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MULTI-MODE BEAM THEORY WITH CONSTRAINED CROSS-SECTIONAL WARPING

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The paper presents a generalized model of the bending of beams (with arbitrary compact cross-sections) due to the effect of constrained cross-sectional warping. The longitudinal displacement is given in the form of a finite series of power functions, satisfying the appropriate boundary conditions imposed on the free surface of a beam. The assumption of functional relationship between the bending angle, shearing angles, and slope of the deformed centroid of the beam has been abandoned. Taking into consideration the influence of non-uniform and constrained shearing, self-balanced shearing forces are derived. Using the example of some elementary problems, it was possible to investigate the effect of the assumptions made on free vibration frequencies and phase velocities of wave propagation. The assumed model enables frequency branches of a simply supported beam, and a finite number of phase velocity modes to be determined. Some more accurate numerical values of the shear thickness mode parameter have also been calculated.

Key words: beams, constrained warping, dynamics

1. Introduction

The application of some generalized equations of elasticity theory, when solving practical problems relevant to beams of finite length and variable cross-sections is, in practice, difficult and very often impossible. However, exact solutions relating to particular cases of beam geometry provide valuable criteria for assessment of the usability and accuracy of various approximated models. On the basis of the Pochhammer-Chree theory (Chree, 1889; Pochhammer, 1876) of the wave propagation in an infinitely long bar of circular cross-section, it is known (Abramson, 1957) that:

- Dispersion of the waves propagated occurs
- There are numerous branches of phase velocities
- In the case of very small wavelengths, all the phase velocity branches attain distorsion velocities, except for the lowest branch, which achieves Rayleigh's surface wave velocity
- Shape of the cross-sectional warping depends on the wave length.

The above qualitative data are used for a modification of technical theories of beams with arbitrary cross-sections. Timoshenko (1921) in his pioneering work, took into consideration the influence of the shearing forces upon bending vibrations of a beam when it had a uniform rectangular cross-section. The introduction of this correction to Rayleigh's model (Reyleigh, 1945) has made it possible to foresee the wave dispersion and the existence of two forms of wave propagation. The Timoshenko model is connected with free warping of beam cross-sections. In many other papers e.g. in Aalami and Atzori (1974), Cowper (1966), Levinson (1981), Stephen and Levinson (1979), Volterra and Zachmanoglou (1957) more improved theories were formulated of beams with rectangular or arbitrary cross-sections. None of them differ in their nature from the Timoshenko theory, providing qualitatively the same results. Quite new formulations have also appeared describing the vibrations of short beams of narrow rectangular cross-section (Janecki, 1977; Murty, 1970). Apart from the bending moment and transverse force, some additional self-balanced internal forces have been introduced. Expanding the beam longitudinal displacement, by appropriate finite power series, it was possible to obtain the equations enabling determination of any arbitrary finite number of free vibration frequency branches. In these equations the constrained warping of the beam cross-sections was taken into account. Janecki (1977) introduced a new additional parameter defining the shape of warping. Bickford (1982) basing on the assumptions made by Levinson (1981), using a variational method, derived the equations of motion of the bent beam, taking into consideration the constrained warping of the cross-sections. Unexpectedly, we obtained some incorrect results, in particular, those relating to phase velocities of the wave propagation (Levinson, 1985). Some imperfections of his considerations were explained by Janecki (1998). In this paper, making use of the assumed displacements it was possible to derive the equations of motion of a beam with arbitrary cross-sections. It has been shown that in the case of constrained cross-sectional warping, we have an additional equation for self-balanced shearing moment. There also appears an extra, third branch of free vibration frequencies for a simply supported beam or the third branch of the wave propagation phase velocity. Ewing (1990), making use of similar assumptions related to the displacement field, analyzed the effect of the constrained warping of cross-sections upon the free vibration frequencies of the cantilever beam. However, he did not go into the essence of bending theory for a beam affected by constrained cross-sectional warping.

In this paper a generalized model is given of the bent beam, where the effect of the constrained shearing is taken into consideration. Using examples of some elementary problems, the influence of the assumptions made was analyzed with regard to free vibration frequencies and phase velocities of elastic wave propagation.

2. Basic assumptions

In the determination of the beam model, definition of the internal forces, and derivation of the motion equation, the following assumptions will be made:

- Straight beam of uniform, bisymmetrical cross-sections is cosidered
- Material of the beam is homogeneous and isotropic, and subject to Hooke's law
- Beam moves in one plane
- Deformations and displacements are small
- Tangential stresses acting within the cross-sectional plane, and the direct stresses acting in a normal direction to the cross-section, will be regarded as significant, with the remaining ones as secondary, which will be neglected.

In compliance with these assumptions, the displacement components of an arbitrary material point are taken in the form of

$$u_1 = 0 \qquad\qquad u_2 = u_0(x_3, t)$$
$$u_3 = x_2\omega(x_3, t) + \sum_{m=1}^{M} \chi_m(x_1, x_2)\gamma_m(x_3, t) \tag{2.1}$$

where

ω, γ_m	–	unknown functions defining the rigid rotation and the cross-sectional warping
χ_m	–	known functions satisfying appropriate boundary conditions and characterizing the warping distribution in a specified beam cross-section
x_1, x_2	–	are the coordinates of the material point in the cross-section.

Confining ourselves to the linearized theory, the components of the strain tensor are

$$\varepsilon_{13} = \frac{1}{2}\sum_{m=1}^{M} \frac{\partial \chi_m}{\partial x_1}\gamma_m \qquad \varepsilon_{23} = \frac{1}{2}\Big(u_0' + \omega + \sum_{m=1}^{M} \frac{\partial \chi_m}{\partial x_2}\gamma_m\Big)$$
$$\varepsilon_{33} = x_2\omega' + \sum_{m=1}^{M} \chi_m\gamma_m' \tag{2.2}$$

where $(\cdot)' = \partial/\partial x_3$. The remaining components of strain are equal to zero. Under the action of external loads internal forces appear in the material of the beam. In the assumptions made with regard to the beam material, some significant components of the stress tensor are

$$\tau_{13} = 2G\varepsilon_{13} \qquad \tau_{23} = 2G\varepsilon_{23} \qquad \tau_{13} = E\varepsilon_{33} \tag{2.3}$$

where G and E are the elasticity constants of the beam material.

