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UAV PROGRAM MOTION AND CONTROL IN PRESCRIBED MISSION

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The paper deals with the inverse simulation problem of aircraft general motion. The aircraft is constrained to follow a specified 3D trajectory (two *program constraints*), and two other demands, are imposed on airframe attitude with respect to the trajectory and on flying speed. The guidelines for effective modelling of aircraft prescribed trajectory flight are discussed, and a method for computing time histories of state variables and program control (that ensures the programmed motion realization) is developed. Some results of numerical simulations are reported.

Key words: aerospace dynamics and control, inverse simulation, control system design

1. Introduction

Inverse simulation techniques are computational methods for determining the control inputs to a dynamic system that produce desired system outputs. Such techniques can be powerful tools for the analysis of problems associated with manoeuvring flight (cf Azam and Singh, 1994; Blajer and Parczewski, 1990; Hess et al., 1991; Kato and Sugiura, 1986; Lane and Stengel, 1988; Thomson and Bradley, 1990). The problem addressed herein involves four *program constraints* imposed on aircraft motion; i.e., a specified space trajectory (two constraints), a condition on airframe attitude with respect to the trajectory, and a specified time-history of flying speed. The aircraft is controlled by aileron, elevator and rudder deflections, and by thrust value changes.

The number of motion specifications is smaller than the number of airframe degrees of freedom. The problem at hand may thus be considered as a partly specified motion. This is not the case, however, since the realization of trajectory constraints is *tangent* (cf Blajer, 1997; Blajer and Parczewski, 1990). The control surface deflections influence primarily the aerodynamic moments, and give small (if any) effects on aerodynamic forces. The induced control reactions cannot thus directly govern the balance of the applied and inertial forces in the orthogonal to the trajectory (constrained) directions, and cannot therefore directly ensure the trajectory realization. Nevertheless, since the aerodynamic forces depend on airframe attitude with respect to the trajectory, the force balances can be assured by suitable fitting the attitude, which can directly be governed by the appropriate control surface deflections. That means two additional requirements imposed on airframe attitude or, in other words, the trajectory constraints are "doubled" consequent upon their tangent realization. This provides also an explanation of why at most four program constraints (including two "doubled" trajectory constraints) can be imposed on the aircraft in order to *fully specify* its motion, and why the six-degree-of-freedom system can explicitly be controlled by four control inputs. Since the realization of the other program constraints is orthogonal (cf Blajer, 1997; Blajer and Parczewski, 1990), we deal thus with a mixed *tangent-orthogonal* realization of the program of motion as introduced above.

In the paper, the guidelines for effective modelling of the aircraft prescribed trajectory flight are discussed, and a method for computing time histories of state variables and control in the programmed motion is developed. The solution gives an opportunity to study the simulated control strategies and evaluate feasibility of the modelled manoeuvres. Such analyses may be useful in analysing extreme flight conditions (e.g. aerobatic manoeuvres) or may be used for planning missions of pilotless vehicles. In the latter case, the computed program control can be applied as feedforward control, while the predicted time-variations of state variables can serve as the reference for the monitored actual flight state variables. The obtained differences can then be used in a feedback control loop added to the feedforward control.

2. Mathematical model

2.1. Coordinate systems

The following coordinate systems are used:

(I) Inertial $O_Ix_Iy_Iz_I$, with O_Iz_I axis pointed vertical and downward

(G) Gravitational $Ox_Gy_Gz_G$, with the origin at the aircraft mass centre O and parallel to $O_Ix_Iy_Iz_I$

(A) Aerodynamic $Ox_Ay_Az_A$, with Ox_A axis pointed along the flying speed vector $\vec{\boldsymbol{v}}$

(B) Body-fixed $Ox_By_Bz_B$, with Ox_Bz_B plane being the airframe symmetry plane.

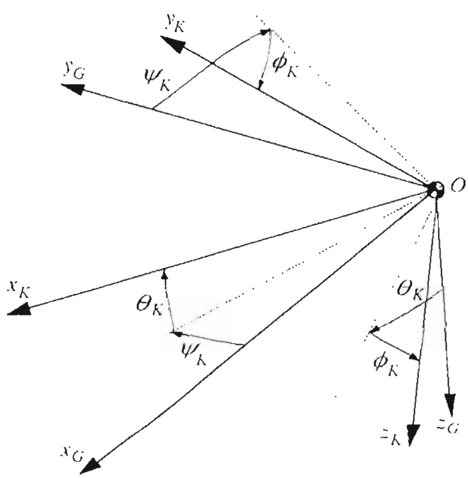

Fig. 1. Orientation of **(K)** and **(G)** reference systems

The three angles that orientate **(A)** and **(B)** reference systems with respect to **(G)** are traditionally Bryant's angles ϕ_K, θ_K and ψ_K, for $K = A, B$ (see Fig.1), and the respective transformation matrix is

