GRADIENT GENERALIZATION TO INTERNAL STATE VARIABLES AND A THEORY OF SUPERFLUIDITY

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Main properties of liquid helium regarded as starting points of the model developed in the paper are presented. Thermomechanical balance equations together with the second law of thermodynamics are discussed in the framework of a gradient generalization of the internal state variable approach. A comparison is made between the present model and the Landau's one; moreover some main physical properties of helium II are analyzed in the framework of the present approach.

Key words: super fluid helium, internal state variable, thermodynamics

1. Introduction

Superfluid helium II is the only known fluid that is able to remain liquid at temperatures close to zero degrees. Moreover it exhibits some other fascinating properties due to quantum effects which become apparent at the macroscopic scale. Such properties have been first observed in pioneering experiments by Kapitza (1938), Peshkov (1944), Allen and Jones (1938), Allen and Misener (1938) and then investigated, from the theoretical point of view by Landau (1941), London (1939), and Tisza (1938). It is generally accepted that the most fundamental aspect so of the behaviour of this liquid can be explained by using the so-called Landau two fluids model. According to this representation helium II is regarded as a mixture of two different liquids: a normal fluid with mass density ρ_n, velocity v_n and normal viscosity, and a superfluid

one, having density ρ_s, velocity $\boldsymbol{v}_s$, vanishing viscosity and extremely high thermal conductivity. The following relations are assumed to hold

$$\rho = \rho_s + \rho_n$$

$$\rho \boldsymbol{v} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n$$

(1.1)

where ρ and $\boldsymbol{v}$ stand for the mass density and velocity field of the mixture regarded as a whole, respectively. At the λ-point (2.17 K) we have only normal fluid so that $\rho_n/\rho = 1$ while at 0 K all the helium is in the superfluid state and $\rho_n/\rho = 0$. Some motivations of Landau's idea can be found by regarding helium II as a degenerate Bose gas in which the Bose-Einstein condensation has set in (Greco and Müller, 1984).

Dozens of papers have been written on that subject but here we consider only the approaches directly related to the principles of continuum mechanics. Let us start with the paper by Greco and Müller (1984), who approached the problem in the spirit of the extended thermodynamics. Their aim was to show that the extended thermodynamics of a degenerate gas offers an alternative to Landau's model of superfluid. However, their results are only partially consistent with the experiments. The reason for this discrepancy must be sought in the fact that superfluid helium is only a first crude approximation of a Bose-Einstein condensate. On the same line is the paper by Mongiovi (1991), where it is shown that at least the main properties of superfluid helium can be found by imposing the entropy conservation condition in the extended thermodynamics of a single fluid. Let us quote now two papers in which the Landau idea of two fluids is accepted. Atkin and Fox (1984), proposed a set of constitutive equations for a class of materials in which the independent variables were density, temperature and some generalized velocity field together with its gradient. Their theory encompasses the Landau's one as a special case. Finally, one should mention the non-local theory of superfluidity by Fabrizio and Gentili (1987). Those authors accepted exactly the Landau model as it is. However, for the superfluid part of helium they introduce a particular constitutive equation of the differential type, which is equivalent to a non-local relation between stress and density.

Another method used Bekarevich and Khalatnikov (1961), who presented a phenomenological deduction of the equations of motion in He II based on the conservation laws. In their derivation two main assumptions concerning the forms of flux of energy and momentum are accepted. Moreover, dissipative processes are taken into account. As one of consequences an evolution equation governing the changes of the vorticity $\boldsymbol{\omega} = \mathrm{rot}\boldsymbol{v}_v$ has been derived.

In the present paper we propose a continuum model of the liquid helium II based on the model with internal state variables and with the vanishing vorticity. In fact, as pointed out by Dreyer (1983), any mixture of two fluids may have some thermodynamic properties of helium II. Its behaviour may be described by a two fluid hydromechanics whenever some additional quantities appear as an argument of state functions (see also Fiszdon, 1992).

According to such a point of view, we introduce in the constitutive equations a vector quantity h as the spatial gradient of a scalar internal variable γ, is such a way that the so-called counterflow velocity $w = v_n - v_s$ is related to h. Indeed, to such a velocity are related most of the characteristic properties of superfluid helium II. Our model occurs to be less general than that by Atkin and Fox. However, it is wide enough to contain the Landau's one. Furthermore, it describes the properties of liquid helium in the whole range of temperatures below the λ-point, taking into account also the possibility of dissipative effects.

To make the contents of the paper better understandable for more readers we are giving some general introductory ideas concerning the thermodynamic model used in the present paper. The model with internal state variables may been known in the thermodynamics of continua for some 30 years (cf Coleman and Gurtin, 1967; Valanis, 1967) and was firstly introduced in order to describe a response of a visco-elastic material. In general the model is obtained by postulating (or better to say – by enriching) the description of the state, given in terms of the so-called conservative state variables (in our case ρ, ϑ, c_s) by additional quantities called *internal state variables*. Besides the constitutive relations, certain kinematic equation for the internal variables are postulated. The additional equations are evolutionary differential equations of order one. The right-hand side of that equation has to be postulated in a way similar to the that used for constitutive response functions. In this way governed by the constitutive relation the response of the material becomes not only a function of the (conservative) state variable but also a functional of their histories, since solutions of those additional kinematic equations depends not only on the initial conditions but also on the history of the state variables.

