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I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 34, 1996

REGULARIZATION IN THE GOVERNING EQUATIONS OF RAILWAY WHEEL MODELS [1]

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In this article it is shown for the formulation of contact kinematics that continuity requirements are necessary. To guarantee the continuity, material damping (relaxation time τ) terms must be taken into account, i.e., the corresponding equations are regularized. Furthermore, it is shown that a boundary layer occurs always behind the point of contact (velocity v) and that the characteristic length of the boundary layer is τv. Because of the little influence of the boundary layer it can be concluded that the material damping can be dropped as a mean of regularization if the nonsmooth velocities are calculated in front of the point of contact. Thus the numerical treatment of railway wheel models can be simplified.

1. Introduction

In investigations of railway wheels running on rails their bending is frequently modelled by an elastically supported beam (e.g., the Bernoulli-Euler beam), their longitudinal deflection and their torsion by elastically supported bars (cf e.g., Bogacz et al. (1991); Brommundt (1991); Grassie et al. (1982); Grassie (1992); Knothe et al. (1994), Meywerk and Brommundt (1993); Ostermeyer (1987), (1989a,b,c); Triantafullidis and Prange (1994)). The interaction between wheel and rail is described by a contact theory, e.g., the theory of Kalker (1990), cf Gross-Thebing (1993). These descriptions permit calculating of the forces and the moment acting between wheel and rail in the contact patch which is idealized as a contact point. For these calculations it is necessary to know the creepages and the spin at the point of contact. When the

[1]The paper was presented during the First Workshop on Regularization Methods in Mechanics and Thermodynamics, Warsaw, April 27-28, 1995

point of contact moves along the rail the velocities of the deflections become discontinuous there. Thus, the creepages and the spin which depend on these velocities are not defined. Ostermeyer introduced material damping to avoid these difficulties (cf Ostermeyer (1989a,b,c)). For very small material damping there arise problems in the numerical calculations due to nearly singular (stiff) partial differential equations (the coefficient of the highest order derivative with respect to the space variable is very small).

In this article it is shown, that the regularization is necessary to calculate the spin (cf Ostermeyer (1989b)), the longitudinal creepage for longitudinal deflection (cf Ostermeyer (1989c)) and the lateral creepage for a twistable rail. Furthermore, for the three cases a perturbation technique shows that the material damping always affects a boundary layer behind the point of contact. It is demonstrated by a simple model with a moving load and a moving moment on the Bernoulli-Euler beam, that the spin at the point of contact in the damped rail is nearly the same as the spin in front of the point of contact in the undamped rail. Knowing that the boundary layer is always behind the point of contact, and that small material damping for small velocities have little influence on the results (cf Ostermeyer (1989b)), one can avoid the material damping and the numerical difficulties by calculating the creepage and spin taking the velocities in front of the point of contact.

2. Procedure of solution

To demonstrate the procedure of solution we model the rail as the Bernoulli-Euler beam which is flexible in the e_2-direction (Fig.1, bending stiffness EI, mass density μ, stiffness of the Winkler foundation k_v). The deflection of the beam in the e_2-direction is $\overline{v}_r(x,t)$. The same deflection is denoted by $v_r(\xi,t)$ as a function of the moving coordinate ξ, $\xi = x - vt$ (Fig.1). Between $\overline{v}_r$ and v_r hold the following relations

$$\overline{v}_r(x,t) \equiv v_r(\xi,t) \qquad \xi = x - vt \tag{2.1}$$

We assume the whole model to be governed by linear equations. Thus, the time can be split off by $\overline{v}_r(x,t) = \tilde{\overline{v}}_r(x)\mathrm{e}^{\lambda t}$, and we assume λ to be a given value[2]. A load $F_{c2} = \widehat{F}_{c2}\mathrm{e}^{\lambda t}$ and a moment $M_{c3} = \widehat{M}_{c3}\mathrm{e}^{\lambda t}$ move at a constant velocity v along the beam. The force F_{c2} and the moment M_{c3} represent the actions of the wheel upon the rail.