$$\mathbf{A}_{KG} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + c_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix} \tag{2.1}$$

where $s_\phi = \sin\phi_K$, $c_\theta = \cos\theta_K$, ... The absolute angular velocity of **(K)**, $\vec{\boldsymbol{\omega}}_K$, expressed in **(K)** is

$$\boldsymbol{\omega}_K^{(K)} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & s_\phi c_\theta \\ 0 & -s_\phi & c_\phi c_\theta \end{bmatrix} \begin{bmatrix} \dot{\phi}_K \\ \dot{\theta}_K \\ \dot{\psi}_K \end{bmatrix} = \mathbf{B}_K \begin{bmatrix} \dot{\phi}_K \\ \dot{\theta}_K \\ \dot{\psi}_K \end{bmatrix} \tag{2.2}$$

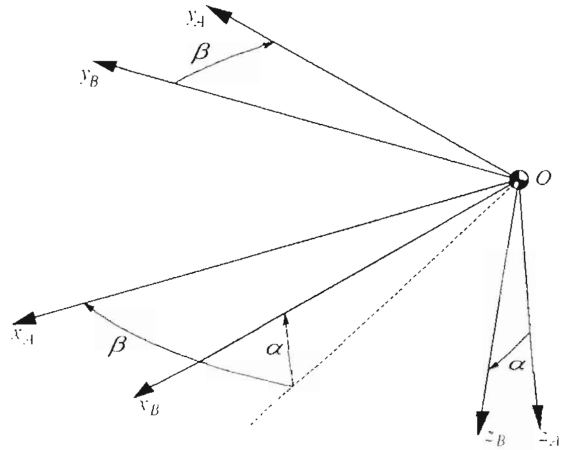

Fig. 2. Orientation of **(B)** and **(A)** reference systems

The angular orientation of **(B)** and **(A)** reference frames is described by the angles of attack α and sideslip β (Fig.2), and the transformation matrix is

$$\mathbf{A}_{BA} = \begin{bmatrix} c_\alpha c_\beta & -c_\alpha s_\beta & -s_\alpha \\ s_\beta & c_\beta & 0 \\ s_\alpha c_\beta & -s_\alpha s_\beta & c_\alpha \end{bmatrix} \tag{2.3}$$

while the angular velocity of **(A)** with respect to **(B)**, $\vec{\boldsymbol{\omega}}_{A/B}$, expressed in **(A)** is

$$\boldsymbol{\omega}_{A/B}^{(A)} = \begin{bmatrix} -s_\beta & 0 \\ 0 - c_\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \mathbf{B}_{A/B} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \tag{2.4}$$

Based on Eqs (2.1) $\div$ (2.4) two other useful relations can be derived. The first one is a matrix representation of the vector formula for the aircraft absolute angular velocity $\vec{\boldsymbol{\omega}}_B = \vec{\boldsymbol{\omega}}_A + \vec{\boldsymbol{\omega}}_{B/A} = \vec{\boldsymbol{\omega}}_A - \vec{\boldsymbol{\omega}}_{A/B}$. Expressed in **(B)** the formula reads

$$\boldsymbol{\omega}_B^{(B)} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{A}_{BA}\Big(\boldsymbol{\omega}_A^{(A)} - \boldsymbol{\omega}_{A/B}^{(A)}\Big) =$$
$$= \mathbf{A}_{BA}(\alpha,\beta)\Big(\mathbf{B}_A(\phi_A,\theta_A) \begin{bmatrix} \dot{\phi}_A \\ \dot{\theta}_A \\ \dot{\psi}_A \end{bmatrix} - \mathbf{B}_{A/B}(\beta) \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}\Big) \tag{2.5}$$

and serves to determine p, q and r (the roll, pitch and yaw rates, respectively) in terms of α, β, ϕ_A, θ_A, and $\dot{\alpha}$, $\dot{\beta}$, $\dot{\phi}_A$, $\dot{\theta}_A$, $\dot{\psi}_A$. The other relation is

$$\mathbf{A}_{BG}(\phi_B,\theta_B,\psi_B) = \mathbf{A}_{BA}(\alpha,\beta)\mathbf{A}_{AG}(\phi_A,\theta_A,\psi_A) \tag{2.6}$$

and can be used to determine the classical airframe roll, pitch and yaw angles (ϕ_B,θ_B,ψ_B) in terms of α, β, ϕ_A, θ_A, ψ_A.