3. Equations of beam bending

With the above assumptions, the equations of beam bending will be derived by applying some general equations of slender bodies described by an arbitrary one-dimensional model of the continuum (Janecki, 1998). They are expressed in the following form

$$\frac{\partial \boldsymbol{H}}{\partial x_3} - \boldsymbol{Q} + \boldsymbol{h} = \boldsymbol{0} \tag{3.1}$$

Vectors of the internal forces, with the assumption of small deformation, are defined by the relations

$$\boldsymbol{H} = \int_A (\mathbf{T}\boldsymbol{e})\mathbf{U}\; dA \qquad \boldsymbol{Q} = \int_A \mathop{\mathrm{tr}}_{(1,3)}(\mathbf{T}\nabla\mathbf{U})\; dA \tag{3.2}$$

The vector of body loadings due to the beam motion is

$$\boldsymbol{h} = -\varrho \int_A \mathsf{U}^\top \ddot{\boldsymbol{u}}\, dA \tag{3.3}$$

In the above relationships T, $\boldsymbol{e}$ and ϱ are the stress tensor, the unit vector tangent to the beam axis, and specific density of the beam material, respectively. Moreover, matrix $\mathsf{U} = \partial \boldsymbol{u}/\partial \boldsymbol{q}$, where $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{q}$ is the vector of the generalized coordinates. Assuming, in the case under consideration, that

$$\boldsymbol{q} = [u_0, \omega, \gamma_1, \gamma_2, ..., \gamma_M]^\top \tag{3.4}$$

we have

$$\mathsf{U} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & \chi_1 & \dots & \chi_M \end{bmatrix} \tag{3.5}$$

Using Eqs (3.2) and (3.3) we arrive at the definitions of the internal forces components

$$Q = \int_A \tau_{23}\, dA \qquad M = \int_A x_2 \tau_{33}\, dA$$
$$H_n = \int_A \chi_n \tau_{33}\, dA \qquad G_n = \int_A \frac{\partial \chi_n}{\partial x_2} \tau_{23}\, dA \qquad n = 1, 2, \dots, M \tag{3.6}$$

and the body loadings

$$q = -\varrho \int_A \ddot{u}_2\, dA \qquad m = -\varrho \int_A x_2 \ddot{u}_3\, dA \qquad h_n = -\varrho \int_A \chi_n \ddot{u}_3\, dA \tag{3.7}$$

where Q and M are the shear force and the bending moment, H_n and G_n are moments and forces, due to the constrained shearing.

The general equations (3.1) provide us with the scalar equations of the beam motion

$$Q' + q = 0 \qquad M' - Q + m = 0 \qquad H_n' - G_n + h_n = 0 \tag{3.8}$$

Making use of Eqs (2.2), (2.3) and (3.6), for internal forces we have

$$
\begin{aligned}
Q &= GA\Big[\Big(\frac{\partial u}{\partial \xi} + \omega + \sum_{m=1}^{M} \gamma_m\Big) - \sum_{m=1}^{M} \kappa_m \gamma_m\Big] \\
M &= \frac{EJ}{L}\frac{\partial}{\partial \xi}\Big(\omega + \sum_{m=1}^{M} \eta_m \gamma_m\Big) \\
H_n &= \frac{EJ}{L}\frac{\partial}{\partial \xi}\Big(\eta_n \omega + \sum_{m=1}^{M} \varepsilon_{mn} \gamma_m\Big) \\
G_n &= GA\Big[(1-\kappa_n)\Big(\frac{\partial u}{\partial \xi} + \omega + \sum_{m=1}^{M} \gamma_m\Big) + \sum_{m=1}^{M} (k_{mn} - \kappa_m)\gamma_m\Big]
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
\kappa_m &= \frac{1}{A}\int_A \Big(1 - \frac{\partial \chi_m}{\partial x_2}\Big)\, dA \qquad\qquad \eta_m = \frac{1}{J}\int_A x_2 \chi_m\, dA \\
\varepsilon_{mn} &= \frac{1}{J}\int_A \chi_m \chi_n\, dA \\
k_{mn} &= \frac{1}{A}\int_A \Big[\frac{\partial \chi_m}{\partial x_1}\frac{\partial \chi_n}{\partial x_1} + \Big(1 - \frac{\partial \chi_m}{\partial x_2}\Big)\Big(1 - \frac{\partial \chi_n}{\partial x_2}\Big)\Big]\, dA
\end{aligned}
\tag{3.10}
$$

are the cross-sectional characteristics connected with shearing and L, A and J are the lenght of the beam, area and second moment of inertia of the cross-section, respectively.

With regard to the body loadings we have

$$
\begin{aligned}
q &= -\varrho A \frac{C_E^2}{L}\frac{\partial^2 u}{\partial \tau^2} \qquad\qquad m = -\varrho J \frac{C_E^2}{L^2}\frac{\partial^2}{\partial \tau^2}\Big(\omega + \sum_{m=1}^{M} \eta_m \gamma_m\Big) \\
h_n &= -\varrho J \frac{C_E^2}{L^2}\frac{\partial^2}{\partial \tau^2}\Big(\eta_n \omega + \sum_{m=1}^{M} \varepsilon_{mn} \gamma_m\Big)
\end{aligned}
\tag{3.11}
$$

where $\xi = x_3/L$, $\tau = (C_E/L)t$, $u = u_0/L$, $C_E^2 = E/\varrho$, and ϱ is the specific density of the beam material.

In order to interpret Eqs (3.9) and (3.11), for bending and shearing moments, the following averaged values will be introduced (Janecki, 1998)

$$
\Phi = \frac{1}{J}\int_A x_2 u_3\, dA \qquad\qquad \Psi_n = \frac{1}{J}\int_A \chi_n u_3\, dA
\tag{3.12}
$$

Making use of Eq (2.1) we obtain

$$\Phi = \omega + \sum_{m=1}^{M} \eta_m \gamma_m \qquad\qquad \Psi_n = \eta_n \omega + \sum_{m=1}^{M} \varepsilon_{mn} \gamma_m \tag{3.13}$$

The first relation is the mean rotation angle of freely warping cross-section of the beam introduced by Cowper (1966). The second relation describes the mean rotation angles of the constrained warping cross-sectional surface. If $M = 1$, we obtain some relationships given by Janecki (1998).