[2]The whole procedure of solution, which includes the wheel, is given by Meywerk and Brommundt (1993) or Ostermeyer (1989a)

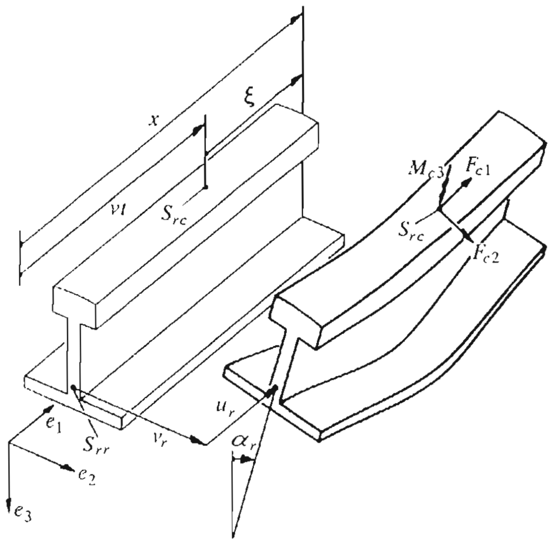

Fig. 1. The model

The equations of motion are established via the Hamilton-Ostrogradsky principle

$$\tilde{\delta} S = \int\limits_{t=t_1}^{t_2} (\delta T - \delta U + \tilde{\delta} W)\, dt = 0 \tag{2.2}$$

where

T	–	kinetic energy
U	–	potential energy
δ	–	variational operator
$\tilde{\delta} W$	–	virtual work of the non-potential forces.

Here we have

$$T = \frac{1}{2} \int\limits_{-\infty}^{\infty} \mu \dot{\overline{v}}_r^2 \, d\xi \qquad\qquad U = \frac{1}{2} \int\limits_{-\infty}^{\infty} \Big(EI \overline{v}_r''^2 + k_v \overline{v}_r^2 \Big)\, d\xi$$

$$\tilde{\delta} W = F_{c2} \delta \overline{v}_r + M_{c3} \delta \overline{v}_r' \tag{2.3}$$

and obtain for $\tilde{\delta} S$ after integration by parts and transformation[3] to the moving

[3]For details of the transformation see Brommundt (1991) or Ostermeyer (1989a)

coordinate ξ (Fig.1)

$$\tilde{\delta S} = \int\limits_{t=t_1}^{t_2} \int\limits_{-\infty}^{\infty} \Big\{ \Big(-\mu(\ddot{v}_r - 2v\dot{v}_r' + v^2 v_r'')\delta v_r - EI v_r^{IV} \delta v_r - k_v v_r \delta v_r \Big) \, d\xi + \\ + \quad EI[v_r''\delta v_r']_-^+ - EI[v_r'''\delta v_r]_-^+ + F_{c2}\delta v_r + M_{c3}\delta v_r' \Big\} \, dt = 0 \tag{2.4}$$

where

$$[f]_-^+ := \lim_{\xi \to 0+} f(\xi) - \lim_{\xi \to 0-} f(\xi) \tag{2.5}$$

From Eq (2.4) we get the jump and smoothness conditions

$$[v_r]_-^+ = 0 \qquad [v_r']_-^+ = 0 \tag{2.6}$$

$$EI[v_r'']_-^+ = -M_{c3} \qquad EI[v_r''']_-^+ = F_{c2} \tag{2.7}$$

and the field equation

$$EIv_r^{IV} + k_v v_r + \mu(\ddot{v}_r - 2v\dot{v}_r' + v^2 v_r'') = 0 \tag{2.8}$$