Some other important features of the transformation matrices introduced in Eqs (2.1) and (2.3) are

$$\begin{aligned} \mathbf{A}_{KL} &= \mathbf{A}_{LK}^{-1} = \mathbf{A}_{LK}^{\top} \\ \dot{\mathbf{A}}_{KL} &= \widetilde{\boldsymbol{\omega}}_{K/L}^{(K)}\mathbf{A}_{LK} = \mathbf{A}_{KL}\widetilde{\boldsymbol{\omega}}_{L/K}^{(L)} \end{aligned} \tag{2.7}$$

for $K = G, A$; $L = K, B$, and $\widetilde{(\cdot)}$ in Eq $(2.7)_2$ denotes a skew-symmetric matrix, which for a vector $\boldsymbol{a} = [a_x, a_y, a_z]^{\top}$ is defined as

$$\widetilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = -\widetilde{\mathbf{a}}^{\top} \tag{2.8}$$

2.2. Dynamic equations

For the sake of simplicity of the mathematical description of prescribed trajectory restrictions imposed on aircraft, it is convenient to use the dynamic equations of translatory motions in **(A)**, while using the angular motion equations expressed traditionally in **(B)**. The matrix form of the equations reads

$$\begin{aligned} m\dot{\mathbf{v}}_A^{(A)} + m\widetilde{\boldsymbol{\omega}}_A^{(A)}\mathbf{v}_A^{(A)} &= \mathbf{F}^{(A)} \\ \mathbf{J}\dot{\boldsymbol{\omega}}_B^{(B)} + \widetilde{\boldsymbol{\omega}}_B^{(B)}\mathbf{J}\boldsymbol{\omega}_B^{(B)} &= \mathbf{N}^{(B)} \end{aligned} \tag{2.9}$$

where

m	–	mass of the aircraft
$\mathbf{v}_A^{(A)}$	–	flying speed vector representation in **(A)**, $\mathbf{v}_A^{(A)} = [1,0,0]^{\top}v$
$\boldsymbol{\omega}_A^{(A)}$	–	representation in **(A)** of the absolute angular velocity of **(A)** frame
$\boldsymbol{\omega}_B^{(B)}$	–	defined in Eq (2.5)
$\mathbf{J}$	–	matrix of aircraft moments of inertia in **(B)**.

Assumed the jet thrust T is pointed along the Ox_B axis, the components of applied forces $\mathbf{F}^{(A)}$ and torques $\mathbf{N}^{(B)}$, respectively in the **(A)** and **(B)** reference frames, are

$$\mathbf{F}^{(A)} = -\frac{1}{2}\rho S v^2 \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} + T \begin{bmatrix} c_\alpha c_\beta \\ -c_\alpha s_\beta \\ -s_\alpha \end{bmatrix} + mg \begin{bmatrix} -s_\theta \\ s_\phi c_\theta \\ c_\phi c_\theta \end{bmatrix}$$

$$\mathbf{N}^{(B)} = \frac{1}{2}\rho S v^2 \begin{bmatrix} b c_l \\ c_a c_m \\ b c_n \end{bmatrix} \tag{2.10}$$

where

- ρ – air density
- S – lifting surface
- g – acceleration of gravity
- b – wingspan
- c_a – mean aerodynamic chord.

The matrices appearing by T and mg in Eq $(2.10)_1$ are, respectively, the first and third columns of $\mathbf{A}_{AB} = \mathbf{A}_{BA}^\top$ and $\mathbf{A}_{AG}$ defined in Eqs (2.3) and (2.1). The drag, side and lift force coefficients $\mathbf{c}_F^{(A)} = [c_x, c_y, c_z]^\top$, and the coefficients $\mathbf{c}_N^{(B)} = [c_l, c_m, c_n]^\top$ of rolling, pitching and yawing moments, have been assumed as follows

$$\begin{aligned} c_x &= c_x(\alpha, \beta, \delta_e, \mathrm{Ma}) \qquad & c_l &= c_l(\alpha, \beta, p, r, \delta_a, \delta_r) \\ c_y &= c_y(\alpha, \beta, p, r, \delta_r) \qquad & c_m &= c_m(\alpha, q, \delta_e) \\ c_z &= c_z(\alpha, q, \delta_e) \qquad & c_n &= c_n(\alpha, \beta, p, r, \delta_a, \delta_r) \end{aligned} \tag{2.11}$$

where Ma is the Mach number.