Applying the formulae for the internal forces (3.9) and loadings (3.11), the equations of motion (3.8) can be presented in the form of

$$\begin{aligned}
&\frac{\partial}{\partial \xi}\Big[\frac{\partial u}{\partial \xi} + \omega + \sum_{m=1}^{M}(1-\kappa_m)\gamma_m\Big] - \frac{E}{G}\frac{\partial^2 u}{\partial \tau^2} = 0\\
&\frac{\partial^3}{\partial \xi^3}\Big[\omega + \sum_{m=1}^{M}\eta_m\gamma_m\Big] - \frac{\partial^2}{\partial \tau^2}\Big[\lambda^2 u + \frac{\partial}{\partial \xi}\Big(\omega + \sum_{m=1}^{M}\eta_m\gamma_m\Big)\Big] = 0\\
&\frac{\partial^3}{\partial \xi^3}\Big[\eta_n\omega + \sum_{m=1}^{M}\varepsilon_{mn}\gamma_m\Big] - \lambda^2\frac{G}{E}\frac{\partial}{\partial \xi}\sum_{m=1}^{M}(k_{mn} - \kappa_m\kappa_n)\gamma_m +\\
&-\frac{\partial^2}{\partial \tau^2}\Big[\lambda^2(1-\kappa_n)u + \frac{\partial}{\partial \xi}\Big(\eta_n\omega + \sum_{m=1}^{M}\varepsilon_{mn}\gamma_m\Big)\Big] = 0
\end{aligned} \tag{3.14}$$

after eliminating the shearing force Q from Eqs (3.8) for moments M and H_n. Here, $\lambda = L\sqrt{A/J}$ is the slenderness ratio of the beam and $(n = 1, 2, \dots, M)$.

To solve the above equations it is still necessary to have some boundary conditions.

$$\begin{array}{ll}
\text{At the free end :} & Q = M = H_n = 0\\
\text{At the fixed end:} & u = \omega = \gamma_n = 0\\
\text{At the simply supported end:} & u = M = H_n = 0
\end{array} \tag{3.15}$$

Janecki (1998) has shown that in the case of bending, if $M = 1$ the relationship $\Gamma = \partial u/\partial \xi + \omega + \gamma = 0$ is fulfilled with some approximation, exept in the nearest vicinity of the fixed beam's end. As a result of the above, let us investigate the case where the following relationship occurs

$$\frac{\partial u}{\partial \xi} + \omega + \sum_{m=1}^{M} \gamma_m = 0 \tag{3.16}$$

Making use of (3.16) in Eqs (3.8) defining the shearing forces Q and G_n, and proceeding as in the case of Eqs (3.14), we obtain the equations for the bending and the shearing moments in the form

$$\frac{\partial^4}{\partial\xi^4}\Big(\omega + \sum_{m=1}^{M}\eta_m\gamma_m\Big) + \lambda^2\frac{\partial^2}{\partial\tau^2}\Big(\omega + \sum_{m=1}^{M}\gamma_m\Big) - \frac{\partial^4}{\partial\xi^2\partial\tau^2}\Big(\omega + \sum_{m=1}^{M}\eta_m\gamma_m\Big) = 0$$

$$\frac{\partial^4}{\partial\xi^4}\Big(\eta_n\omega + \sum_{m=1}^{M}\varepsilon_{mn}\gamma_m\Big) - \lambda^2\frac{G}{E}\frac{\partial^2}{\partial\xi^2}\sum_{m=1}^{M}k_{mn}\gamma_m + \lambda^2\frac{\partial^2}{\partial\tau^2}\Big(\omega + \sum_{m=1}^{M}\gamma_m\Big) + \tag{3.17}$$

$$-\frac{\partial^4}{\partial\xi^2\partial\tau^2}\Big(\eta_n\omega + \sum_{m=1}^{M}G_{mn}\gamma_m\Big) = 0 \qquad n = 1,2,\dots,M$$

Janecki's equations (cf Janecki, 1977, Eq 41) are given in a similar form. Inserting $\varepsilon_{mn} = 0$ and $\eta_m = 0$ we shall obtain the Timoshenko type equations with free cross-sectional warping. If $M = 1$, the classical Timoshenko equation will be obtained.

4. Warping functions and shear parameters

In Eqs (3.14) and (3.16) the shear parameters defined by Eqs (3.10) appear. To determine them it is necessary to be familiar with the warping functions χ_m of a beam with arbitrary cross-sections. For the bending of the beam in one plane, it will be assumed that

$$\chi_m = k_m\frac{GA}{EJ}\psi_m + x_2 \qquad m = 1,2,\dots,M \tag{4.1}$$

where k_m are parameters that have not yet been defined, and which are possible to determine after introduction of some additional conditions. It is desired that the functions ψ_m satisfy the following equations and conditions:

$$\begin{array}{lll} 1. & \Delta\psi_1 = 2x_2 & (x_1,x_2)\in A \\ 2. & \dfrac{\partial\psi_m}{\partial n} = 0 & (x_1,x_2)\in\partial A \\ 3. & \psi_m(-x_2) = -\psi_m(x_2) & \\ 4. & \kappa_m = k_m & \end{array} \tag{4.2}$$

where

$A, \partial A$ – area and boundary of the beam cross-section

n – unit normal to ∂A.

In the case of a narrow, rectangular cross-section and $\nu = 0$ we have

$$\psi_m = -\frac{h^3}{24}\xi\Big[2 + \sum_{n=1}^{m}\frac{(1-\xi^2)^n}{n!}\Big] \tag{4.3}$$

where $\xi = 2x_2/h$ and h is the height of the cross-section.

A similar situation arises in the case of a beam of a circular cross-section of radius a. We have

$$\psi_m = -\frac{a^3}{4}\varrho\Big[2 + \sum_{n=1}^{m}\frac{(1-\varrho^2)^n}{n!}\Big]\sin\theta \tag{4.4}$$

in the polar coordinates (r,θ), where $\varrho = r/a$.

In the general case, the shear parameters can be presented in the following form

$$\eta_m = 1 - A_m k_m \qquad k_{mn} = B_{mn}k_m k_n \qquad \varepsilon_{mn} = \eta_m\eta_n + C_{mn}k_m k_n \tag{4.5}$$

where

$$A_m = -\frac{GA}{EJ}\Big(\frac{1}{J}\int\limits_A x_2\psi_m\,dA\Big) \qquad B_{mn} = \Big(\frac{GA}{EJ}\Big)^2\Big(\frac{1}{A}\int\limits_A \nabla\psi_m\nabla\psi_n\,dA\Big)$$
$$C_{mn} = \Big(\frac{GA}{EJ}\Big)^2\Big[\Big(\frac{1}{J}\int\limits_A \psi_m\psi_n\,dA\Big) - \Big(\frac{1}{J}\int\limits_A x_2\psi_m\,dA\Big)\Big(\frac{1}{J}\int\limits_A x_2\psi_n\,dA\Big)\Big] \tag{4.6}$$

