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ON A NUMERICAL SOLUTION OF RAIL-WHEEL CONTACT PROBLEMS [1]

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We assume rail and wheel to be undeformable bodies. The motion is described by 6 variables. We postulate the existence of at least one contact point at each instant, thus introducing a nonsmooth constraint manifold. We study jump conditions for trajectories crossing the singularities in the constraint on the basis of a geometrical regularization. Numerical solutions for a wheel-set are presented.

1. Introduction

There exists a wide variety of models which describe, on different levels of resolution, the motion of railway vehicles (cf e.g., Dang van Ky (1994); Frischmuth et al. (1994); Netter and Arnold (1993)). Simple models consider just a concentrated force moving along an elastic beam, more complicated ones use the Finite Element Method for solution of impulse and energy equations in the contact zone between wheels and rails.

For certain applications – especially within simulation packages – the emphasis is placed mainly on the rigid body components of the motion. On the other hand, sufficient speed for on-line calculations is required. For this reason there are attempts to treat rail-wheel problems within the framework of multy-body systems (MBS).

For conical wheel profiles and a wheel-set as basic element of the description this approach turned out to be a good compromise between accuracy and efficiency. However, for more realistic profiles and a single wheel as basic element, we encounter practical as well as theoretical difficulties.

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Both are due to lacking regularity. The standard profiles of rails and wheels are defined piecewise as polynomials and arcs of circles. They are differentiable, but only once. This affects, of course, the performance of all solvers for the resulting ordinary differential equations or for the algebraic-differential systems.

But worse, the curvatures of realistic wheels are made to fit the rails (on the tread) and to confine the lateral motion (flange). This entails a discontinuous dependence of the geometrical contact point between both bodies on their relative placement. As a consequence, the manifold of admissible placements is nonsmooth. Moreover, the jumps of the geometrical contact point result in discontinuous changes in geometrical parameters appearing in the friction law, thus accounting for discontinuous forces.

It is easily seen that a dynamical problem on a non-differentiable constraint manifold has in general no classical solution, and there is no unique way of defining generalized (weak) solutions. On the other hand, for a deformable body, say an elastic one, with a given local friction law on the boundary, we expect unique solvability. Of course, that elastic solution depends on material constants which are not present on the rigid body level of modelling.

The aim of the present paper is to determine approximately the rigid body components of the elastic solution without accept calculating this latter solution. To this end we exploit the idea that due to the elastic deformations the elastic solution yields differentiable generalized velocities for the rigid body components. However, because of the large values of the elasticity constants, the elastic trajectories remain within a very small neighborhood of the admissible manifold for the rigid body motion. Hence we substitute for the original manifold a neighboring smooth (regularized) manifold. We prove, that this procedure leads in the limit to a unique generalized solution within a certain class of regularization methods. Moreover, in some special cases we characterize the generalized solutions by jump conditions for the velocities. This allows us to calculate numerical solutions very effectively by solving a sequence of smooth initial value problems.

2. Kinematics

Let us introduce the following notations

$\mathcal{R}$ – rail, $\mathcal{R} \subset \mathbb{E} = \mathbb{R}^3$

$\mathcal{W}$ – wheel, $\mathcal{W} \subset \mathbb{E}$

$\mathcal{W}_t$ – rail at the time t, $\mathcal{W}_t \subset \mathbb{E}$

κ – position of particle X, $\kappa : \mathcal{W} \to \mathbb{E}$, $X \mapsto x = \kappa(X)$
K – set of all placements
t – motion, $t \mapsto \kappa_t \in K$.

By definition we have

$$\mathcal{W}_t = \kappa_t(\mathcal{W})$$

Now, our approach is based on the following six assumptions.

Postulate 1. *The rail is motionless.*

Postulate 2. *The wheel is rigid.*

Hence, the actual position of a particle is given by

$$x(X,t) = \kappa_t(X) = s(t) + Q(t)(X - X_0)$$

where
s – translation of the particle X_0, $s = s(t) \in \mathbb{E}$
Q – rotation, $Q = Q(t) \in SO(\mathbb{E})$.