2.3. Equations of program constraints

2.3.1. *Prescribed trajectory*

The most convenient representation of a desired trajectory in space is

$$\mathbf{r}^{(I)} = \widehat{\mathbf{r}}^{(I)}(s) \equiv \begin{bmatrix} \widehat{x}(s) \\ \widehat{y}(s) \\ \widehat{z}(s) \end{bmatrix} \tag{2.12}$$

where $\widehat{(\cdot)}$ denotes *specified*, and s is the arc length parameter (Fig.3). For the purpose of this model, a continuous second derivative of $\widehat{\mathbf{r}}^{(I)}(s)$ with respect to s must be assured. In this application, the trajectory is first sketched by successive points in space, and then interpolated/approximated by spline functions of appropriate order. The foundations of such a procedure were given by Blajer and Parczewski (1990).

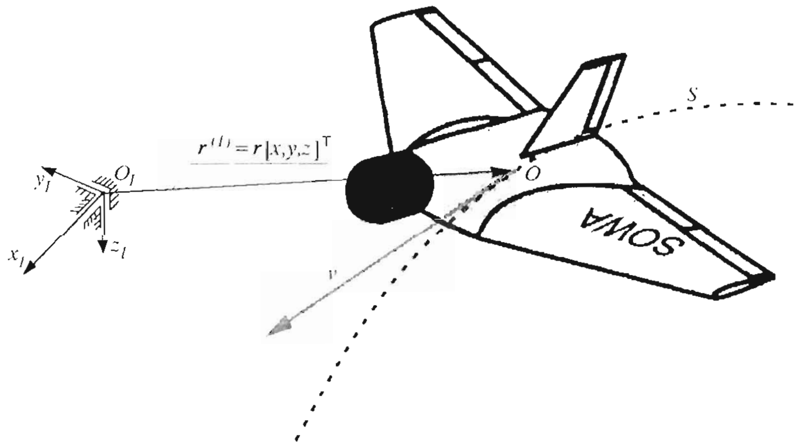

Fig. 3. Aircraft prescribed trajectory flight

2.3.2. Airframe attitude specification

The following two optional requirements imposed on airframe attitude with respect to the trajectory are considered

$$\beta = 0 \qquad \text{or} \qquad \phi = \widehat{\phi}(s) \tag{2.13}$$

The first one specifies the coordinated turn flight conditions, while the other is useful in modelling of such aerobatic manoeuvres like roll or bunt.

2.3.3. Specified flying speed

A reasonable specification of this type is $v = \widehat{v}(s)$. For the purpose of this model the constraint should however be modified to

$$s = \widehat{s}(t) \tag{2.14}$$

In some simple cases, e.g. $\widehat{v} =$ const, the formulation (2.14) is evident. In a more general case, Eq (2.14) comes as a solution to $ds/v(s) = dt$. Herein it is assumed that Eq (2.14) is given, and the flying speed constraint is equivalent to $\widehat{v} = \dot{\widehat{s}}$.

3. Equations of program motion

3.1. Trajectory constraint conditions

Differentiating with respect to time the trajectory constraint (2.12) we obtain $\dot{\hat{\mathbf{r}}}^{(I)} = \hat{\mathbf{r}}'^{(I)}\dot{s} = \mathbf{v}^{(I)} = \mathbf{v}^{(G)}$, which means that the aircraft linear velocity $\vec{v}$ is tangent to the trajectory path. Since $\mathbf{v}^{(G)} = \mathbf{A}_{GA}\mathbf{v}^{(A)}$, $\mathbf{v}^{(A)} = [1,0,0]^\top$, and $v = \dot{s}$, the condition can be transformed to

$$\begin{bmatrix} \cos\theta_A \cos\psi_A \\ \cos\theta_A \sin\psi_A \\ -\sin\theta_A \end{bmatrix} = \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} \tag{3.1}$$

where $(\cdot)'$ denotes differentiation with respect to s. As $\theta_A \in \langle -\pi/2, \pi/2\rangle$ and $\psi_A \in \langle 0, 2\pi\rangle$, Eq (3.1) serves for the explicit determination of θ_A and ψ_A angles that orientate $\vec{v}$ with respect to the $\mathbf{(G)}$ frame.