Taking advantage of the properties of the function ψ_1 and using Green's formula, we have

$$k_{1m} = 2\frac{G}{E}(1-\eta_m)k_1 \tag{4.7}$$

It is evident that if $\eta_m = 0$, then k_{1m} is the same for each $m = 1, 2, ..., M$. For a narrow, rectangular cross-section and Possion ratio $\nu = 0$, we obtain

$$\eta_1 = 1 - \frac{6}{5}k_1 \qquad \eta_2 = 1 - \frac{44}{35}k_1 \qquad \kappa_1 = k_1$$
$$\kappa_2 = k_2 \qquad \varepsilon_{11} = 1 - \frac{12}{5}k_1 + \frac{51}{35}k_1^2$$
$$\varepsilon_{12} = 1 - \frac{44}{35}k_2 - \frac{6}{5}k_1 + \frac{161}{105}k_1k_2 \qquad \varepsilon_{22} = 1 - \frac{88}{35}k_2 + \frac{1867}{1155}k_2^2 \tag{4.8}$$
$$k_{11} = \frac{6}{5}k_1^2 \qquad k_{12} = \frac{44}{35}k_1k_2 \qquad k_{22} = \frac{422}{315}k_2^2$$

Whereas, for a circle

$$\eta_1 = 1 - \frac{7}{6}k_1 \qquad \eta_2 = 1 - \frac{29}{24}k_2 \qquad \kappa_1 = k_1$$

$$\kappa_2 = k_2 \qquad \varepsilon_{11} = 1 - \frac{7}{3}k_1 + \frac{11}{8}k_1^2$$

$$\varepsilon_{12} = 1 - \frac{7}{6}k_1 - \frac{29}{24}k_2 + \frac{343}{240}k_1k_2 \qquad \varepsilon_{22} = 1 - \frac{29}{12}k_2 + \frac{119}{80}k_2^2 \tag{4.9}$$

$$k_{11} = \frac{7}{6}k_1^2 \qquad k_{12} = \frac{29}{24}k_1k_2 \qquad k_{22} = \frac{19}{15}k_2^2$$

Parameters k_m, which have not yet been identified, appearing in the warping functions, can be determined from the condition $\eta_m = 0$. Then the mean angle of rotation of the warping cross-section is equal to the bending angle (3.13), as it takes place in Cowper's theory (Cowper, 1966). The parameters $k_m = k_{Cm}$ introduced in this way will be called – Cowper's shear coefficients.

For successive modes we shall have, in the case of narrow rectangle: $k_{C1} = 0.833333$, $k_{C2} = 0.795456$, $k_{C3} = 0.787500$, $k_{C4} = 0.786071$, and with respect to a circle: $k_{C1} = 0.857143$, $k_{C2} = 0.827587$, $k_{C3} = 0.821918$, $k_{C4} = 0.820981$.

5. Vibrations of a simply supported beam

The solution of the equations of the beam bending vibration, satisfying appriopriate boundary conditions, will be looked for in the form

$$u = U\sin\alpha_j\xi\cos p\tau \qquad \omega = \Omega\cos\alpha_j\xi\cos p\tau$$
$$\gamma_m = \Gamma_m\cos\alpha_j\xi\cos p\tau \qquad j = 1,2,\dots \qquad m = 1,2,\dots,M \tag{5.1}$$

where $\alpha_j = \pi j$ and $p = 2\pi fL/C_E$ is a non-dimensional, circular free vibration frequency. The equations of motion (3.14), with regard to constrained warping of the beam cross-sections can be expressed in the form of the matrix system

$$(\mathbf{A} - p^2\mathbf{B})\boldsymbol{x} = \mathbf{0} \tag{5.2}$$

where

$$\boldsymbol{x} = [U, \Omega, \Gamma_1, \Gamma_2, \dots, \Gamma_M]^\top \tag{5.3}$$

The elements of the **A** and **B** matrices are given by the formulae

$$\begin{array}{lll}
a_{11} = \alpha_j^2 & a_{12} = \alpha_j & a_{1,m+2} = \alpha_j(1-\kappa_m) \\
a_{r1} = 0 & a_{22} = \alpha_j^3 & a_{2,m+2} = a_{m+2,2} = \alpha_j^3\eta_m \\
\multicolumn{3}{l}{a_{2+m,2+n} = \alpha_j\Big[\alpha_j^2\varepsilon_{mn} + \lambda^2\frac{G}{E}(k_{mn} - \kappa_m\kappa_n)\Big]} \\
b_{11} = \frac{E}{G} & b_{1s} = 0 & b_{21} = \lambda^2 \\
b_{22} = \alpha_j & b_{2m,1} = -\lambda^2(1-\kappa_m) & b_{2,2+m} = b_{m+2,2} = \lambda_j\eta_m \\
b_{2+m,2+n} = \lambda_j\varepsilon_{mn} & r,s = 2,3,\dots,M+2 & m,n = 1,2,\dots,M
\end{array} \tag{5.4}$$

From the above set of equations, it is possible to compute the nondimensional free vibration frequency $k_j = p/p_{EB} = p/(\alpha_j^2/\lambda)$ of the simply supported beam, taking into account the constrained cross-sectional warping. Confining our attention to the case $M = 1$, from Eqs (5.2) and (5.4) we arrive at the equation which was given by Janecki (1998).

The set of equations (3.17) formulated on the symplifying assumption (3.16) can be expressed in the form (5.2), where

$$\boldsymbol{x} = [\Omega, \Gamma_1, \Gamma_2, \dots, \Gamma_M]^\top \tag{5.5}$$

Matrices **A** and **B** in this case are symmetrical and their elements are given as follows

$$\begin{array}{ll}
a_{11} = \alpha_j^4 & a_{m+1,1} = a_{1,m+1} = \eta_m\alpha_j^4 \\
\multicolumn{2}{l}{a_{m+1,n+1} = \alpha_j^2\Big(\alpha_j^2\varepsilon_{mn} + \lambda^2\frac{G}{E}k_{mn}\Big)} \\
b_{11} = \alpha_j^2 + \lambda^2 & b_{m+1,1} = b_{1,m+1} = \eta_m\alpha_j^2 + \lambda^2 \\
\multicolumn{2}{l}{b_{m+1,n+1} = \alpha_j^2\varepsilon_{mn} + \lambda^2}
\end{array} \tag{5.6}$$

Tables 1, 2 present the calculation results of the non-dimensional, relative frequencies $k_j^{(m)}$ of simply supported beams of narrow, rectangular crosssections. On their basis it is evident that:

- An increase of the number M of the series expansion components of the longitudinal displacement, in the case of constrained warping crosssections of the bent beam, results in
  - appearance of additional branches of vibration frequencies
  - improvement of the accuracy of the vibration frequency calculations.

Along with the increase of the number M, the frequencies of specified branch have smaller values. This is consistent with the results of Murty (1970).