In 1989 the first author Kosiński (1989) introduced, for the first time, a gradient of a scalar internal state variable β as a state variable in response functions of a thermoelastic material. In the course of derivation of consequences of the laws of thermodynamics he obtained a modified Fourier's type law and finite speeds of propagation of thermal and thermomechanical waves. Postulated by Kosiński (1989) the right-hand side of the evolution equation for the scalar internal state variable was a real-valued function of the temperature ϑ and β. In this way, the internal state variable has represented a history of

the (thermodynamic) temperature.

The paper is organized as follows. In Section 2 we give a short resumé of the main properties of liquid helium, in order to point out the experimental starting points of the theory. Section 3 is devoted to introduction of meaningful physical quantities and then to writting the thermomechanical balance equations together with the second law of thermodynamics. Then in Section 4 the second law of thermodynamics is exploited, in order to derive some analytical restrictions on the constitutive equations. Finally, in Sections 5 and 6 a comparison is made between the present model and the Landau's one; moreover some main physical properties of helium II are analyzed in order to determine material functions.

2.   Some properties of helium II

Now we shall give a short review of the main properties of helium below the temperature $\vartheta_\lambda = 2.17\,\mathrm{K}$. We refer to classical books on superfluid hydrodynamics, as Wilks (1967), and Putterman (1974). Helium liquefies at $4.2\,\mathrm{K}$ and between that temperature and $2.17\,\mathrm{K}$ (λ-point) it behaves as an ordinary fluid, called helium I. However, below the λ-point helium passes to a new phase, helium II, which presents some unusual thermomechanical properties. For instance, it is able to flow through a narrow capillary ($\simeq 10^{-4}\,\mathrm{cm}$ in diameter) without measurable resistance. This phenomenon is known as the *fountain effect*. This property can be observed by connecting with a narrow capillary, called superleak, the two vessels – both containing helium II. Heating one of those leads to a flow of the liquid from one to another vessel and in consequence to *cooling* the second container. Then the temperature of the fluid in the first vessel is found to be higher. This apparent paradox can be explained by supposing that helium is capable ofo flowing in the capillary without carrying entropy. The fluid remaining behind has the same total entropy distributed over smaller amount of a substance; hence it has a higher temperature. In the same way the fluid in the second vessel conserves the same total entropy distributed over more mass and then it is cooled.

3.   Balance equations and the second law of thermodynamics

From now on we will regard our material as a binary mixture of fluids.

Hence, we postulate balance laws for mass, linear momentum and energy for the mixture as a whole together with a law of mass balance for each constituent, (Gurtin and Vargas, 1984). Eqs (1.1) are supposed to be valid. Due to gradient of concentrations of each constituent, diffusion of matter takes place in the fluid. In the classical theory of mixtures, account is taken of this diffusion, either by supposing that the entropy flux may differ (Müller, 1968). from q/ϑ, where q and ϑ mean the heat flux and the absolute temperature, respectively, or by introducing an energy extra flux l. which allows for an additional mechanical energy (Gurtin and Vargas, 1971). Since these choices are mathematically equivalent, we may adopt the second one without loosing the generality. Let B_t represent the moving fluid an the actual configuration. The vector position of the points of B_t (in the Eulerian description) will be denoted by x while $v = dx/dt$ denotes the velocity field defined on B_t. Moreover, let $\phi(x,t)$ be a scalar density defined on B_t, $i(x,t)$ and $R(x.t)$ be two fields defined on ∂B_t and B_t, representing the influx and supply of ϕ, respectively. We accept the following integral balance equation

$$\frac{d}{dt}\int_{B_t}\phi\,dv + \int_{\partial B_t}i\cdot n\,ds - \int_{B_t}R\,dv = 0 \qquad (3.1)$$

where n means the outer normal to the boundary ∂B_t. The localization procedure, in the case of differentiable fields ϕ and i, leads to

$$\dot{\phi} + \phi\,\mathrm{div}\,v + \mathrm{div}\,i - R = 0 \qquad (3.2)$$

where div is the divergence operator calculated with respect to Eulerian coordinates and

$$\dot{\phi} = \frac{\partial\phi}{\partial t} + v\cdot\mathrm{grad}\phi \qquad (3.3)$$

Owing to Eq (3.2), the complete set of local balance laws for the whole mixture, regarded as a single body, is

$$\dot{\rho} + \rho\,\mathrm{div}\,v = 0$$
$$\rho\dot{v} = \mathrm{div}\,\mathsf{T} + \rho b \qquad (3.4)$$
$$\rho\frac{d}{dt}\left(\epsilon + \frac{1}{2}v\cdot v\right) - \mathrm{div}(v\mathsf{T}) + \mathrm{div}(q + l) - \rho b\cdot v - \rho r = 0$$

where
ρ	–	mass density of B_t
T	–	Cauchy stress tensor
b	–	body force
ϵ	–	specific internal energy
q	–	heat flux vector
r	–	body heat supply.