Differentiating once more the above trajectory constraint at the "velocity level" $(\mathbf{A}_A\mathbf{v}^A = \hat{\mathbf{r}}'\dot{s})$, and then using Eq $(2.7)_2$, we obtain the trajectory constraint condition at the "acceleration level" in the following form

$$\mathbf{A}_{GA}\Big(\dot{\mathbf{v}}^{(A)} + \tilde{\boldsymbol{\omega}}_A^{(A)}\mathbf{v}^{(A)}\Big) = \hat{\mathbf{r}}'^{(I)}\ddot{s} + \hat{\mathbf{r}}''^{(I)}\dot{s}^2 \tag{3.2}$$

The right-hand side of Eq (3.2) expresses the representation in $\mathbf{(G)}$ of tangent $\vec{a}_\tau$ $(\mathbf{a}_\tau^{(G)} = \hat{\mathbf{r}}'^{(I)}\ddot{s})$ and normal $\vec{a}_n$ $(\mathbf{a}_n^{(G)} = \hat{\mathbf{r}}''^{(I)}\dot{s}^2)$ accelerations imposed on the aircraft by the trajectory constraint (2.12). Premultiplying Eq (3.2) by $\mathbf{A}_{AG}$, and using the dynamic equation $(2.9)_1$, we finally arrive at

$$-\mathbf{F}^{(A)} + m\mathbf{A}_{AG}\Big(\hat{\mathbf{r}}'^{(I)}\ddot{s} + \hat{\mathbf{r}}''^{(I)}\dot{s}^2\Big) = \mathbf{0} \tag{3.3}$$

According to Eq (3.1), $\hat{\mathbf{r}}'^{(I)}$ is equivalent to the first row of $\mathbf{A}_{AG}$ (the first column of $\mathbf{A}_{GA}$). It is thus the representation in $\mathbf{(I)}/\mathbf{(G)}$ of a (unit) vector pointed at $\vec{v}$ direction (along the first axis of $\mathbf{(A)}$ frame). Then, $\hat{\mathbf{r}}''^{(I)}$ represents a vector pointed at the centre of trajectory curvature, contained in the orthogonal to trajectory plane. It is then

$$\mathbf{A}_{AG}\hat{\mathbf{r}}'^{(I)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}_{AG}\hat{\mathbf{r}}''^{(I)} = \begin{bmatrix} 0 \\ \times \\ \times \end{bmatrix} \tag{3.4}$$

where $(\times)$ denotes a non-zero entry. The first scalar equation of Eq (3.3), expressing balance of the applied and inertial forces at the tangent to the trajectory direction, is therefore

$$\frac{1}{2}\rho S v^2 c_x - T\cos\alpha\cos\beta + mg\sin\theta_A + m\ddot{s} = 0 \tag{3.5}$$

The other two scalar equations of Eq (3.3), which express balance of the applied and inertial forces in the plane orthogonal to the trajectory (along the second and third axes of **(A)** reference frame), are then

$$\begin{aligned}
&\frac{1}{2}\rho S v^2 c_y + T\sin\beta - mg\sin\phi_A\cos\theta_A + \\
&+m\dot{s}^2\Big[\widehat{x}''(\sin\phi_A\sin\theta_A\cos\psi_A - \cos\phi_A\sin\psi_A) + \\
&+\widehat{y}''(\sin\phi_A\sin\theta_A\sin\psi_A + \cos\phi_A\cos\psi_A) + \widehat{z}''\sin\phi_A\cos\theta_A\Big] = 0 \\
&\frac{1}{2}\rho S v^2 c_z + T\sin\alpha\cos\beta - mg\cos\phi_A\cos\theta_A \\
&+m\dot{s}^2\Big[\widehat{x}''(\cos\phi_A\sin\theta_A\cos\psi_A + \sin\phi_A\sin\psi_A) + \\
&+\widehat{y}''(\cos\phi_A\sin\theta_A\sin\psi_A - \sin\phi_A\cos\psi_A) + \widehat{z}''\cos\phi_A\cos\theta_A\Big] = 0
\end{aligned} \tag{3.6}$$

Setting $\ddot{s} = \ddot{\widehat{s}}$, Eqs (3.5) and (3.6) express the conditions imposed on aircraft dynamics by the acceleration forms of the velocity, and trajectory constraints defined in Eqs (2.14) and (2.12), respectively. Only the first condition, regulating the demanded variations of $\widehat{v}(s) \to \widehat{s}(t)$, can be directly and explicitly governed by appropriate changes of T control parameter. The two other conditions, however, denoting the condition of vanishing motion in the plane orthogonal to the trajectory, are not explicitly governed by the available control reactions (T has already been "used" and the control surface deflections induce small/vanishing effects on c_y and c_z). The realization of flying speed constraint (2.14) is thus *orthogonal*, while the realization of trajectory constraint (2.12) is *tangent* (refer to Blajer (1997) for more details). In consequence, Eqs (3.6) denote two additional constraints imposed on the airframe attitude – the trajectory constraints are "doubled". As the realization of the airframe attitude constraint (2.13) is orthogonal (the condition imposed on aircraft dynamics by the constraint acceleration form involve control reactions induced by control surface deflections), we deal thus with a *fully specified motion* of six-degree-of-freedom aircraft – the four constraints (2.12) $\div$ (2.14) are supplemented by the additional constraints (3.6). We can also state that there are three constraints imposed on airframe attitude, Eqs (2.13) and (3.6), governed by coordinated deflections of δ_a, δ_e and δ_r, and one constraint imposed on the flying speed, Eq (2.14), regulated by T changes. The realization of constraints (3.6) ensures then the realization of trajectory constraints (2.12). In this way, the six-degree-of-freedom aircraft can be controlled explicitly by the four control parameters.