- The assumption of the relationship $u' + \omega + \gamma_m = 0$ for the transverse displacement slope, bending angle and shearing angles, causes a stiffening effect of the vibrating beam. The vibration frequencies are, in this situation, slightly larger than in a general case. It refers, in particular, to free vibration frequencies of higher branches and higher modes, as well as larger beam slenderness-ratios (see also Janecki (1977), Table 2).

Table 1. Nondimensional vibration frequencies $k_j^{(m)}$ of a simply supported beam of narrow, rectangular cross-section (for $\lambda = 10$, $\nu = 0.25$)

j \ M	1	2	3	4	5
1	0.853544 6.818758 24.352601	0.853541 6.816989 17.963735 50.388645	0.853541 6.816988 17.736413 30.886771 85.744939	0.853541 6.816988 17.732802 29.333358 46.129306 130.503853	0.853541 6.816988 17.732784 29.236076 41.461010 64.248250 184.679196
3	0.506204 1.288489 2.896523	0.505991 1.288254 2.239926 5.690516	0.505991 1.288254 2.216778 3.577066 9.580704	0.505991 1.288254 2.216403 3.411169 5.223304 14.535381	0.505991 1.288254 2.216401 3.400771 4.714938 7.209049 20.544522
5	0.343473 0.692567 1.166443	0.342729 0.692482 0.956743 2.112994	0.342723 0.692488 0.949426 1.386174 3.487231	0.342723 0.692488 0.949305 1.330487 1.948865 5.257810	0.342723 0.692488 0.949039 1.326991 1.772645 2.645171 7.413717

Table 2. Nondimensional vibration frequencies $k_j^{(m)}$ of a simply supported beam of narrow, rectangular cross-section. Case $u'+\omega+\sum\gamma_i=0$ (for $\lambda=10$, $\nu=0.25$)

M / j	1	2	3	4	5
1	0.853548 6.821073	0.853541 6.816989 18.093901	0.853541 6.816988 17.738133 31.436128	0.853541 6.826988 17.732810 29.362046 47.361976	0.853541 6.816988 17.732783 29.236775 41.584467 66.361056
3	0.506605 1.288788	0.505992 1.288254 2.253165	0.505991 1.288254 2.216956 3.635860	0.505991 1.288254 2.216404 3.414234 5.357859	0.505991 1.288254 2.216401 3.400846 4.728356 7.441585
5	0.345326 0.692666	0.342735 0.692488 0.960922	0.342723 0.692488 0.949483 1.405977	0.342723 0.692488 0.949305 1.331517 1.995706	0.342723 0.692488 0.949304 1.327016 1.777285 2.727398

Putting $\eta_m=0$ and $\varepsilon_{mn}=0$ into Eqs (5.6) we obtain the matrix elements of the set of the equations for the Timoshenko model. It is possible to prove that the set can be reduced to one equation of the form

$$\alpha_j^2(\alpha_j^2-p^2)-p^2(\alpha_j^2-p^2)\frac{E}{\widehat{\kappa}_M G}-\lambda^2p^2=0 \tag{5.7}$$

where

$$\widehat{\kappa}_M=\frac{\det\mathbf{K}}{\sum\limits_{m=1}^{M}\det\mathbf{K}_m} \qquad \mathbf{K}=[k_{rs}] \qquad r,s=1,2,\dots,M \tag{5.8}$$

Matrices $\mathbf{K}_m$ are formed by substituting the unit vector $\mathbf{1}=[1,1,...,1]^\top$ into the mth column of matrix $\mathbf{K}$. In this way we have obtained the Timoshenko type equation possessing only two branches of frequencies. With regard to a beam of a narrow, rectangular cross-section, we have $\widehat{\kappa}_M=5/6$ for each M, in the case when $\eta_m=0$, $(m=1,2,...,M)$. A similar situation arises with a circular cross-section beam. This means that the vibration frequencies do not depend on m, the number of the warping functions χ_m, existing in longitudinal displacement (2.1). Hence, for free cross-sectional warping, there is no need to expand the longitudinal displacement into a series.

It should be pointed out that by inserting $\eta_m = 0$ and $\varepsilon_{mn} = 0$ into Eqs (5.4), we also arrive at the equations in the form of Eq (5.7), in which

$$\widehat{\kappa}_M = \frac{\det \overline{\mathsf{K}}}{\sum\limits_{m=1}^{M} \det \overline{\mathsf{K}}_m} \tag{5.9}$$

where $\overline{\mathsf{K}} = [k_{mn} - \kappa_m \kappa_n]$ and $\overline{\mathsf{K}}_m$ is obtained from K by substituting vector $\overline{\boldsymbol{\kappa}} = [1 - \kappa_1, ..., 1 - \kappa_M]^\top$ for the mth column. Putting $\alpha_j = 0$ into Eqs (5.2) and (5.4), after some transformations, we have the algebraic equation

$$(s-1)\det \mathsf{A} + s \sum_{m=1}^{M} \overline{\kappa}_m \det \mathsf{A}_m = 0 \tag{5.10}$$

for the determination of the roots $s = Ep^2/G\lambda^2$ for the thickness shear mode on the assumption that $\eta_m = 0$. If $\eta_m \neq 0$, the equation is more complicated. In the above equation

$$\mathsf{A} = [\overline{k}_{rs} - s\varepsilon_{rs}] \qquad \overline{k}_{rs} = k_{rs} - \kappa_r\kappa_s \qquad r,s = 1,2,\dots,M \tag{5.11}$$

Matrices A_m are derived from the matrix A by substitution vector $\overline{\boldsymbol{\kappa}} = [1 - \kappa_1, ..., 1 - k_M]^\top$ for the mth column of the matrix. For $M = 2$ this equation can be written in the form

$$\alpha s^3 - (\alpha + \beta + \varepsilon)s^2 + \Big(\varepsilon + \frac{1}{\kappa}\Big)s - 1 = 0 \tag{5.12}$$

where

$$\begin{array}{lll} \alpha = \dfrac{\det \boldsymbol{E}}{\det \mathsf{K}} & \beta = \alpha \overline{\boldsymbol{\kappa}}^\top \mathsf{E}^{-1} \overline{\boldsymbol{\kappa}} & \varepsilon = \operatorname{tr} \mathsf{E}\mathsf{K}^{-1} \\ \dfrac{1}{\kappa} = 1 + \overline{\boldsymbol{\kappa}}^\top \mathsf{K}^{-1} \overline{\boldsymbol{\kappa}} & \mathsf{E} = [\varepsilon_{rs}] & \mathsf{K} = [\overline{k}_{rs}] \\ \overline{\boldsymbol{\kappa}} = [1-\kappa_1, 1-\kappa_2]^\top & r,s = 1,2 & \end{array} \tag{5.13}$$

For example, in the case of a beam of a circular cross-section, Eq (5.12) will be

$$s^3 - 66s^2 + 480s - 360 = 0 \tag{5.14}$$

For a beam of a narrow, rectangular cross-section we have

$$s^3 - 70s^2 + 525s - 385 = 0 \tag{5.15}$$

Table 3 presents values of the parameter s for $M = 1, 2$ and 3.