As far as the second law is concerned, Eq (3.2) must be understood as an inequality; then we get the unilateral differential constraint

$$\rho\dot{\eta} + \operatorname{div}\frac{\boldsymbol{q}}{\vartheta} - \frac{\rho r}{\vartheta} \geq 0 \tag{3.5}$$

where η means the specific entropy.

Besides the above equations we postulate the balance of mass for each constituent in the form

$$\rho\dot{c}_s + \operatorname{div}[\rho_s(\boldsymbol{v}_s - \boldsymbol{v})] = m_s$$
$$\rho\dot{c}_n + \operatorname{div}[\rho_n(\boldsymbol{v}_n - \boldsymbol{v})] = m_n \tag{3.6}$$

In Eqs (3.6) $c_s \equiv \rho_s/\rho$ and $c_n \equiv \rho_n/\rho$ represent the mass concentrations of the superfluid and normal components, respectively, while m_s and m_n are the mass productions of the same components. In the theory of reacting fluid mixtures m_s and m_n should take into account the mass production of each constituent due to chemical reactions. Although in liquid helium II there are no chemical reactions, the fractions of normal and super components are varying with temperature so that an actual mass production takes place. Since the total mass remains constant, we have $m_s = -m_n$.

One should note that Eqs (3.6) are not independent since Eq $(3.6)_2$ follows from Eqs $(3.4)_1$, $(3.6)_1$ and $(1.1)_1$. Henceforth, to describe our fluid besides the variables ρ and $\boldsymbol{v}$, we have introduced two additional scalar fields, ρ_s, ρ_n and two vectors $\boldsymbol{v}_s$, $\boldsymbol{v}_n$, but only five additional equations given by Eqs (1.1) and $(3.6)_1$. Usually, in theories on superfluidity an extra balance equation for the linear momentum of the superfluid component is postulated. This equation is of the type

$$\frac{D_s}{Dt}\boldsymbol{v}_s = \boldsymbol{f} + \boldsymbol{b} \tag{3.7}$$

where

$$\frac{D_s}{Dt}\boldsymbol{v}_s = \frac{\partial \boldsymbol{v}_s}{\partial t} + \boldsymbol{v}_s \cdot \operatorname{grad}\boldsymbol{v}_s \tag{3.8}$$

denotes the convected (material) time derivative based on the velocity field $\boldsymbol{v}_s$. The vector $\boldsymbol{f}$ represents a driving force acting on the atoms of superfluid matter and its form must be determined by a constitutive equation. For instance, the Landau conjecture is that $\boldsymbol{f}$ is due to the gradient of chemical potential while Fabrizio and Gentili postulate the non-local relation

$$\operatorname{div}\boldsymbol{f} = l\rho \tag{3.9}$$

where l is a suitable material parameter.

In the present paper we follow a slightly different procedure. First of all, we postulate the existence of a superfluid velocity field $\boldsymbol{v}_s$. Moreover, if

$$x_i' = Q_{ij}(t)x_j + c_i(t) \tag{3.10}$$

where $Q_{ij}(t)$ are the Cartesian components of an orthogonal tensor-valued function and $c_i(t)$ are the components of a vector-valued function of time t, that represent a change of the frame of reference, then $\boldsymbol{v}_s$ transforms as an absolute velocity, i.e.

$$v_{si}' = Q_{ij}v_{sj} + \Omega_{ij}(x_j - c_j) + \dot{c}_i \tag{3.11}$$

with

$$\Omega_{ij} = \dot{Q}_{ik}Q_{jk} \tag{3.12}$$

Furthermore, the evolution of the scalar field $\gamma(\boldsymbol{x}, t)$ regarded as an internal state variable (and at the same fine a potential for $\boldsymbol{h}$) is goverend by an initial value problem

$$\dot{\gamma} = H(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) \qquad \gamma(t_0) = \gamma_0 \tag{3.13}$$

where $H(\cdot)$ is a suitable smooth function to be specified in a similar way to other constitutive functions appearing in the model. Finally, let $\boldsymbol{h} \equiv \mathrm{grad}\gamma$. Our main assumptions are the following:

1. There exists, at least local in time, a solution $\gamma(\boldsymbol{x}, t)$ to Eq (3.13) (for any γ_0 from an open set of initial conditions to Eq (3.13))

2. The function γ and its spatial gradient $\boldsymbol{h}$ appear as state variables in constitutive equations of the model

3. The counterflow velocity $\boldsymbol{w}$ is related to the gradient $\boldsymbol{h}$ by a suitable constitutive law which will be specified later.

What actually we are doing is to considering helium II as a first gradient fluid with a scalar internal state variable, playing the role of the potential.

For the first time a consistent thermodynamic model of a gradient generalization of the internal state variable approach has been recently developed for a rigid heat conductor by Kosiński and Wojno (1995). On the other hand, it is worthwhile to notice that the concept of potential, related to the counterflow velocity has already appeared in the literature, (see e.g. Lhuillier, 1975). Moreover, we will see that such a model is too general for our scope and hence we can confine ourselves to analysis a particular case.