3.2. Recapitulation

Let us introduce the following vectors of algebraic variables $\mathbf{y}$, differential variables $\mathbf{z}$, and control variables $\mathbf{u}$

$$\mathbf{y} = [\phi_A, \theta_A, \psi_A, \alpha, \beta]^\top \qquad \mathbf{z} = [p, q, r]^\top \qquad \mathbf{u} = [\delta_a, \delta_e, \delta_r, T]^\top \tag{3.7}$$

After setting $s = \widehat{s}(t)$, $\dot{s} = v = \dot{\widehat{s}}(t)$ and $\ddot{s} = \ddot{\widehat{s}}(t)$, and then $\widehat{\mathbf{r}} = \mathbf{r}[\widehat{s}(t)]$ and $\widehat{\mathbf{r}}' = \mathbf{r}'[\widehat{s}(t)]$, the six algebraic equations: (2.13), (3.1) (which stand for the two requirements on θ_A and ψ_A) and (3.5) and (3.6) can be symbolically rewritten as

$$\mathbf{F}(\mathbf{y}, \mathbf{z}, \mathbf{u}, t) = \mathbf{0} \tag{3.8}$$

Note that Eqs (2.13) and (3.1) can be solved separately for $\widehat{\beta}(t)$, $\widehat{\theta}_A(t)$ and $\widehat{\psi}_A(t)$ or $\widehat{\phi}_A(t)$, $\widehat{\theta}_A(t)$ and $\widehat{\psi}_A(t)$, depending on the requirement of Eq (2.13) which is in use. The kinematic relation (2.5) can then be rewritten as three equations of the following symbolic form

$$\mathbf{z} = \mathbf{G}(\mathbf{y}, \dot{\mathbf{y}}) \tag{3.9}$$

Finally, the dynamic equation $(2.9)_2$ constitutes three differential equations of the symbolic form

$$\dot{\mathbf{z}} = \mathbf{H}(\mathbf{y}, \mathbf{z}, \mathbf{u}, t) \tag{3.10}$$

Taken together, Eqs (3.8) $\div$ (3.10) form twelve differential-algebraic equations (DAEs) in twelve variables $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{u}$, called *equations of program motion*. The index of DAEs (3.8) $\div$ (3.10) is three (Brenan et al., 1989).

3.3. Numerical procedure

The solution to DAEs (3.8) $\div$ (3.10) is formed by program variations of the state variables $\widehat{y}(t)$ and $\widehat{z}(t)$, and control $\widehat{u}(t)$ that ensures realization of the program. In order to obtain this solution, Gear's approach can be used (Brenan et al., 1989). Given $\mathbf{y}_n$, $\mathbf{z}_n$ and $\mathbf{u}_n$ at time t_n, using the Euler backward method, the values $\mathbf{y}_{n+1}$, $\mathbf{z}_{n+1}$ and $\mathbf{u}_{n+1}$ at time $t_{n+1} = t_n + \Delta t$, can be found as a solution to the following set of non-linear algebraic equations

$$
\begin{aligned}
&\mathbf{F}(\mathbf{y}_{n+1},\mathbf{z}_{n+1},\mathbf{u}_{n+1},t_{n+1}) = \mathbf{0}\\
&\mathbf{z}_{n+1} - \mathbf{G}\Big(\mathbf{y}_{n+1},\frac{\mathbf{y}_{n+1}-\mathbf{y}_n}{\Delta t}\Big) = \mathbf{0}\\
&\frac{\mathbf{z}_{n+1}-\mathbf{z}_n}{\Delta t} - \mathbf{H}(\mathbf{y}_{n+1},\mathbf{z}_{n+1},\mathbf{u}_{n+1},t_{n+1}) = \mathbf{0}
\end{aligned}
\tag{3.11}
$$

In this way the solution can be advanced from time t_n to t_{n+1}. If necessary, in order to improve numerical accuracy, higher-order backward difference approximation methods (Brenan et al., 1989) can be applied.