Hence the space of all placements K is parametrized by a set of six numbers $y \in Y = \mathbb{R}^6$

$$\kappa_t = \kappa_{y(t)} \qquad\qquad y(t) = [s_1, s_2, s_3, \varphi, \psi, \theta]^\top$$

For the velocity vector it holds

$$v(X,t) = \frac{\partial}{\partial t} x(X,t) = \dot{s}(t) + \dot{Q}(t)(X - X_0) = \dot{s}(t) + \omega \times (X - X_0)$$

with

$$\omega = \omega(\varphi, \psi, \theta, \dot{\varphi}, \dot{\psi}, \dot{\theta}) \qquad\qquad \omega_i = \frac{1}{2}\epsilon_{ijk}\dot{Q}_{kj}$$

Postulate 3. *Rail and wheel do not penetrate each other*

$$\forall t \quad \mathrm{int}\mathcal{W}_t \cap \mathrm{int}\mathcal{R} = \emptyset$$

Postulate 4. *There is a continuous contact between rail and wheel*

$$\forall t \quad \mathcal{W}_t \cap \mathcal{R} \neq \emptyset$$

Postulate 5. $\forall t$ *the set* $\mathcal{W}_t \cap \mathcal{R}$ *is finite.*

Practically $\quad \mathcal{W}_t \cap \mathcal{R} = \{x_c\}$ or $\mathcal{W}_t \cap \mathcal{R} = \{x_c, x_c'\}$

We denote by $x_c, x_c', \ldots$ the geometrical contact points.

Definition 1. *Let* $\mathcal{W}_y = \kappa_y(\mathcal{W})$. *The set*

$$M = \left\{ y \in Y \,:\, \emptyset \neq \mathcal{W}_y \cap \mathcal{R} \subset \partial\mathcal{W}_y \cap \partial\mathcal{R} \right\}$$

is called the **admissible manifold**.

We introduce the multi-valued function

$$\begin{aligned} X_c \,:\, M &\to 2^{\partial\mathcal{R}} \quad \text{contact point (or set of contact points)} \\ y &\mapsto \{x_c^l \,:\, l = 1, \ldots\} \subset \partial\mathbb{R}\,, \end{aligned}$$

which assigns the set of all geometrical contact points to a given configuration.

Now, the main problem is that M is not smooth, i.e., it is a C^0, but not C^1-manifold.

In general, it holds

$$|X_c(y)| > 1 \Rightarrow \nexists \quad T_y M$$

Thus, we define the **singular manifold** $S \subset M$ as follows:

Definition 2. $S = \left\{ y \in M \,:\, |X_c(y)| > 1 \right\}$.

There are two different reasons for the existence of singularities of the above type. First, real wheels have a flange (for theoretical purposes, sometimes conical wheels are considered), and hence the contact point can change from the tread to the flange and back. Secondly, since the wheel is designed to fit to the rail in the tread region, the distance function takes very small values in the neighborhood of the actual contact point, and is not globally convex. Thus small changes in the position y can cause sudden changes of the contact point, even within the tread. However, this second type of singularities is connected with small changes of the tangent direction.

For application to railway dynamics, the vertical direction is distinguished in the sense that we usually have

$$M = \left\{ y \in Y \,:\, s_3 = \max\{h^l(s_1, s_2, \varphi, \psi),\ l = 1, \ldots\} \right\}$$

with $h^l \in C^1$, $(l = 1, 2, \ldots)$. Indeed, for our main application, the geometry is as follows

$$\mathcal{R} = \left\{ X \in \mathbb{R}^3 \,:\, X_3 \leq r(X_2) \right\}$$

$$r : [-0.03716, 0.03716] \to \mathbb{R}$$

— rail profile, e.g., UIC60-ORE, usually rotated around X_1-axis

$$\mathcal{W} = \left\{X \in \mathbb{R}^3 : \; X_1^2 + X_3^2 \leq [0.5 + w(X_2)]^2\right\}$$
$$w : \; [-0.07, 0.06] \to \mathbb{R}$$

— wheel profile, e.g., S1002

$$\Rightarrow \quad M = \left\{y \in \mathbb{R}^6 : \; s_3 = h(s_2, \varphi, \psi)\right\}$$

Note that s_1 and θ are cyclical coordinates, hence

$$M = \widetilde{M} \times \mathbb{R}^2 \qquad \widetilde{M} = \left\{ \begin{bmatrix} s_2 \\ s_3 \\ \varphi \\ \psi \end{bmatrix} \in \mathbb{R}^4 : \; s_3 = h(s_2, \varphi, \psi) \right\}$$

For visualization we often use the ψ-cuts $\widetilde{M}_\psi$

$$\widetilde{M}_\psi = \left\{ \begin{bmatrix} s_2 \\ s_3 \\ \varphi \end{bmatrix} \in \mathbb{R}^3 : \; s_3 = h(s_2, \varphi, \psi) \right\}$$

3. Smoothing concepts

In this section we discuss several concepts for the smooth approximation of the nonsmooth admissible manifold M. Such concepts are required both for practical reasons (computational speed) and theoretical reasons (for the well-posed initial value problem to be dealt with).

