(\xi,\chi,n)$ into a series in the neighbourhood of the points $\chi_3 = 0$, $\chi_\pm$ and using the Felsen-Marcuvitz theorem (cf Felsen and Marcuvitz, 1973) of asymptotical behaviour of the inverse transforms as $\tau \to \infty$ we obtain the following asymptotical representation

$$G_j^{1,3}(\xi,\tau,n) \sim Q_j^{1,3}(\tau) \pm R_j(\tau) \qquad \begin{cases} j = \overline{1,14} \\ \xi = \text{const} \\ \tau \to \infty \end{cases} \tag{4.7}$$

where

$$\begin{gathered} Q_j^{1,3}(\tau) = a_{j,0}^{1,3} H_{v_j^{1,3}}(\tau) \\ R_j(\tau) = \frac{1}{\sqrt{\tau}} \exp(\chi_0^I\tau - \xi \operatorname{Im} a_0^+) P_j \left[{\sin \atop \cos} \{\chi_0^R \tau - \xi \operatorname{Re} a_0^+\} \right] \end{gathered} \tag{4.8}$$

$P_j[\cdot]$ – some functions which are neglected; $a_{j,0}^{1,3}$, $v_j^{1,3}$, a_0^+ – are represented by following formulae

$$\begin{gathered} a_{1,0}^{1,3} = \pm\frac{1}{b} \qquad a_{2,0}^{1,3} = 1 \qquad a_0^+ = \sqrt{\frac{\chi_+}{2}(\chi_+ a + ib)} \\ v_1^{1,3} = n \qquad v_2^{1,3} = n+1 \end{gathered} \tag{4.9}$$

The analysis of Eqs (4.7), (4.8) shows that in the case when $\operatorname{Im}\chi_\pm = \chi_0^I > 0$, the functions $G_j^{1,3}(\xi,\tau,n)$ increase as $\tau \to \infty$ according to the exponential law. For $\chi_0^I < 0$ the term $R_j(\tau)$ decreases exponentially and the behaviour of the functions $G_j^{1,3}(\xi,\tau,n)$ is determined by the term $Q_j^{1,3}(\tau)$.

The asymptotical representation of the functions $G_j(\xi,\tau,n)$ as $\tau\to\infty$, $\xi=\text{const}$ can be obtained basing on the theorem (Felsen and Marcuvitz, 1973) of the asymptotical behaviour of the inverse transforms as $\tau\to\infty$. Finally, it can be written down in the form

$$G_j(\xi,\tau,n)\sim Q_j(\tau) \qquad j=\overline{1,14} \qquad \xi=\text{const} \qquad \tau\to\infty \tag{4.10}$$

where

$$Q_j(\tau)=a_{j,0}H_{v_j}(\tau)$$

$a_{j,0}$, v_j are given by the formulae

$$\begin{array}{llll}
v_1=n-\frac{1}{2} & a_{1,0}=a^1_{1,1} & v_2=n+1 & a_{2,0}=2\\
v_3=n+\frac{1}{2} & a_{3,0}=a^3_{3,0} & v_4=n+\frac{3}{2} & a_{4,0}=a^3_{4,0}\\
v_5=n+\frac{1}{2} & a_{5,0}=a^1_{5,0} & v_6=n+\frac{3}{2} & a_{6,0}=a^1_{6,0}\\
v_7=n & a_{7,0}=a^3_{7,0} & v_8=n+1 & a_{8,0}=a^3_{8,0}\\
v_9=n & a_{9,0}=a^1_{9,1} & v_{10}=n+\frac{3}{2} & a_{10,0}=2a^1_{10,1}\\
v_{11}=n-1 & a_{11,0}=a^1_{11,1} & v_{12}=n+\frac{1}{2} & a_{12,0}=2a^1_{12,0}\\
v_{13}=n & a_{13,0}=a^1_{13,0} & v_{14}=n+1 & a_{14,0}=a^1_{14,0}
\end{array}$$

The analysis of the functions $G_j(\xi,\tau,n)$ behaviour as $\tau\to\infty$ shows that in contrast to their components $G_j^{1,3}(\xi,\tau,n)$ which increase exponentially for $\mathrm{Im}\chi_\pm>0$, they do not increase exponentially for any value of $\mathrm{Im}\chi_\pm$. Their behaviour at infinity is determined by $Q_j(\tau)$, i.e. $G_j(\xi,\tau,n)$ increases if $v_j-1>0$, tends to some constant value if $v_j=1$ and decreases if $v_j<1$.