By taking the spatial gradient of (3.13) we get

$$\boldsymbol{h} = -h\mathsf{L} + \operatorname{grad} H \qquad (3.14)$$

where $\mathsf{L} = \operatorname{grad}\boldsymbol{v}$. Once the relation between $\boldsymbol{w}$ and $\boldsymbol{h}$ is specified, the above equation furnishes the additional kinematic relation necessary for describing a two-fluid model. This relation plays a similar role as Eq (3.9) or (3.5) do. However, at the moment our approach is more general and could be applied not only to a superfluid but to any fluid with an internal state variable the gradient of which enters the constitutive equations.

4.   Exploitation of the second law of thermodynamics

Our aim here is to develop a thermodynamic theory of the fluid under consideration. To this end we need to characterize its thermodynamic states and processes. According to our approach a *thermodynamic state* is defined by the values of the following 13 functions of $\boldsymbol{x}$ and t:
1) total velocity field $\boldsymbol{v}$, 2) counterflow velocity $\boldsymbol{w}$, 3) symmetric Cauchy stress tensor T, 4) specific body force $\boldsymbol{b}$, 5) specific internal energy ϵ, 6) heat flux vector $\boldsymbol{q}$, 7) heat supply r, 8) energy extra flux $\boldsymbol{l}$, 9) total mass density ρ, 10) superfluid mass density ρ_s, 11) specific entropy η, 12) absolute temperature ϑ, 13) scalar internal state variable γ.

Such a set of 13 fields, defined for all $\boldsymbol{x}$ in B_t and all t in some interval $[t_0, t_0 + \tau]$ of an amplitude τ, will be called a *thermodynamic process of duration* τ if and only if it is compatible with the local balance laws of mass, linear momentum and energy together with the evolution (kinetic) equation (3.13). Furthermore, a thermodynamic process is called *admissible* if it satisfies the Clausius-Duhem inequality (3.5) for all $\boldsymbol{x}$ in B_t and all t in the interval $[t_0, t_0 + \tau]$. Here, only the differentiable processes are considered, the case of non-smooth processes needs more elaborated arguments. Now, we postulate the constitutive equations

$$\begin{aligned} \psi &= \psi^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) & \eta &= \eta^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) \\ \mathsf{T} &= \mathsf{T}^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) & \boldsymbol{q} &= \boldsymbol{q}^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) \\ \boldsymbol{l} &= \boldsymbol{l}^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) & & \end{aligned} \qquad (4.1)$$

$$H = H^*(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) \qquad \boldsymbol{w} = \boldsymbol{W}(\rho, \vartheta, c_s, \gamma, \boldsymbol{h}) \qquad (4.2)$$

where ψ is the Helmoholtz free energy given by

$$\psi = \epsilon - \vartheta\eta \tag{4.3}$$

These equations require some comments. It is rather a classical result of the *continuum theory of binary mixtures that free energy, pressure and entropy* depend only on ρ and ϑ, together with the concentration of one of the constituents, (see Gurtin and Vargas, 1971). Besides the above mentioned variables, however, due to a superficial properties of liquid helium we suppose the constitutive equations to depend on the vector variable $\boldsymbol{h}$, related to the scalar internal state variable γ. Moreover, some of the properties peculiar to the superfluid state, namely those related to the assumed existence of the counterflow velocity has to be described by an additional dependence of the state function on the gradient of γ.

Let us suppose we are given at an initial time t_0, an initial value of the internal state vector $\gamma_0(\boldsymbol{x})$ for any $\boldsymbol{x}$ in B_{t_0}, and moreover the motion $\chi : B_{t_0} - B_t$, some temperature distribution, mass density and superfluid concentration on B_t for all $t \in [t_0, t_0 + \tau]$. Suppose further that $\gamma_0(\boldsymbol{x}, t)$, $\rho(\boldsymbol{x}, t)$, $c_s(\boldsymbol{x}, t)$ are smooth enough to ensure the existence of a unique solution $\gamma(\boldsymbol{x}, t)$, to Eq (3.13) for all t in $[t_0, t_0 + \tau]$ and such that $\gamma(\boldsymbol{x}, t_0) = \gamma_0$. Then we can compute $\boldsymbol{h} = \mathrm{grad}\gamma(\boldsymbol{x}, t)$ and $\boldsymbol{v} = d\chi/dt$. Moreover, owing to the balance laws together with the constitutive equation in $\boldsymbol{w}$ we can compute the fields ρ_n, $\boldsymbol{v}_n$ and $\boldsymbol{v}_s$. Hence, from the constitutive equations we get the fields η, $\epsilon = \psi - \vartheta\eta$, $\boldsymbol{\mathsf{T}}$, $\boldsymbol{q}$ and $\boldsymbol{l}$. If the constitutive laws are compatible with the Clausius-Duhem inequality then the corresponding thermodynamic process will be admissible. Thus, to any sufficiently smooth choice of γ_0, χ, ϑ, ρ, and c_s there corresponds a unique admissible thermodynamic process of duration τ, provided the constitutive laws are such that the entropy inequality is satisfied.

Let us now investigate the conditions under which our constitutive laws are compatible with the second law of thermodynamics.