It may be worth noting that, in the developed model, there is only a weak coupling between Eqs $(3.11)_1$ and $(3.11)_3$ through the control parameters $\mathbf{u}$. The jet thrust T is represented only in Eqs *(3.5)* and *(3.6)*, *included in Eq* $(3.11)_1$, while the effect of control surface deflections is manifested mainly in the dynamic equations of rotational motions $(2.9)_1$, included in Eq $(3.11)_3$, and their effect on aerodynamic forces (aerodynamic coefficients) in Eqs (3.5) and (3.6), and thus in Eq $(3.11)_1$, is usually small or even vanishing (as it was assumed in Blajer and Parczewski (1990)). The above procedure results then in the exact solutions for $\widehat{\mathbf{y}}(t)$ and $\widehat{T}(y)$, based mainly on Eq $(3.11)_1$, and approximated solutions for $\widehat{\mathbf{z}}(t)$ and $\widehat{\delta}(t)$, obtained from Eqs $(3.11)_{2,3}$. Note also that neither the numerical error accumulates in time nor the approximations influence the solution. For these reasons, the simplest Euler backward difference method, indicated in Eqs (3.11), was used to solve the problem at hand.

4. Case study

The proposed mathematical model has been used for digital simulation of a prescribed trajectory flight of an Unmanned Aerial Vehicle (UAV). The mission is to pass successively through a set of points. For large scale missions, the points are specified by their geographic coordinates and QNH heights (above the sea level). Taking into account the ellipsoidal shape of the Earth, the points are then identified in a geocentric cartesian coordinate system, and in this frame the trajectory is interpolated/approximated by spline functions. For a given UAV position, the functions $\widehat{\mathbf{r}}^{(I)}(s)$, $\widehat{\mathbf{r}}'^{(I)}(s)$ and $\widehat{\mathbf{r}}''^{(I)}(s)$ are thus referred to a local **(I)** frame, with $O_I x_I y_I$ plane tangent to the Earth ellipsoid.

For small scale manoeuvres, the modelling and interpolation/approximation of the desired trajectory can directly be completed in one **(I)** frame.

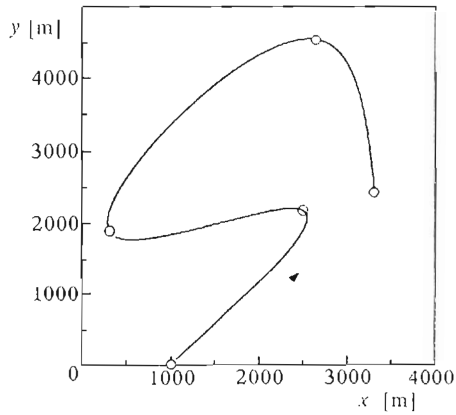

Fig. 4. Specified trajectory

In the considered case, the trajectory was sketched by five *way points* (shown in Fig.4), and then interpolated through these points by using cubic spline functions (see Blajer and Parczewski (1990) for details). The aircraft was then required to fly along the modelled trajectory at a constant speed $v_c = 180\,\mathrm{km/h}$, and the condition of coordinated turns $(\widehat{\beta} = 0)$ was assumed.

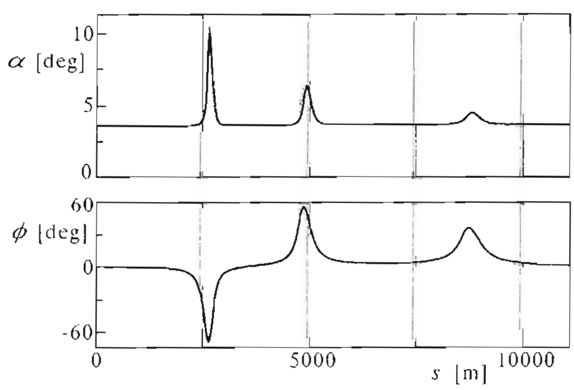

Fig. 5. Angle of attack and bank angle variations

The results of numerical simulation of the modelled UAV mission are demonstrated in Fig.5 ÷ Fig.7. As seen in the graphs, the simulated flight is characterized by five phases of practically steady motion, separated by three