To solve Eqs (2.6) to (2.8) we assume

$$v_r(\xi, t) = \widehat{v}_r e^{\kappa\xi} e^{\lambda t} \tag{2.9}$$

By substituting Eq (2.9) into Eq (2.8) we obtain four characteristic values κ_k ($k = 1, \ldots, 4$) which depend on λ, $\kappa_k = \kappa_k(\lambda)$. Since the boundary conditions, $\lim_{\xi \to \pm\infty} v_r = 0$, we have to distinguish between κ_k with a positive real part and κ_k with a negative real part. Thus the solution can be expressed as

$$v_r(\xi, t) = e^{\lambda t} \begin{cases} \sum\limits_{\substack{k=1 \\ \mathrm{Re}(\kappa_k)<0}}^{4} \widehat{v}_{rk} e^{\kappa_k \xi} & \text{for} \quad \xi \geq 0 \\ \sum\limits_{\substack{k=1 \\ \mathrm{Re}(\kappa_k)>0}}^{4} \widehat{v}_{rk} e^{\kappa_k \xi} & \text{for} \quad \xi < 0 \end{cases} \tag{2.10}$$

For $\xi \geq 0$ we have to sum over the terms $\widehat{v}_{rk} e^{\kappa_k \xi}$ with real part of κ_k lower than zero and for $\xi < 0$ vice versa.

We substitute Eq (2.10) into Eqs (2.6) and (2.7) and obtain an inhomogeneous system of four linear equations for the constants $\widehat{v}_{rk}$, ($k = 1, \ldots, 4$).

3. Spin and a beam model

The angular velocity ω_c of the rail at the point of contact S_{rc} (cf Fig.1) is given by

$$\omega_c = (\dot{v}_r' - vv_r'')(0,t) \tag{3.1}$$

It is not defined if v_r'' is discontinuous. The angular velocity is needed to calculate the spin which is proportional to the difference between angular velocities of the rail and the wheel, respectively at the point of contact. Eq $(2.7)_1$ shows that the spin is not defined for an undamped beam.

To overcome this difficulty the material damping is introduced via the Voigt model (relaxation time τ (cf Fung (1965))

$$\sigma = E\Big(1 + \tau\frac{\partial}{\partial t}\Big)\varepsilon \tag{3.2}$$

The equations of motion for the damped Bernoulli-Euler beam are

$$[v_r]_-^+ = 0 \qquad [v_r']_-^+ = 0 \qquad [v_r'']_-^+ = 0 \tag{3.3}$$

$$\tau vEI[v_r''']_-^+ = M_{c3} \qquad EI[v_r''' + \tau\dot{v}_r''' - \tau v v_r^{IV}]_-^+ = F_{c2} \tag{3.4}$$

$$-\tau vEIv_r^V + EIv_r^{IV} + \tau EI\dot{v}_r^{IV} + k_v v_r + \mu(\ddot{v}_r - 2v\dot{v}_r' + v^2 v_r'') = 0 \tag{3.5}$$

where $F_{c2} = \widehat{F}_{c2}e^{\lambda t}$ and $M_{c3} = \widehat{M}_{c3}e^{\lambda t}$.

The differences between the equations for the damped and undamped beams, respectively, are:

- The second derivative is continuous at the point of contact instead of being discontinuous
- The order of field equation with respect to the space variable ξ is five instead of four.

To solve the equations we assume $v_r(\xi,t) = \widehat{v}_r e^{\kappa\xi}e^{\lambda t}$. Then, we get from Eq (3.5) a polynomial of fifth degree in κ

$$0 = -\tau vEI\kappa^5 + EI\kappa^4 + \tau EI\lambda\kappa^4 + k_v + \mu(\lambda^2 - 2v\lambda\kappa + v^2\kappa^2) \tag{3.6}$$