To this end we follow the classical procedure introduced by Kosiński and Perzyna (1972) and by Valanis (1967) or Coleman and Gurtin (1967). First of all we may rewrite Eq (3.5) as follows

$$- \rho\dot{\psi} - \rho\eta\dot{\vartheta} + \boldsymbol{\mathsf{T}} \cdot \boldsymbol{\mathsf{L}} - \mathrm{div}\boldsymbol{l} - \frac{1}{\vartheta}\boldsymbol{q} \cdot \mathrm{grad}\vartheta \geq 0 \tag{4.4}$$

Moreover, owing to the constitutive equation (4.2), the prolongated evolution (kinetic) equation yields

$$h = -h\mathsf{L} + \operatorname{grad}H = -h\mathsf{L} + \frac{\partial H}{\partial \rho}\operatorname{grad}\rho +$$

$$+ \frac{\partial H}{\partial \vartheta}\operatorname{grad}\vartheta + \frac{\partial H}{\partial c_s}\operatorname{grad}c_s + \frac{\partial H}{\partial h}\operatorname{grad}h \tag{4.5}$$

inequality (4.4) may now be rewritten in the form

$$-\rho\left(\frac{\partial \psi}{\partial \vartheta} + \eta\right)\dot{\vartheta} - \rho\left(\frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial c_s}\dot{c}_s + \frac{\partial \psi}{\partial h}\cdot\dot{h} + \frac{\partial \psi}{\partial \gamma}H\right) + \mathsf{T}\cdot\mathsf{L} +$$

$$-\frac{\partial l}{\partial \rho}\cdot\operatorname{grad}\rho - \frac{\partial l}{\partial \vartheta}\cdot\operatorname{grad}\vartheta - \frac{\partial l}{\partial c_s}\cdot\operatorname{grad}c_s - \left(\frac{\partial l}{\partial h}\right)^{\mathsf{T}}\cdot\operatorname{grad}h - \frac{q}{\vartheta}\cdot\operatorname{grad}\vartheta \geq 0 \tag{4.6}$$

On the other hand, the balance of mass together with the prolongated kinetic equation may be used to compute the material time derivative of ρ, c_s and h; then (4.6) may be rearranged as follows

$$-\rho\left(\frac{\partial \psi}{\partial \vartheta} + \eta\right)\dot{\vartheta} + \left[\rho^2\frac{\partial \psi}{\partial \rho}\mathsf{I} - \rho_s c_s \mu_s \mathsf{I} + \rho h \otimes \frac{\partial \psi}{\partial h} + \mathsf{T}\right]\cdot\mathsf{L} +$$

$$+\left[-\rho\frac{\partial \psi}{\partial h}\frac{\partial H}{\partial \rho} + c_s\mu_s(v_s - v) - \frac{\partial l}{\partial \rho}\right]\cdot\operatorname{grad}\rho +$$

$$+\left[-\rho\frac{\partial \psi}{\partial h}\frac{\partial H}{\partial c_s} + \rho\mu_s(v_s - v) - \frac{\partial l}{\partial c_s}\right]\cdot\operatorname{grad}c_s + \tag{4.7}$$

$$-\left[\rho\frac{\partial \psi}{\partial h}\frac{\partial H}{\partial \vartheta} + \frac{\partial l}{\partial \vartheta} + \frac{q}{\vartheta}\right]\cdot\operatorname{grad}\vartheta - \left[\rho\frac{\partial H}{\partial h}\otimes\frac{\partial \psi}{\partial h} + \left(\frac{\partial l}{\partial h}\right)^{\mathsf{T}}\right]\cdot\operatorname{grad}h +$$

$$-\rho\frac{\partial \psi}{\partial \gamma}H + \mu_s m_s + \rho\mu_s c_s\operatorname{div}v_s \geq 0$$

In Eq (4.7) the symbol $\otimes$ stands for the tensor product, I is the identity tensor while $\mu_s \equiv \partial\psi/\partial c_s$ is the reduced chemical potential due to the superfluid constituent. Let $s_0 \equiv (\rho_0, \vartheta_0, c_{s0}, h_0)$ be an arbitrary point of the domain of the *response functions* ψ, η, T, q, l and H, while v_0 an arbitrary value of the velocity field. Moreover, for a given point x_0 and initial time t_0 let $\rho(x, t)$, $\vartheta(x, t)$, $c_s(x, t)$, $v(x, t)$, $h(x, t)$ be a solution to the field equations such that: $\rho(x_0, t_0) = \rho_0$, $\vartheta(x_0, t_0) = \vartheta_0$, $c_s(x_0, t_0) = c_{s0}$, $v(x_0, t_0) = v_0$, $h(x, t) = h_0$. Then the Taylor expansion of such a solution around the initial point yields

$$\rho(x, t) = \rho_0 + b(t - t_0) + r_0\cdot[x - x_0] + b\cdot[x - x_0](t - t_0) + \ldots$$

$$\vartheta(x, t) = \vartheta_0 + a(t - t_0) + g_0\cdot[x - x_0] + a\cdot[x - x_0](t - t_0) + \ldots$$

$$c_s(\boldsymbol{x}, t) = c_{s0} + c(t - t_0) + \boldsymbol{s}_0 \cdot [\boldsymbol{x} - \boldsymbol{x}_0] + \boldsymbol{c} \cdot [\boldsymbol{x} - \boldsymbol{x}_0](t - t_0) + \ldots \quad (4.8)$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_0 + \boldsymbol{V}(t - t_0) + \mathsf{L}_0[\boldsymbol{x} - \boldsymbol{x}_0] + \mathsf{A}[\boldsymbol{x} - \boldsymbol{x}_0](t - t_0) + \ldots$$