Table 3. Shear thickness vibration parameter s of a simply supported beam

M	Circle	Rectangular
1	0.847933	0.822925
	14.152067	14.177075
2	0.847490	0.822467
	7.348721	7.602121
	57.803789	61.575411
3	0.847489	0.822467
	7.111097	7.405316
	20.738504	22.980809
	161.302909	178.791407

For a beam of narrow, rectangular cross-section, Leung (1990) gave the value $s = 0.8225$. Let us investigate what influence the assumption (3.16) has upon the values of parameter s. It will be easy to observe this in the simplest case when $M = 1$. Then Eq (5.10) can be written as

$$\widehat{\varepsilon} s^2 - \Big(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\Big) s + 1 = 0 \tag{5.16}$$

However, on assumption (3.16) it is

$$\Big(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\Big) s - 1 = 0 \tag{5.17}$$

For example, in the case of a beam with circular cross-section $(\widehat{\varepsilon} = 1/12$, $\widehat{\kappa} = 6/7)$ from Eq (5.17) $s_1 = 0.8479$, while from Eq (5.18) $s_1 = 0.8$. The assumption (3.16) results, in this situation, in a decrease in the value of the thickness shear mode parameter.

6. Wave dispersion in the beam

Making use of Eqs (3.8), (3.9) and (3.11), let us analyze the phase velocity variation of the wave propagation in the beam. Attention will be concentrated on the cases of simple shearing, pure shearing and the general case of wave propagation.

6.1. Simple shearing ($\omega = 0$)

In this situation Eqs (3.8) for the transverse force and the shearing moments, after taking into account the appropriate relations given by Eqs (3.9) and (3.11), can be expressed in the following form

$$\begin{gathered} \Box^* u + \sum_{m=1}^{M} \overline{\kappa}_m \frac{\partial \gamma_m}{\partial \xi} = 0 \\ \frac{\partial}{\partial \xi} \sum_{m=1}^{M} \varepsilon_{mn} \Box \gamma_m - \lambda^2 \frac{G}{E} \sum_{m=1}^{M} \overline{k}_{mn} \frac{\partial \gamma_m}{\partial \xi} - \lambda^2 \overline{\kappa}_m \frac{\partial^2 u}{\partial \tau^2} = 0 \end{gathered} \tag{6.1}$$

for $n = 1, 2, \dots, M$, where $\overline{\kappa}_m = 1 - \kappa_m$, $\overline{k}_{mn} = k_{mn} - \kappa_m \kappa_n$ and

$$\Box^* = \frac{\partial^2}{\partial \xi^2} - \frac{E}{G} \frac{\partial^2}{\partial \tau^2}$$

The solution to the problem is sought in the form

$$u = \mathrm{i}U \exp\Big[\mathrm{i}\Big(\frac{2\pi L}{\Lambda}\Big)(\xi - c\tau)\Big] \qquad \gamma_m = \Gamma_m \exp\Big[\mathrm{i}\Big(\frac{2\pi L}{\Lambda}\Big)(\xi - c\tau)\Big] \tag{6.2}$$

where $c = c_p/c_E$, $c_E = \sqrt{E/\varrho}$; c_p – phase velocity, Λ – wavelength.

Then the nondimensional phase velocity c can be determined from a type (5.2) equation, where

$$\boldsymbol{x} = [U, \Gamma_1, \Gamma_2, \dots, \Gamma_M]^\top \tag{6.3}$$

The non-zero elements of matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\begin{array}{lll} a_{11} = 1 & a_{1,1+m} = \overline{\kappa}_m & a_{1+m,1+n} = \varepsilon_{mn} R^2 + \frac{G}{E} \overline{k}_{mn} \\ b_{11} = \frac{E}{G} & b_{m+1,1} = -\overline{\kappa}_m & b_{1+m,1+n} = \varepsilon_{mn} R^2 \end{array} \tag{6.4}$$

where $m, n = 1, 2, \dots, M$, $R = 2\pi r/\Lambda$, $r = \sqrt{J/A}$.

For $M = 1$ we have the equation

$$\widehat{\varepsilon} \frac{E}{G} R^2 c^4 - \Big[\widehat{\varepsilon}\Big(1 + \frac{E}{G}\Big) R^2 + \frac{1}{\widehat{\kappa}}\Big] c^2 + \Big(\frac{G}{E} + \widehat{\varepsilon} R^2\Big) = 0 \tag{6.5}$$

where $\widehat{\varepsilon} = \varepsilon_{11}/\overline{k}_{11}$, $1/\widehat{\kappa} = 1 + \overline{\kappa}_1^2/\overline{k}_{11}$. For very long waves ($R = 0$) the nondimensional phase velocity $c = \sqrt{\widehat{\kappa} G/E}$, i.e., it is approximately equal to the Rayleigh wave propagation velocity. In the case of very short waves ($R = \infty$) the first branch of the phase velocity has a horizontal asymptote $c = \sqrt{G/E}$, while with regard to the second phase velocity its horizontal asymptote is $c = 1$. Neglecting the effect of the constrained warping of the beam cross-sectional ($\widehat{\varepsilon} = 0$), from Eq (6.5) we have $c = \sqrt{\widehat{\kappa} G/E}$.