In the following we approximate the eigenvalues κ by an expansion with respect to the small parameter ε, $\varepsilon := \tilde{\tau}\tilde{v}$ ($\tilde{\tau}$ and $\tilde{v}$ are the nondimensional relaxation time and velocity, respectively). To do this we choose realistic values of the parameters

$$
\begin{aligned}
&EI = 5 \cdot 10^6 \text{ Nm}^2 && k_v = 1 \cdot 10^8 \frac{\text{N}}{\text{m}^2} && \mu = 60 \frac{\text{kg}}{\text{m}} \\
&v = 50 \frac{\text{m}}{\text{s}} && \tau = 1 \cdot 10^{-5} \text{ s} && \lambda = 4\pi \cdot 10^2 \frac{1}{\text{s}}
\end{aligned}
\tag{3.7}
$$

and scale Eq (3.6) using the following reference quantities

$$
\begin{aligned}
&\ell_0 = 0.5 \text{ m} \approx \sqrt[4]{\frac{EI}{k_v}} && t_0 = 5 \cdot 10^{-3} \text{ s (200Hz)} \\
&F_0 = 5 \cdot 10^4 \text{ N} && m_0 := \frac{F_0 t_0^2}{\ell_0} = 2.5 \text{ kg}
\end{aligned}
\tag{3.8}
$$

Tildes mark nondimensional parameters. Having done this all the coefficients in Eq (3.6) are either of the magnitude[4] $1/(\tilde{\tau}\tilde{v}) = 1000$ or of the magnitude $(\tilde{\tau}\tilde{v})^0 = 1$

$$
0 = -\underbrace{\tilde{\tau}\tilde{v}\widetilde{EI}}_{\approx 0.5}\kappa^5 + \underbrace{\widetilde{EI}}_{\approx 500}\kappa^4 + \underbrace{\tilde{\tau}\widetilde{EI}\tilde{\lambda}}_{\approx 5}\kappa^4 + \underbrace{\tilde{k}}_{\approx 400} v_r + \underbrace{\tilde{\mu}\tilde{\lambda}^2}_{\approx 470} - \underbrace{\tilde{\mu}2\tilde{v}\tilde{\lambda}}_{\approx 38}\kappa + \underbrace{\tilde{\mu}\tilde{v}^2}_{\approx 3}\kappa^2 \tag{3.9}
$$

We do not apply the usual approach of a perturbation technique, $\kappa = \kappa_0 + \varepsilon\kappa_1 + \ldots$, but use an expansion similar to the Laurent series

$$
\kappa = \frac{1}{\varepsilon^n}\kappa_{-n} + \frac{1}{\varepsilon^{n-1}}\kappa_{-n+1} + \ldots + \frac{1}{\varepsilon}\kappa_{-1} + \kappa_0 + \varepsilon\kappa_1 + \ldots \tag{3.10}
$$

We put κ from Eq (3.10) into Eq (3.9) and collect the terms of equal powers ε^m. The conditions that the coefficient of ε^m must vanish for each m lead sucessively to

$$
\kappa_{-n} = 0\,, \qquad \kappa_{-(n-1)} = 0\,, \quad \ldots\,, \quad \kappa_{-2} = 0 \tag{3.11}
$$

by looking at the coefficients $1/\varepsilon^{5n}, 1/\varepsilon^{5(n-1)}, \ldots, 1/\varepsilon^{10}$ one after another.

From the coefficient of $(1/\varepsilon^5)$-term we obtain five solutions for κ_{-1}

$$
\kappa_{-1,k} = \begin{cases} 0 & \text{for} \quad k = 1, \ldots, 4 \\ 1 & \text{for} \quad k = 5 \end{cases} \tag{3.12}
$$

[4]We call a coefficient α to be of magnitude β^n if $|\alpha/\beta^n| \in [1/\sqrt{\beta}, \sqrt{\beta}]$

The following comparisons of terms with the same magnitude yield successive terms for $\kappa_{n,5}$, $(n = 0, \dots)$. The first value n for not vanishing $\kappa_{n,k}$, $(k = 1, \dots, 4)$ is $n = 0$ and these $\kappa_{0,k}$ are the eigenvalues of the undamped system. The solution is of similar structure as given in Eq (2.10). By substituting it into Eqs (3.3) and (3.4) we obtain the constants $\widehat{v}_{rk}$ $(k = 1, \dots, 5)$. The constant $\widehat{v}_{r5}$ for the boundary layer $e^{\kappa_5 \xi}$ (characteristic length τv) is very small and its contribution to the deflection is small. As an example the deflection v_r is depicted in Fig.2 (parameters EI, k_v, v, μ see Eq (3.7) and $\lambda = 0$, i.e., a moving static load and static moment), $\widehat{F}_{c2} = 1 \cdot 10^5$ N, $\widehat{M}_{c3} = 1 \cdot 10^2$ Nm.