$$\operatorname{grad}\gamma = \boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{h}_0 + \mathsf{M}(t - t_0) + \mathsf{H}_0[\boldsymbol{x} - \boldsymbol{x}_0] + \mathsf{P}[\boldsymbol{x} - \boldsymbol{x}_0](t - t_0) + \ldots$$

where the meaning of the symbols is obvious. Hence for the process under consideration, Clausius-Duhem inequality at the point $\boldsymbol{x}_0$ and time t_0 yields

$$-\rho\left[\frac{\partial\psi}{\partial\vartheta} + \eta\right]_{s_0} a + \left[\rho^2\frac{\partial\psi}{\partial\rho}\mathsf{I} - \rho_s c_s \mu_s \mathsf{I} + \rho h \otimes \frac{\partial\psi}{\partial\boldsymbol{h}} + \mathsf{T}\right]_{s_0} \cdot \mathsf{L}_0 +$$

$$+\left[-\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial\rho} + c_s\mu_s(\boldsymbol{v}_s - \boldsymbol{v}) - \frac{\partial\boldsymbol{l}}{\partial\rho}\right]_{s_0} \cdot \boldsymbol{r}_0 +$$

$$+\left[-\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial c_s} + \rho\mu_s(\boldsymbol{v}_s - \boldsymbol{v}) - \frac{\partial\boldsymbol{l}}{\partial c_s}\right]_{s_0} \cdot \boldsymbol{s}_0 + \quad (4.9)$$

$$-\left[\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial\vartheta} + \frac{\partial\boldsymbol{l}}{\partial\vartheta} + \frac{q}{\vartheta}\right]_{s_0} \cdot \boldsymbol{g}_0 - \left[\rho\frac{\partial H}{\partial\boldsymbol{h}} \otimes \frac{\partial\psi}{\partial\boldsymbol{h}} + \left(\frac{\partial\boldsymbol{l}}{\partial\boldsymbol{h}}\right)^{\mathsf{T}}\right]_{s_0} \cdot \mathsf{H}_0 +$$

$$-\rho\frac{\partial\psi}{\partial\gamma}H + \mu_s m_s + \rho\mu_s c_s \operatorname{div}\boldsymbol{v}_s \geq 0$$

Here the subscript s_0 means that the response functions are evaluated at the point s_0 of their domain of definition. Due to the arbitrariness in the choice of the points $(\boldsymbol{x}_0, t_0)$ and s_0, the previous inequality must hold for any value of $a, \boldsymbol{h}_0, \mathsf{L}_0, \mathsf{H}_0, \boldsymbol{r}_0, \boldsymbol{s}_0, \boldsymbol{g}_0$, if and only if

$$\eta = -\frac{\partial\psi}{\partial\vartheta} \qquad\qquad \mathsf{T} = -\rho^2\frac{\partial\psi}{\partial\rho}\mathsf{I} + \rho_s c_s \mu_s \mathsf{I} - \rho h \otimes \frac{\partial\psi}{\partial\boldsymbol{h}}$$

$$\frac{\partial\boldsymbol{l}}{\partial\rho} = -\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial\rho} + c_s\mu_s(\boldsymbol{v}_s - \boldsymbol{v}) \qquad \frac{\partial\boldsymbol{l}}{\partial c_s} = -\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial c_s} + \rho\mu_s(\boldsymbol{v}_s - \boldsymbol{v})$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.10)$$

$$\frac{\partial\boldsymbol{l}}{\partial\vartheta} = -\rho\frac{\partial\psi}{\partial\boldsymbol{h}}\frac{\partial H}{\partial\vartheta} - \frac{q}{\vartheta} \qquad\qquad \frac{\partial\boldsymbol{l}}{\partial\boldsymbol{h}} = -\rho\left[\frac{\partial\psi}{\partial\boldsymbol{h}} \otimes \frac{\partial H}{\partial\boldsymbol{h}}\right]$$

$$-\rho\frac{\partial\psi}{\partial\gamma}H + \rho_s\mu_s\operatorname{div}\boldsymbol{v}_s - \mu_s m_s \geq 0$$

Although a number of severe restrictions has been placed on the constitutive equations by the preceding analysis, the theory is still too general and some simplification is desirable. Hence, we pursue our analysis under the additional constitutive assumptions

$$\psi = \psi_1(\rho, \vartheta, c_s) + \frac{1}{2}\psi_2(\rho, \vartheta, c_s, \gamma)|\boldsymbol{h}|^2$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.11)$$

$$H = H_1(\rho, \vartheta, c_s, \gamma) + \frac{1}{2}H_2(\rho, \vartheta, c_s, \gamma)|\boldsymbol{h}|^2$$

Finally, we specify the constitutive equation in $\boldsymbol{w}$. As a first approximation let us suppose for a while $\boldsymbol{w}$ to be proportional to $\boldsymbol{h}$, i.e.