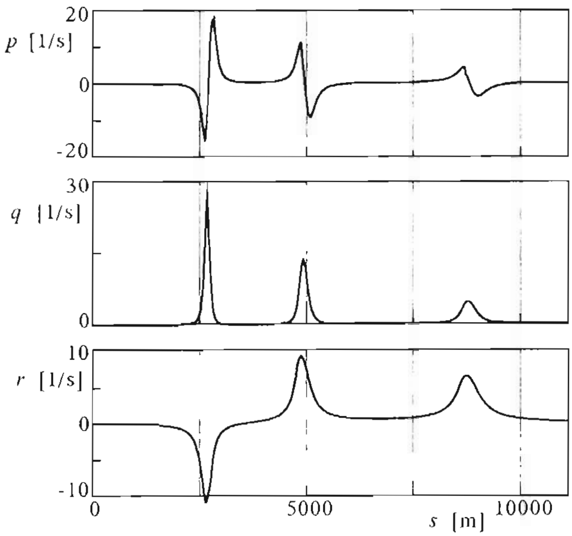

Fig. 6. Angular velocity variations

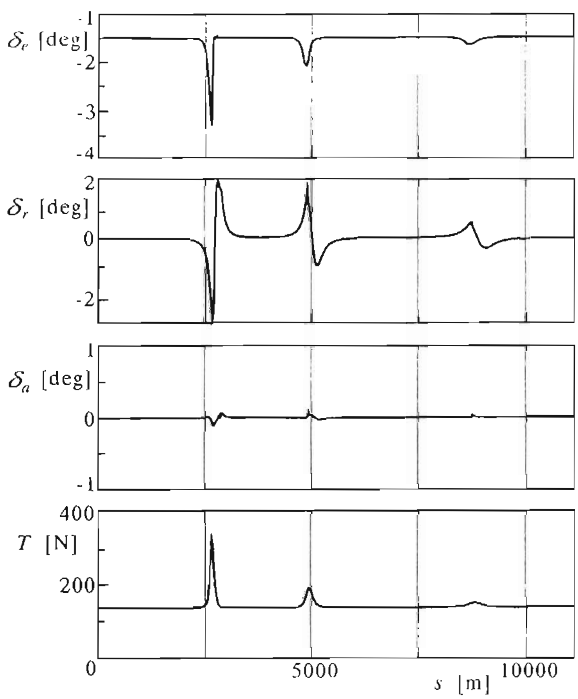

Fig. 7. Changes of control parameters

highly non-linear phases in the neighbourhood of the inner *way points* indicated in Fig.4. The extreme flight conditions at those places are, in part, due to that cubic splines were used to interpolate the trajectory. The produced changes of the trajectory curvature $(\sqrt{x''^2 + y''^2 + z''^2})$ are shown in Fig.8, and the maximal values appear at the inner way points. Using five-order splines to construct the trajectory would probably smooth the curvature variations and, consequently, the changes of state variables and control parameters. Similar effects could possibly be achieved by setting additional points to sketch the trajectory more "rounded", and using additional points as approximation points (Blajer and Parczewski, 1990). This problem will not however be discussed here.

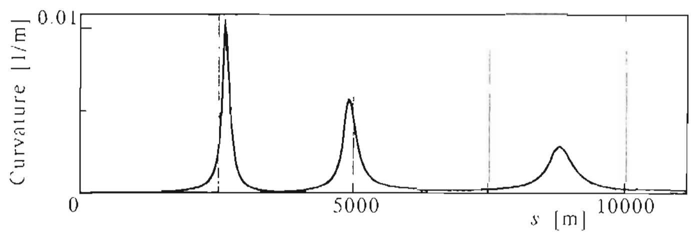

Fig. 8. Changes of trajectory curvature

5. Conclusion

A unified and general model of aircraft prescribed trajectory flight is developed. Given program of motion defined in Eqs (2.12) ÷ (2.14), consequent variations of aircraft state variables in the specified motion as well as the demanded control can be determined. Such results can be useful for at least two reasons:

- To study the nature of an UAV mission and to examine its feasibility. On this basis the mission model can be improved/optimized. In future research, the mathematical model will be generalized by considering the wind drift.

- To develop flight control systems based upon the non-linear inverse dynamics, which may provide an improved level of safety and performance over the conventional designs. The obtained variations of program (nominal) control can serve as feedforward control, while in future work it is planned to construct a feedback control loop to stabilize the prescribed trajectory path flight.

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Programowany ruch i sterowanie UAV w zadanej misji

Streszczenie

Praca dotyczy zagadnienia symulacji odwrotnej ruchu samolotu. Żąda się, by samolot wykonywał lot po zadanej trajektorii przestrzennej (dwa więzy programowe) a dodatkowymi wymaganiami są narzucone zmiany konfiguracji płatowca względem trajektorii oraz prędkości lotu. Dyskutuje się sposób modelowania tak określonego ruchu programowego oraz przedstawia metodę wyznaczania przebiegów w czasie zmiennych stanu tego ruchu oraz wymaganego sterowania samolotem. Zamieszcza się wyniki symulacji numerycznych.

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