Basing on the expansions of functions $\widetilde{G}_j^{1,3}(\xi,\chi,n)$ into a series at infinity we obtain the asymptote

$$G_j^{1,3}(\xi,\tau,n)\sim R_j^{1,3}\mathrm{e}^{-\xi\eta_\pm}H_{n+1}\Big(\tau-\frac{\xi}{c_\pm}\Big) \qquad j=\overline{1,14} \qquad \tau-\frac{\xi}{c_\pm}\to 0 \tag{4.11}$$

where $R_j^{1,3}$, $\eta_\pm$ are defined by the formulae

$$R_j^{1,3}=\sum_{m=1}^{7}\Big(\delta_{j(2m)}\pm\frac{c_+^2c_-^2}{c_-^2-c_+^2}\delta_{j(2m-1)}B_m^{1,3}\Big)$$

$$\eta_\pm=\frac{c_\pm(b\mp\chi_0^I\sqrt{p_1})}{4} \qquad B_1^{1,3}=1 \qquad B_7^{1,3}=\frac{1}{B_2^{1,3}}=c_\mp$$

$$B_4^{1,3}=\frac{1}{B_3^{1,3}}=c_\pm \qquad B_6^{1,3}=\frac{1}{B_5^{1,3}}=c_+c_-$$

Note that the dimensionless expansion coefficients η_+, η_- can be represented in terms of dimensionless velocities c_+, c_-, c_0 as follows

$$\eta_\pm = \pm\frac{c_0^2 - c_\pm^2}{2\eta c_\pm(c_-^2 - c_+^2)} \tag{4.12}$$

where

$$c_0 = \sqrt{\frac{b}{d}} \qquad c_- < c_0 < c_+ \qquad \eta = \frac{c}{d} \tag{4.13}$$

We apply the method of crossing the top (Felsen and Marcuvitz, 1973) to calculation of $G_j^{1,3}(\xi,\tau,n)$, $j = \overline{1,14}$

$$G_j^{1,3}(\xi,\tau,n) \sim a_{j,0}^3 E\Big(\tau - \frac{\xi}{c_0}\Big) * H_{v_j^3}(\tau) \qquad \tau - \frac{\xi}{c_\pm} \to 0 \qquad \frac{\tau}{\mu_0^2} \to \infty \tag{4.14}$$

where

$$E\Big(\tau - \frac{\xi}{c_0}\Big) = \frac{\exp\Big[-\frac{(\tau-\xi/c_0)^2}{\tau_1^2}\Big]}{\tau_1\sqrt{\pi}} \qquad \tau_1^2 = \xi\mu_0^2$$
$$\mu_0^2 = \frac{2\eta}{c_0^3}(c_0^2 - c_+^2)(c_-^2 - c_0^2) \tag{4.15}$$

and coefficients $a_{j,0}^3$, v_j^3 are defined by Eqs (4.9).

According to Whitham (1959) the waves which propagate at the speeds $c_\pm$ and c_0 are called higher and lower order waves, respectively.

The above analysis of the functions $G_j^{1,3}(\xi,\tau,n)$, $G_j(\xi,\tau,n)$, $j = \overline{1,14}$, as $\tau \to \infty$, $\xi = \text{const}$ as $\tau \to \infty$, $\xi/\tau = \text{const}$ and for $\tau - \xi/c_\pm \to 0$ allows for formulation of their characteristic properties.

The diagrams of functions $G_j(\xi,\tau,n)$ can be easily shown (see Fig.4). The equations of wave fronts $|\xi| - c_\pm\tau = 0$ as well as the equation of main disturbance $|\xi| - c_0\tau = 0$ motion are also sketched in Fig.4. In Fig.4 for Eqs(4.11) either the sign "-" or "+" should be taken, while Eqs (4.14) and (4.10) hold.