$$h = f(\rho, \vartheta, c_s)\boldsymbol{w} \qquad (4.12)$$

Under the above hypotheses we get the following thermodynamic restrictions

$$\mathsf{T} = -\rho^2 \frac{\partial \psi}{\partial \rho} \mathsf{I} + \frac{1}{2}\rho_s c_s \mu_s \mathsf{I} - \rho \psi_2 \boldsymbol{h} \otimes \boldsymbol{h}$$

$$\frac{\partial l}{\partial \rho} = -\rho \frac{\partial H_1}{\partial \rho}\psi_2 \boldsymbol{h} - \frac{1}{2}\rho \frac{\partial H_2}{\partial \rho}|\boldsymbol{h}|^2 \psi_2 \boldsymbol{h} + c_s \mu_s (\boldsymbol{v}_s - \boldsymbol{v}) + \rho c_s \mu_s f^{-1}\frac{\partial f}{\partial \rho}\boldsymbol{w}$$

$$\frac{\partial l}{\partial c_s} = -\frac{1}{2}\rho \frac{\partial H_2}{\partial c_s}|\boldsymbol{h}|^2 \psi_2 \boldsymbol{h} + \rho \mu_s (\boldsymbol{v}_s - \boldsymbol{v}) + \rho c_s \mu_s f^{-1}\frac{\partial f}{\partial c_s}\boldsymbol{w} \qquad (4.13)$$

$$\frac{\partial l}{\partial \vartheta} = -\rho \frac{\partial H_1}{\partial \vartheta}\psi_2 \boldsymbol{h} - \frac{\boldsymbol{q}}{\vartheta} + \rho c_s \mu_s f^{-1}\frac{\partial f}{\partial \vartheta}\boldsymbol{w}$$

$$\frac{\partial l}{\partial \boldsymbol{h}} = -\rho \psi_2 H_2 \boldsymbol{h} \otimes \boldsymbol{h} - \rho c_s \mu_s f^{-1}\mathsf{I}$$

together with the reduced entropy inequality

$$\rho_s \mu_s \mathrm{div}\,\boldsymbol{v}_n - \mu_s m_s - \rho \frac{\partial \psi}{\partial \gamma}H \geq 0 \qquad (4.14)$$

Although the theory conserves a certain generality, the obtained relations allow Landau's results to be discussed satisfactory. This will be the subject of the next Section.

5. Comparison with Landau's theory

The main problem related to Landau's theory concerns the vorticity of the superfluid state. Indeed, on the basis of some microscopic considerations, (see Landau, 1941), Landau postulated the condition

$$\mathrm{rot}\boldsymbol{v}_s = \boldsymbol{0} \qquad (5.1)$$

having the meaning of a kinematic constraint. In the framework of continuum mechanics such a requirement seems to be questionable. On the other hand the relation $\mathrm{rot}\boldsymbol{w} = \boldsymbol{0}$ was adapted by Atkin and Fox (1984), who showed that it is frame-invariant (objective). In the present theory, as a consequence of our main assumption, we get

$$\mathrm{rot}\boldsymbol{h} = \boldsymbol{0} \qquad (5.2)$$

It is easily proved that owing to the property of transformation of vector v_s, the above condition results to be frame invariant. It is enough to observe that. due to (3.11)

$$w' = Qw \tag{5.3}$$

so that

$$\mathrm{rot}h' = \mathrm{rot}[f'w'] = \mathrm{rot}[fw'] = \mathrm{rot}[fQw] = Q\mathrm{rot}[fw] = Q\mathrm{rot}h \tag{5.4}$$

Finally, if f satisfies the differential equation

$$f^{-1}\mathrm{grad}f \times w + \mathrm{rot}v_n = 0 \tag{5.5}$$

then, Eq (5.1) is fulfilled. Such a condition is verified in the case considered below, if the fluid is in the superfluid state. Indeed let us pursue our analysis under the hypothesis

$$h = w \tag{5.6}$$

Note that there are still some free parameters in the theory, represented by the material functions ψ and H. To get the constitutive equations of Landau's non-dissipative theory we have to assume

$$H = \psi_1 \qquad\qquad \psi_2 = c_s c_n \tag{5.7}$$

Due to Eq (5.7), together with the thermodynamic restrictions we get

$$\psi = \psi_1(\rho, \vartheta, c_s) + \frac{1}{2}c_n c_s |w|^2$$

$$\mathsf{T} = -p\mathsf{I} - \frac{\rho_s \rho_n}{\rho} w \otimes w$$

$$p = \rho^2 \frac{\partial \psi_1}{\partial \rho} - \frac{1}{2}\frac{\rho_s^2 \rho_n}{\rho^2}|w|^2 \tag{5.8}$$

$$q = \rho_s c_n \vartheta \eta w - \vartheta \frac{\partial l}{\partial \vartheta}$$

These relations represent just Landau's two fluid model unless for the last equation (5.8) an additional term due to the vector l does not appear. Since Landau regards helium II as a non-conventional mixture, (in the sense that he does not apply the canonical balance equations of mixture theory) this term is absent in his theory.

6.   On the experimental determination of the evolution equation

In this section we will discuss determination of the RHS of the evolution equation (3.13), the most crucial property of the theory. For the class of fluids analyzed above, function H should be assigned on the basis of the experimental results at hand. Indeed, as noted by Maugin and Muschik (1994), any physically meaningfull internal state variable should be related to some detectable macroscopic quantities and, in this way, it must be considered measurable.