6.2. Pure shearing $(u = 0)$

In this case Eqs (3.8) related to the bending moment and shearing moments can be presented as

$$
\begin{aligned}
&\Box\omega + \sum_{m=1}^{M} \eta_m \Box\gamma_m - \lambda^2 \frac{G}{E}\omega - \lambda^2 \frac{G}{E} \sum_{m=1}^{M} (1-\kappa_m)\gamma_m = 0 \\
&\eta_n \Box\omega + \sum_{m=1}^{M} \varepsilon_{mn} \Box\gamma_m - \lambda^2 \frac{G}{E}(1-\kappa_n)\omega + \\
&-\lambda^2 \frac{G}{E} \sum_{m=1}^{M} (k_{mn} - \kappa_n - \kappa_m + 1)\gamma_m = 0
\end{aligned}
\tag{6.6}
$$

Assuming that

$$
\omega = \Omega \exp\Big[\mathrm{i}\Big(\frac{2\pi L}{\Lambda}\Big)(\xi - c\tau)\Big] \qquad \gamma_m = \Gamma_m \exp\Big[\mathrm{i}\Big(\frac{2\pi L}{\Lambda}\Big)(\xi - c\tau)\Big] \tag{6.7}
$$

it is possible to determine the non-dimensional phase velocity from the equation in the form of (5.2), where

$$
\boldsymbol{x} = [\Omega, \Gamma_1, \Gamma_2 \ldots, \Gamma_M] \tag{6.8}
$$

The non-zero elements of matrics **A** and **B** are given in the formulae

$$
\begin{aligned}
&a_{11} = 1 + \frac{E}{G}R^2 \qquad a_{1,1+m} = a_{1+m,1} = (1-\kappa_m) + \eta_m \frac{E}{G}R^2 \\
&a_{1+m,1+n} = (k_{mn} - \kappa_m - \kappa_n + 1) + \varepsilon_{mn}\frac{E}{G}R^2 \\
&b_{11} = \frac{E}{G}R^2 \qquad b_{1,1+m} = b_{1+m,1} = \eta_m \frac{E}{G}R^2 \\
&b_{1+m,1+n} = \varepsilon_{mn}\frac{E}{G}R^2 \qquad m,n = 1,2,\ldots,M
\end{aligned}
\tag{6.9}
$$

Hence, for $M = 1$ we arrive at the equation

$$
\widehat{\varepsilon}\Big(\frac{E}{G}\Big)^2 R^4(1-c^2) + \Big(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\Big)\frac{E}{G}R^2(1-c^2) + 1 = 0 \tag{6.10}
$$

where

$$
\widehat{\varepsilon} = \frac{\varepsilon_{11} - \eta_1^2}{k_{11} - \kappa_1^2} \qquad \frac{1}{\widehat{\kappa}} = 1 + \frac{(1-\kappa_1-\eta_1)^2}{k_{11} - \kappa_1^2} \tag{6.11}
$$

In this way, we obtain the two branches of phase velocities for pure shearing

$$c_{1,2} = \sqrt{1 + \frac{G}{2\widehat{\varepsilon}E}\Big[\Big(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\Big) \mp \sqrt{\Big(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\Big)^2 - 4\widehat{\varepsilon}}\Big]\frac{1}{R^2}} \tag{6.12}$$

with the horizontal asymptote $c = 1$.

Hence, in the case of unconstrained cross-sectional warping ($\widehat{\varepsilon} = 0$), there is only one branch

$$c = \sqrt{1 + \frac{\widehat{\kappa}G}{E}\frac{1}{R^2}} \tag{6.13}$$

This equation approximates fairly well the phase velocities of the first branch of solutions of Eq (5.2), where we are dealing with pure shearing, taking into account the constrained cross-sectional warping.

Table 4. Nondimensional phase velocity c of the wave propagation in a circular cross-section beam (for $\nu = 0.29$)

$M = 5$	a/Λ					
	0.1	0.2	0.4	0.6	0.8	1.0
Simple shear	0.576961	0.578536	0.583356	0.588484	0.592904	0.596499
	5.354921	2.816647	1.656574	1.334152	1.200327	1.132713
	8.512862	4.344444	2.339812	1.729841	1.456966	1.311390
	11.870360	5.998431	3.122469	2.211727	1.786345	1.550332
	18.385678	9.233860	4.698058	3.220127	2.504612	2.092109
	52.937090	26.482838	13.269966	8.878270	6.691765	5.387227
Pure shear	2.080441	1.353536	1.099097	1.045204	1.025672	1.016505
	5.376499	2.824302	1.656554	1.332361	1.198350	1.130958
	8.518402	4.346354	2.339380	1.728544	1.455395	1.309821
	11.872965	5.999319	3.122172	2.210878	1.785214	1.549088
	18.387665	9.234531	4.697780	3.219327	2.503455	2.090709
	52.937883	26.483105	13.269842	8.877908	6.691202	5.386484
General shear	0.268045	0.407562	0.513494	0.551526	0.570162	0.581107
	2.140447	1.411818	1.130606	1.062727	1.036479	1.023746
	5.379068	2.828883	1.662798	1.338202	1.203182	1.134815
	8.519031	4.347556	2.341408	1.730926	1.457783	1.312033
	11.873257	5.999887	3.123210	2.212232	1.786730	1.550643
	18.387888	9.234967	4.698616	3.220503	2.504898	2.092341
	52.937988	26.483287	13.270191	8.878416	6.691878	5.387315

6.3. The general case of wave propagation

Eqs (3.14) shall be used to describe this case. Their solutions are sought in the form of Eqs (6.2) and (6.7). Then, the nondimensional phase velocities can be calculated from Eq (5.2) with the vector $\boldsymbol{x}$ defined by Eq (5.3). The elements of matrices **A** and **B** in the case under consideration, are expressed by the following formulae

$$
\begin{array}{lll}
a_{11} = R & a_{12} = -1 & a_{1,2+m} = -\overline{\kappa}_m \\
a_{1+m,1} = 0 & a_{22} = R^2 & a_{2,2+m} = a_{2+m,2} = \eta_m R^2 \\
b_{11} = \frac{E}{G} R & b_{1,1+m} = 0 & a_{2+m,2+n} = \varepsilon_{mn} R^2 + \overline{k}_{mn} \frac{G}{E} \\
b_{21} = R & b_{22} = R^2 & b_{2,2+m} = b_{2+m,2} = \eta_m R^2 \\
b_{2+m,1} = \overline{\kappa}_m R & b_{2+m,2+n} = \varepsilon_{mn} R^2 & \overline{\kappa}_m = 1 - \kappa_m \\
\overline{k}_m = k_{mn} - \kappa_m \kappa_n & m, n = 1, 2, \dots, M &
\end{array}
\tag{6.14}
$$

Hence, for $M = 1$ we have the equation in the form (Janecki, 1998)

$$
\begin{aligned}
&\widehat{\varepsilon}\frac{E}{G}(1 - c^2)\Big[\frac{E}{G}c^4 - \Big(1 + \frac{E}{G} + \frac{1}{R^2}\Big)c^2 + 1\Big] + \\
&+\frac{1}{R^2}\Big[\frac{E}{\widehat{\kappa}G}c^4 - \Big(1 + \frac{E}{\widehat{\kappa}G} + \frac{1}{R^2}\Big)c^2 + 1\Big] = 0
\end{aligned}
\tag{6.15}
$$