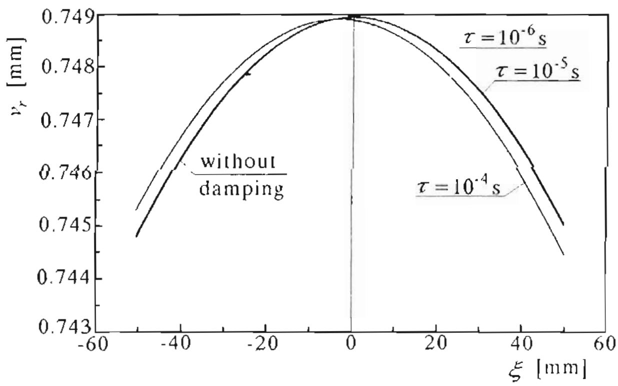

Fig. 2. The deflection of the undamped rail for three different relaxation times

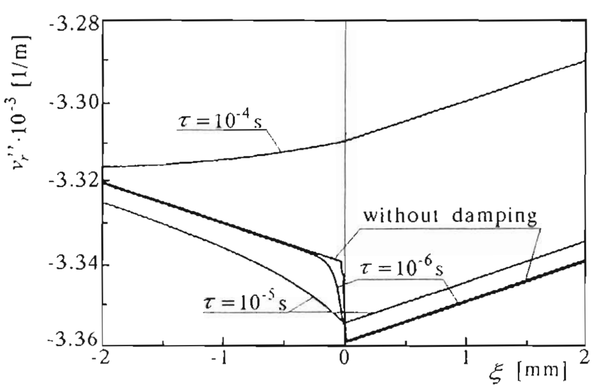

Fig. 3. The second derivative of the deflection v_r of the undamped rail and for three different relaxation times

In Fig.3 we see the smoothing influence of the boundary layer on the second derivative v_r''. Furthermore we see that for a very small relaxation time ($\tau = 10^{-6}$ s) the second derivative in the undamped rail in front of S_{rc} (i.e., $\xi = 0^+$) is about the same as in the damped rail at S_{rc}.

If the rail is modelled as the Timoshenko beam the perturbation technique results in two space eigenvalues $\kappa_{-1,5} = \kappa_{-1,6} = 1/(\tau v) + \ldots$, i.e., two boundary layers.

4. Lateral creepage and torsion

For a bended rail the velocity in lateral direction at the point of contact is smooth, no regularization is necessary. If the rail is modelled as a twistable bar the lateral velocity is discontinuous again. Thus we repeat the procedure given in Section 3. We assume that the rail rotates about the axis through S_{rr}. The distance between S_{rr} and S_{rc} is equal to r (see Fig.1). Then the lateral velocity in S_{rc} is

$$v_{c2} = (r\dot{\alpha}_r - vr\alpha_r')(0,t) \tag{4.1}$$

We establish the equations of motion by the Hamilton-Ostrogradsky principle Eq (2.2) with

$$U = \frac{1}{2}\int_{-\infty}^{\infty} \Big(GI_T\alpha_r'^2 + k_\alpha\alpha_r^2\Big)\, d\xi$$
$$T = \frac{1}{2}\int_{-\infty}^{\infty} J\dot{\alpha}_r^2\, d\xi \qquad \tilde{\delta}W = F_{c2}r\delta\alpha_r \tag{4.2}$$

where

GI_T – torsional stiffness
J – density of the moments of inertia
k_α – Winkler foundation for the torsion.