In the particular case of superfluid helium, by the constitutive equation (4.12) we have related this function to the velocity and density fields and then, at least from the theoretical point of view, its nature is clear. However, it is worthwhile to note that, from the experimental point of view, it could be rather difficult to project suitable measurements. Our aim here is to show how it is possible to assign the evolution equation in a particular case. Suppose for a while that function $\psi_1(\rho, \vartheta)$ could be split as follows

$$\psi_1(\rho, \vartheta) = \psi_{10}(\rho) + \psi_{11}(\vartheta) \tag{6.1}$$

Then to get the heat flux in the form of Eq $(5.8)_4$, it is enough to assume

$$H = \psi_{11}(\vartheta) \tag{6.2}$$

and our problem reduces to determination of function $\psi_{11}(\vartheta)$. This can be done owing to one of the most fascinating superfluid phenomena, the so called *fountain effect*, discovered by Allen and Jones in 1938. They observed that the level of helium contained in a vessel and communicating with a helium bath trough a narrow capillary, were higher than the level of the neighbouring bath, provided the helium in the vessel were suitably heated. According to Landau's conjecture, only the supercomponent flows trough the capillary so that all the helium in the vessel is in the superfluid state. Taking into account that the exterior force in this case is exerted by the gravitational field U, at equilibrium, the equation of motion $(3.4)_2$ yields

$$\text{grad}[p + U] = 0 \tag{6.3}$$

or, equivalently

$$p + U = 0 \tag{6.4}$$

by a suitable choice of the reference level of U. Moreover, since

$$U = \rho a_g y \tag{6.5}$$

where a_g represents the acceleration due gravitational field and y the altitude of the fluid in the vessel, we conclude that

$$\frac{p}{\rho} = -a_g y \qquad (6.6)$$

If μ denotes the chemical potential of the fluid, classical thermodynamic relations yield

$$\frac{\partial H}{\partial \vartheta} = \frac{\partial \psi}{\partial \vartheta} = -\eta = \frac{\partial(p/\rho)}{\partial \vartheta} - \frac{\partial \mu}{\partial \vartheta} = \frac{\partial(p/\rho)}{\partial \vartheta} \qquad (6.7)$$

since in the superfluid state the chemical potential remains constant (see Putterman, 1974). By combining (6.4) and (6.7) we get finally

$$\left(\frac{\partial y}{\partial \vartheta}\right)_{\mu=const} = \frac{\eta}{a_g} = \frac{1}{a_g}\frac{\partial H}{\partial \vartheta} \qquad (6.8)$$
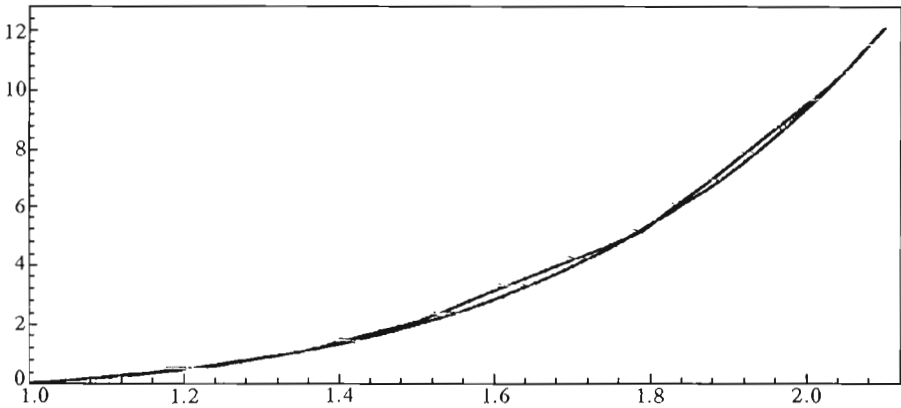

Fig. 1. Curve fitting with the help of the analytical expression
$$\eta/a_g = -0.21817 + 257529\vartheta^{5.195253}$$

Eq (6.8) shows that function H can be obtained by measurements of the altitude of the helium column in a superleak, or, equivalently, by measuring the function η/a_g. Such an experiment were performed by Kramers, Wasscher, Bots and Gorter (cf Zemanski, 1968). We have found a fit of their data by a least squares method and the fitting curve is shown in Fig.1. Its analytical expression is

$$\frac{\eta}{a_g} = \alpha + \beta\vartheta^n \qquad (6.9)$$

with

$$\alpha = -0.21817 \qquad \beta = 0.257529 \qquad n = 5.195253$$

In this way H can be determined by solving a simple differential equation.

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Gradientowe uogólnienie wewnętrznych zmiennych stanu i teoria nadciekłości

Streszczenie

Jako punkt startowy rozważanego w artykule modelu, przedstawiono podstawowe własności ciekłego helu. W ramach gradientowego uogólnienia podejścia przez wewnętrzne zmienne stanu dyskutuje się termomechaniczne równania bilansu wraz z drugim prawem termodynamiki. Porównuje się obecny model z modelem Landau'a, a ponadto analizuje się główne fizyczne własności helu II w ramach rozwiniętego podejścia.