There are three major variants we employ:

- Splines

- Softabs

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} \to \frac{a+b}{2} + \frac{\text{sabs}(a-b)}{2}$$

e.g., with

$$\text{sabs}(a) = \text{sabs}(a, \nu) = \frac{2}{\pi} \int_0^a \arctan \frac{\xi}{\nu} \, d\xi$$

We have: $\text{sabs}(a, \nu) \to |a|$ with $\nu \to 0$.

– Penetration

Postulate 3'. $\forall t \quad \mathrm{mes}(W_t \cap \mathcal{R}) = \epsilon = \mathrm{const}$

In all of the three approaches we are confronted with the following question: *Is there a limit of the corresponding solutions to the equations of motion?*

Here we use the following form of the equations of motion for a smooth manifold M (Euler-Lagrange equations)

$$m(y)\ddot{y} = f(t, y, \dot{y}, \lambda) + \lambda\vec{n} \qquad \lambda \geq 0 \qquad y \in M \subset \mathbb{R}^6$$
$$0 = g(y) = s_3 - h(s_2, \varphi, \psi)$$

with the vector normal to M in y

$$\vec{n} = \nabla g = \begin{bmatrix} 0 \\ -h_{s_2} \\ 1 \\ -h_\varphi \\ -h_\psi \\ 0 \end{bmatrix}$$

(normalized if necessary) and the mass matrix $m \in \mathrm{Sym}^+(\mathbb{R}^6)$.

If M is not differentiable in y the balance of momentum takes the form

$$m(\dot{y}^+ - \dot{y}^-) = \int_{0^-}^{0^+} f(t, y, \dot{y}, \lambda)\, dt + \int_{0^-}^{0^+} \lambda\vec{n}\, dt$$

Here are
λ – Dirac-function-type distribution
$\vec{n}$ – jump function.

The product on the right-hand side of the above equation is not well defined (cf Schwartz (1950); Volpert (1967)). There is a variety of *jump conditions* which are compatible with the impulse conversation:

(a) Ideal plastic impact (without friction, $\int f\, dt = 0$, $\int \lambda\vec{n}\, dt = \Lambda\vec{n}^+$)

$$\dot{y}^+ = \dot{y}^- + \Lambda m^{-1}(y)\vec{n}^+$$

$$\dot{y}^+\vec{n}^+ = 0 \Rightarrow \Lambda = -\frac{\dot{y}^-\vec{n}^+}{(m^{-1}\vec{n}^+, \vec{n}^+)} \Rightarrow \dot{y}^+ = \dot{y}^- - \frac{\dot{y}^-\vec{n}^+}{(m^{-1}\vec{n}^+, \vec{n}^+)} m^{-1}\vec{n}^+$$

(b) Regularization (with friction)

$$\dot{y}^+ = \lim_{t\to 0+} \lim_{\nu\to 0} y(t;\nu)$$

where $y(\cdot;\nu)$ – solution on the smoothed (with parameter ν, corresponding to ϵ in other papers) admissible manifold and initial velocity $\dot{y}^-$

(c) Generally

$$\dot{y}^+ = y^- + m^{-1}\vec{n} \qquad \vec{n} \in P_y M$$

where $P_y M$ – polar cone at y.

Thus we assume:

Postulate 6. *At all regular points of M the Euler-Lagrange equations apply, at all singular points there is given a mapping*

$$(t, y, \dot{y}^-) \mapsto \dot{y}^+$$

4. A test case

Let $M = c$ be a curve in $\mathbb{R}^2$, $m = I_{2\times 2}$, $v := |\dot{y}|$

$$c : y_2 = \gamma(y_1)$$

and $\gamma : \mathbb{R} \to \mathbb{R}$ piecewise smooth, i.e.,

– γ smooth in $(-\infty, 0) \cup (0, +\infty)$

– γ' jumps at $y_1 = 0$.