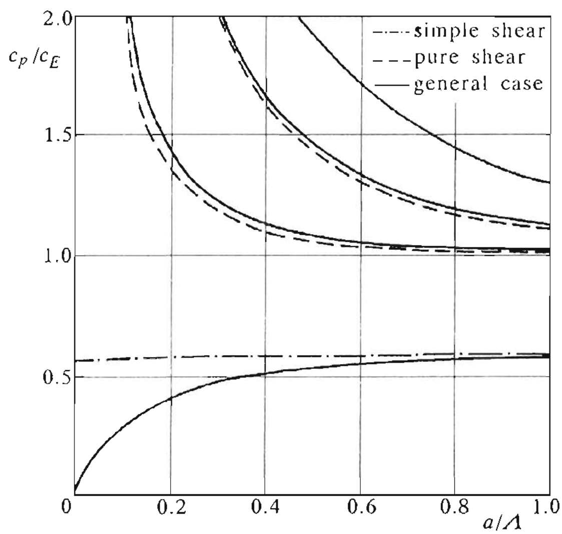

Fig. 1. Phase velocity branches for elastic waves in a circular cross-section beam with the effect of constrained warping

The shear coefficients $\widehat{\kappa}$ and $\widehat{\varepsilon}$ are given by Eq (6.11). Table 4 presents the calculation results of the nondimensional phase velocities c, related to the simple and pure shearing, and in a general case with regard to a beam of circular cross-section and radius equal to a, as well the assumption that $\eta_m = 0$. Some identical data can be obtained when $\eta_m \neq 0$. The calculation results given in Tables 4 are also presented in Fig.1. On their basis it is evident that:

- An increase in the number M of the expansion components in the series of longitudinal displacement, in the case of a constrained cross-sectional warping of the beam, results in
  - appearance of additional branches of wave propagation phase velocities associated with shearing;
  - calculation accuracy improvement of the phase velocities.
- The values of the phase velocities of their corresponding solution branches (of the wave propagation equation for simple and pure shearing), and with respect to the general case, are close to each other. In the latter case the phase velocities have the highest magnitudes.

7. Final remarks

The paper presents a one-dimensional technical theory of beams, enabling the determination of an arbitrary finite number of free vibration frequency branches and phase velocity branches of wave propagation. Obtaining an arbitrary number of solution branches has made it possible to expand the longitudinal displacement into a finite series, and to take into account the constrained cross-sectional warping caused by shearing. The assumed model presents some progress in the construction of one-dimensional technical theories of the beam. Making use of the beam model, described above, in the calculations of the wave propagation it was possible to prove that in the case of very short wavelengths, the phase velocities of branches related to shearing attain the velocities of longitudinal waves, and not the distortion velocities, as in the Pochhammer-Chree theory. Hence, there arises a need to construct a more improved beam theory, that would be consistent with the solution properties of the exact theory.

References

1. AALAMI B., ATZORI B., 1974, Flexural Vibrations and Timoshenko's Beam

Theory, *American Institute of Aeronautics and Astronautics Journal*, **12**, 5, 679-685

2. ABRAMSON H.N., 1957, Flexural Waves in Elastic Beams of Circular Cross Section, *The Journal of the Acoustical Society of America*, **29**, 1, 42-46
3. BICKFORD W.B., 1982, A Consistent Higher Order Beam Theory, *Developments in Theoretical and Applied Mechanics*, **11**, 137-150
4. CHREE C., 1889, The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Coordinates. Their Solution and Application, *Transactions of the Cambridge Philosophical Society*, **14**, 250
5. COWPER G.R., 1966, The Shear Coefficient in Timoshenko's Beam, *Journal of Applied Mechanics*, **33**, 2, 325-340
6. EWING M.S., 1990, Another Second Order Beam Vibration Theory: Explicit Bending Warping Flexibility and Restraint, *Journal of Sound and Vibration*, **137**, 1, 43-51
7. JANECKI S., 1977, Free Vibrations of Beams Influenced by the Shear Effect, (in Polish), *Mechanika Teoretyczna i Stosowana*, **15**, 2, 247-263
8. JANECKI S., 1998, The Effect of Constrained Cross-Sectional Warping on the Bending of Beams, *Journal of Theoretical and Applied Mechanics*, **36**, 1, 121-144
9. LEUNG A.Y.T., 1990, An Improved Third Order Beam Theory, *Journal of Sound and Vibration*, **142**, 3, 527-528
10. LEVINSON M., 1981, A New Rectangular Beam Theory, *Journal of Sound and Vibration*, **74**, 1, 81-87
11. LEVINSON M., 1985, On Bickford's Consistent Higher Order Beam Theory, *Mechanics Research Communications*, **12**, 1, 1-9
12. MURTY A.V.K., 1970, Vibrations of Short Beams, *American Institute of Aeronautics and Astronautics Journal*, **1**, 8, 34-38
13. POCHHAMMER L., 1876, Über die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiszylinder, *Zeitschrift für reine und angewandte Mathematik*, **81**, 324-336
14. RAYLEIGH J.W.S., 1945, *The Theory of Sound*, **1**, 293-294, Dover Publications, New York
15. STEPHEN N.G., LEVINSON M., 1979, A Second Order Beam Theory, *Journal of Sound and Vibration*, **67**, 3, 293-305
16. TIMOSHENKO S.P., 1921, On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars, *Philosophical Magazine*, **41**, 744-746

17. VOLTERRA E., ZACHMANOGLOU E.C., 1957, Free and Forced Vibrations of Straight Elastic Bars According to the Method of Internal Constraints, *Ingenieur Archiv*, **25**, 6, 424-436

Wielomodowa teoria belki ze skrępowaną deplanacją przekrojów poprzecznych

Streszczenie

W pracy przedstawiono uogólniony model zgięcia belek, o jednospójnych przekrojach poprzecznych, uwzględniając wpływ skrępowanego ścinania. Pole przemieszczeń wzdłużnych przedstawiono w postaci skończonej sumy funkcji potęgowych, spełniających odpowiednie warunki brzegowe na powierzchni swobodnej belki. Odstąpiono od założenia funkcyjnego związku pomiędzy kątem zgięcia, kątami ścinania i spadkiem odkształconej centroidy belki. Uwzględniono wpływ nierównomiernego i skrępowanego ścinania, wprowadzając samozrównoważone siły ścinania.

Na przykładach elementarnych zadań zbadano wpływ przyjmowanych założeń na częstości drgań własnych i prędkości fazowe propagacji fal. Przyjęty model umożliwia otrzymanie skończonej, dowolnej ilości gałęzi częstości drgań własnych belki swobodnie podpartej i skończonej liczby modów prędkości fazowych. Obliczono dokładniejsze wartości liczbowe parametru "shear thickness modes".

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