We obtain from Eq (4.2) for the undamped rail the jump condition $F_{c2}r = -GI_T[\alpha_r']_-^+$ and the field equation. We see that the velocity v_{c2} is not defined. Taking material damping into account the equations of motion become

$$[\alpha_r]_-^+ = 0 \qquad [\alpha_r']_-^+ = 0 \tag{4.3}$$

$$\tau v G I_T [\alpha_r'']_-^+ = F_{c2} \tag{4.4}$$

$$\tau v G I_T \alpha_r''' - G I_T \alpha_r'' - \tau G I_T \dot{\alpha}_r'' + k_\alpha \alpha_r + J(\ddot{\alpha}_r - 2v\dot{\alpha}_r' + v^2 \alpha_r'') = 0 \tag{4.5}$$

We choose the following parameters

$$\begin{aligned} &GI_T = 2 \cdot 10^5 \ \mathrm{Nm}^2 \qquad &&k_\alpha = 3 \cdot 10^6 \ \mathrm{N} \qquad &&J = 0.285 \ \mathrm{kg\,m} \\ &v = 50 \ \frac{\mathrm{m}}{\mathrm{s}} &&\tau = 1 \cdot 10^{-5} \ \mathrm{s} &&\lambda = \pi \cdot 10^3 \ \frac{1}{\mathrm{s}} \end{aligned} \tag{4.6}$$

and scale the field equation with respect to the following reference quantities

$$\begin{aligned} &\ell_0 = 0.25 \ \mathrm{m} \approx \sqrt{\frac{GI_T}{k_\alpha}} \qquad &&t_0 = 2 \cdot 10^{-3} \ \mathrm{s} \ (500 \ \mathrm{Hz}) \\ &F_0 = 5 \cdot 10^4 \ \mathrm{N} &&m_0 := \frac{F_0 t_0^2}{\ell_0} = 0.8 \ \mathrm{kg} \end{aligned} \tag{4.7}$$

Let α_r be $\alpha_r = \widehat{\alpha}_r \mathrm{e}^{\kappa\xi} \mathrm{e}^{\lambda t}$. We obtain from Eq (4.5) a polynomial in κ. Multiplying the polynomial by 10 the coefficients are of the magnitude 1 and $1/(\tilde{\tau}\tilde{v}) = 500$

$$0 = \underbrace{\tilde{\tau}\tilde{v}\widetilde{GI}_T}_{\approx 0.3} \kappa^3 - \underbrace{\widetilde{GI}_T}_{\approx 64} \kappa^2 - \underbrace{\tilde{\tau}\widetilde{GI}_T\tilde{\lambda}}_{\approx 2} \kappa^2 + \underbrace{\tilde{k}_\alpha}_{\approx 60} + \underbrace{\tilde{J}\tilde{\lambda}^2}_{\approx 56} - \underbrace{2\tilde{J}\tilde{v}\tilde{\lambda}}_{\approx 18} \kappa + \underbrace{\tilde{J}\tilde{v}^2}_{\approx 1.4} \kappa^2 \tag{4.8}$$

We apply the approach Eq (3.10) and get $\kappa_{-1,1} = \kappa_{-1,2} = 0$ and $\kappa_{-1,3} = 1$. That means that we have the boundary layer $\widehat{\alpha}_r \mathrm{e}^{\kappa_3 \xi}$, $\kappa_3 = 1/(\tau v) + \ldots$, for the deflection α_r behind S_{rc}. The characteristic length τv is the same as for the deflection v_r, cf Section 3. The effects on the deflection and its first derivative are small.

5. Longitudinal Velocity and Elongation

Applying the procedure presented in Sections 3 and 4 we get with

$$\begin{aligned} &U = \frac{1}{2} \int_{-\infty}^{\infty} \left(EAu_r'^2 + k_u u_r^2 \right) d\xi \\ &T = \frac{1}{2} \int_{-\infty}^{\infty} \mu \dot{u}_r^2 \, d\xi \qquad \tilde{\delta} W = F_{c1} \delta u_r \end{aligned} \tag{5.1}$$

from Eq (2.2) a jump of the first derivative of u_r. Thus the velocity in S_{rc}

$$v_{c1} = (\dot{u}_r - v u_r')(0,t) \tag{5.2}$$

is not defined.