Let $\alpha = \arctan(\gamma'^+) - \arctan(\gamma'^-)$.
Due to the concept of plastic impact (a) it holds

$$\dot{y}^+ = \dot{y}^- - v^- \sin\alpha \vec{n}^+ \qquad v^+ = \cos\alpha v^-$$

On the other hand, the regularization technique (b) yields

$$v^+ = v^- e^{-\mu\alpha} ,$$

where μ is Coulomb's coefficient for dry friction and

$$F_t = -\mu |F_n| \frac{\dot{y}}{|\dot{y}|}$$

The latter is preferable since it includes the effect of friction and, furthermore, is more stable. In fact, it is not affected by a misinterpretation of a change in direction during the course of the numerical integration of the equations of motion.

It is worthwhile to mention that the above result can be generalized to cover the multidimensional case. For vanishing forces we have:

Lemma 1. *Each solution to the equations of motion is contained in a geodetic line of the admissible manifold M.*

Furthermore, in the presence of Coulomb friction the following lemma holds:

Lemma 2. *The solution to the equations of motion is a superposition of a monotonous rescaling of the time-scale and the solution for the non-frictional case.*

Now, let $\bar{c}$ be a plane curve, $M = \bar{c} \times \mathbb{R}^d$ and c a trajectory in M. Then

$$k = \bar{k}\cos^2\beta \qquad\qquad \beta = \arccos\frac{\bar{s}}{s}$$

where

s	–	arclength on c	k	–	curvature of c
$\bar{s}$	–	arclength on $\bar{c}$	$\bar{k}$	–	curvature of $\bar{c}$.

Theorem 1. *The kinetic energy after the "wedge" is obtained from*

$$T^+ = \exp(-2\mu\bar{\alpha}\cos\beta)T^-$$

However, for these results it is essential that the friction law is isotropic, and that there is no deviatoric part of the mass matrix.

A realistic description, however, requires the solution to (at least) a quasi-stationary variational inequality. For the sake of computational speed, there are several approaches to the decoupling of a normal and tangential problems. Finally, a local *contact zone* around the *geometrical contact point* is determined and divided into a *slip* and a *stick region*, respectively. Only in the slip region Coulomb's law holds as an equality, hence in macroscopic terms the friction law differs from that considered for the test case. Furthermove, due to the shape of the contact zone (which depends on the local geometry around the contact point), there is also an anisotropy effect.

A numerical solution to the equations of motion and the regularization concept (b) is shown below. Here we adopted data for a wheelset following Kaas-Peterson (1986), Netter and Arnold (1993), Frischmuth et al. (1994).

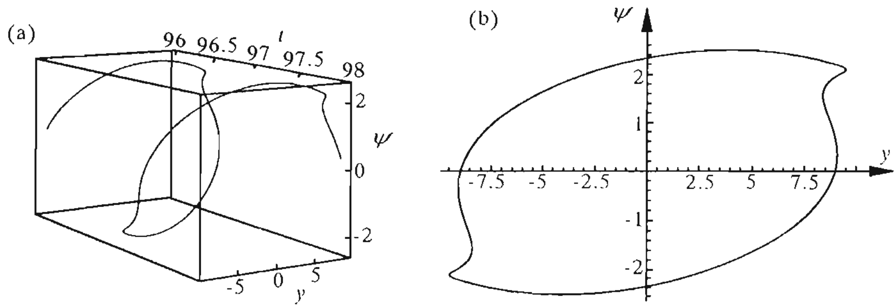

Fig. 1. (a) – Trajectory, (b) – Orbit

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Rozwiązanie numeryczne zagadnień styku koło-szyna

Streszczenie

Zakładamy, że zarówno koło, jak i szyna są ciałami nieodkształcalnymi. Ruch opisany jest 6 zmiennymi. Zakładamy istnienie przynajmniej jednego punktu styku w każdej chwili czasu, wprowadzając w ten sposób niegładką rozmaitość więzów. Badamy warunki skoku przy przejściu trajektorii przez osobliwości, opierając się na regularyzacji geometrycznej. Przedstawiono wyniki numeryczne otrzymane dla zestawu kołowego.

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