The functions $G_j^{1,3}(\xi,\tau,n)$ are equal to zero for $\tau < \xi/c_\pm$. For $n \geqslant 0$ the functions $G_j^{1,3}(\xi,\tau,n)$ are ordinary, for $n > 0$ continuous and for $n < 0$ generalised singular functions with singularities at the points $\tau = \xi/c_\pm$, respectively. They should be considered as n-derivatives of the functions $G_j^{1,3}(\xi,\tau,n)$ for $n = 0$. If $n \geqslant 0$ at the points $\tau = \xi/c_\pm$ the nth derivatives of the functions $G_j^{1,3}(\xi,\tau,n)$ reveal jumps of the value of $R_j^{1,3}\exp(-\xi\eta_\pm)$, which

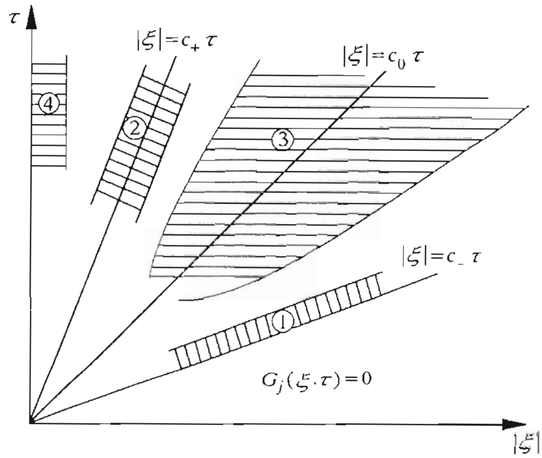

Fig. 4. $(|\xi|, \tau)$ diagramme

decrease when the spatial coordinate ξ increases. For fixed ξ and increasing τ the functions $G_j^{1,3}(\xi, \tau, n)$ behave in accordance with Eqs (4.7).

The functions $G_j(\xi, \tau, n)$, $j = \overline{1, 14}$ are equal to zero if $\tau < \xi/c_-$ and take non-zero values for $\tau > \xi/c_-$. For $n \geqslant 0$ the functions $G_j(\xi, \tau, n)$ are ordinary, and for $n > 0$ – continuous. For $n < 0$ the functions are distributions with singularities at the points $\tau = \xi/c_-$ and $\tau = \xi/c_+$. When passing through the point $\tau = \xi/c_-$ the nth derivative of the functions $G_j(\xi, \tau, n)$ for $n \geqslant 0$ reveal the jump of the value of $R_j^3 \exp(-\xi\eta_-)$. Between the points $\tau = \xi/c_-$ and $\tau = \xi/c_+$ the functions are infinitely differentiable. When passing through the point $\tau = \xi/c_+$ the nth derivatives of the functions $G_j(\xi, \tau, n)$ reveal the jumps of the value of $R_j^1 \exp(-\xi\eta_+)$ for $\geqslant 0$. For $\tau = \xi/c_+$ the functions are infinitely differentiable and behave in accordance with Eqs(4.10) as $\tau \to \infty$, $\xi =$ const. Note that the jump of the nth derivative of the functions $G_j(\xi, \tau, n)$, $n \geqslant 0$ exponentially decreases with the increasing space coordinate ξ. Thus, the last parameter n in the functions $G_j(\xi, \tau, n)$ shows that for $n \geqslant 0$ they are ordinary and the nth derivatives of these functions reveal discontinuities of the first kind.

The characteristic feature of the functions $G_j(\xi, \tau, n)$ is their behaviour between the points $\tau = \xi/c_-$ and $\tau = \xi/c_+$ in the neighbourhood of $\tau = \xi/c_0$ as $\tau \to \infty$ which is determined Eq (4.14).

After differentiation of the functions $G_j(\xi,\tau,n)$, $j=\overline{1,14}$ with respect to time the following equation is true

$$\frac{\partial^m}{\partial\tau^m}G_j(\xi,\tau,n)=G_j(\xi,\tau,n-m) \tag{4.16}$$

5. Conclusions

The components of fundamental solution of the generalised thermoelastic system are represented by a series of functions $G_j(\xi,\tau,n)$. In the general case the representation of these functions in terms of integrals along the segments connecting singular points of the corresponding Laplace transforms with respect to time is obtained. The proposed method allows for investigation into processes represented by this equation for arbitrary fixed times, arbitrary material parameters and selected orders of the singularities of propagation fronts. In the obtained solution there the waves of higher and lower orders appear, which propagate at the corresponding velocities $c_\pm$ and c_0.

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Macierz podstawowa układu równań uogólnionej termosprężystości

Streszczenie

Macierz podstawowa uogólnionej teorii termosprężystości została skonstruowana dla oddziaływań temperatury, przemieszczeń oraz propagacji ciepła. Elementy macierzy są przedstawione w formie transformat Laplace'a odpowiednich funkcji. Właściwości charakterystyczne tych funkcji zostały zbadane.

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