Taking material damping into account the equations of motion become

$$[u_r]_-^+ = 0 \qquad [u_r']_-^+ = 0 \tag{5.3}$$

$$\tau v EA [u_r'']_-^+ = F_{c1} \tag{5.4}$$

$$\tau v EA u_r''' - EA u_r'' - \tau EA \dot{u}_r'' + k_u u_r + \mu(\ddot{u}_r - 2v\dot{u}_r' + v^2 u_r'') = 0 \tag{5.5}$$

With the choice of parameters

$$\begin{aligned} &EA = 1.65 \cdot 10^9 \text{ N} && k_u = 1 \cdot 10^7 \frac{\text{N}}{\text{m}^2} && \mu = 60 \frac{\text{kg}}{\text{m}} \\ &v = 50 \frac{\text{m}}{\text{s}} && \tau = 1 \cdot 10^{-5} \text{ s} && \lambda = 2\pi \cdot 10^2 \frac{1}{\text{s}} \end{aligned} \tag{5.6}$$

and the reference quantities

$$\begin{aligned} &\ell_0 = 13 \text{ m} \approx \sqrt{\frac{EA}{k_u}} && t_0 = 1 \cdot 10^{-2} \text{ s (100Hz)} \\ &F_0 = 5 \cdot 10^4 \text{ N} && m_0 := \frac{F_0 t_0^2}{\ell_0} = 0.8 \text{ kg} \end{aligned} \tag{5.7}$$

we obtain the dimensionless polynomial (see Sections 3 and 4) from Eq (5.5). Multiplying the polynomial by 10^4 the coefficients are of the magnitude 1 and $1/(\tilde{\tau}\tilde{v}) = 2.6 \cdot 10^4$

$$0 = \underbrace{\tilde{\tau}\tilde{v}\widetilde{EA}}_{\approx 0.00013} \kappa^3 - \underbrace{\widetilde{EA}}_{\approx 3.3} \kappa^2 - \underbrace{\tilde{\tau}\widetilde{EA}\tilde{\lambda}}_{\approx 0.02} \kappa^2 + \underbrace{\tilde{k}_u}_{\approx 3.4} + \underbrace{\tilde{\mu}\tilde{\lambda}^2}_{\approx 8} - \underbrace{2\tilde{\mu}\tilde{v}\tilde{\lambda}}_{\approx 0.05} \kappa + \underbrace{\tilde{\mu}\tilde{v}^2}_{\approx 0.0003} \kappa^2 \tag{5.8}$$

The same procedure as in the previous sections yields $\kappa_{-1,1} = \kappa_{-1,2} = 0$ and $\kappa_{-1,3} = 1$. The influence of the boundary layer to the elongation is small.

Acknowledgement

The author thanks the Deutsche Forschungsgemeinschaft for support of this research.

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Regularyzacja równań konstytutywnych modeli kół kolejowych

Streszczenie

W artykule wykazano, że warunki ciągłości są konieczne do sformulowania kinematyki kontaktu. W celu zagwarantowania ciągłości należy uwzględnić tłumienie materiałowe (czas relaksacji τ), to znaczy odpowiednie równania są regularyzowane. Ponadto pokazano, że warstwa brzegowa (wyznaczona przez prędkość v) pojawia się zawsze za punktem kontaktu, a jej długość charakterystyczna wynosi τv. Ze względu na mały wpływ warstwy brzegowej można stwierdzić, że tłumienie materiałowe może być narzędziem regularyzacji przy obliczaniu niegładkich prędkości przed punktem kontaktu. Obliczenia numeryczne modeli kół kolejowych mogą być wówczas uproszczone.

Manuscript received August 7, 1995; accepted for print September 26, 1995