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Schanuel's Lemma in P-Poor Modules

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Abstrak: Modul merupakan perumuman dari ruang vektor aljabar linier yaitu dengan memperumum lapangan skalarnya menjadi ring dengan elemen satuan. Dalam teori modul terdapat konsep modul proyektif, yaitu suatu modul atas ring *R* yang proyektif relatif terhadap semua modul atas *R* Selanjutnya, diperoleh fakta bahwa setiap modul atas *R* adalah modul proyektif relatif terhadap sebarang modul semisederhana atas *R*. Jika *P* adalah suatu modul atas *R* yang proyektif relatif hanya terhadap semua modul semisederhana atas *R* saja, maka *P* disebut modul *p*-miskin. Dalam pembahasan modul proyektif terdapat suatu lemma yang berkaitan dengan keekuivalenan dua buah modul K_1 dan K_2 dengan syarat terdapat dua buah modul proyektif P_1 dan P_2 sedemikian hingga $K_1 \oplus P_2$ isomorfik dengan $K_2 \oplus P_1$. Lemma tersebut dikenal sebagai lemma Schanuel di modul proyektif. Karena modul *p*-poor merupakan kasus khusus dari modul proyektif, maka pada tulisan ini akan dibahas tentang lemma Schanuel di modul *p*-poor.

Kata kunci*: modul proyektif, modul semisederhana, modul p-poor, lemma Schanuel*

Abstract: Modules are a generalization of the vector spaces of linear algebra in which the "scalars" are allowed to be from a ring with identity, rather than a field. In module theory there is a concept about projective module, i.e. a module over ring *R* in which it is projective module relative to all modules over ring *R*. Next, there is the fact that every module over ring *R* is projective module relative to all semisimple modules over ring *R*. If *P* is a module over ring *R* which it's projective relative only to all semisimple modules over ring *R*, then *P* is called *p*-poor module. In the discussion of the projective module, there is a lemma associated with the equivalence of two modules K_1 and K_2 provided that there are two projective modules P_1 and P_2 such that $K_1 \oplus P_2$ is isomorphic to $K_2 \oplus P_1$. That lemma is known as Schanuel's lemma in projective modules. Because the *p*-poor module is a special case of the projective module, then in this paper will be discussed about Schanuel's lemma in *p*-poor modules.

Keywords*: projective module, semisimple module, p-poor module, Schanuel's lemma*

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1. Introduction

Let *M* and *N* are *R*-modules, i.e. modules over a ring *R*. In this paper, *Mod-R* denotes the set of all right *R*-modules and *SSMod-R* the set of all semisimple right *R*-modules. An *R*-module is called a semisimple module if that module is a direct sum of simple modules [5]. A non-zero *R*-module is called a simple module if that module has no non-trivial submodules. In other words, its submodule is only {0} and himself. Following [3], for any *R*-module *M*, $\mathfrak{P}r^{-1}(M) = \{ N \in Mod-R \mid M$ is N - projective module} is called the projectivity domain of *M*. If $\mathfrak{P}r^{-1}(M) = Mod$-*R*, then *M* is called a projective module. Next, Alahmadi et al. [1] which discuss poor-module become the initial idea of the emergence of *p*-poor module concept, which *p*-poor module is dual of poor-module. Furthermore, this *p*-poor module is a special case of the projective module because the projectivity domain of *p*-poor only consists of all semisimple modules over ring *R* [2]. Regarding the existence of the *p*-poor module, it was found that each ring has a *p*-poor module. As for the formation of the *p*-poor module, it was found that an *R*-module, which is the result of the direct sum of all cyclic modules over *R* is a *p*-poor module [2].

This paper is inspired by similar ideas and problems in [4][5], where there is a lemma introduced by Stephen Schanuel in 1958 and known as the Schanuel's lemma in projective modules. That lemma associated with the equivalence of two modules K_1 and K_2 provided that there are two projective modules P_1 and P_2 such that $K_1 \oplus P_2$ is isomorphic to $K_2 \oplus P_1$. The organization of this paper describes as follows: section 2 explains a basic theory about exact sequences of *R*-modules and semisimple module. The explanation about the Schanuel's lemma in projective modules and Schanuel's lemma in *p*-poor modules will be presented in section 3. In section 4, we conclude the discussion.

2. Basic Theory

In this section, we define the external direct sum, the short exact sequence, the split exact sequence, and some properties of the semisimple module.

2.1 External Direct Sum

Before we define the external direct sum, will first be discussed about the direct product.

Definition 2.1. [3] The cartesian product $\times_A X_\alpha$ of the sets $\{X_\alpha\}_{\alpha \in A}$ be the set of all *A*-tuple $(x_\alpha)_{\alpha \in A}$ such that $x_\alpha \in X_\alpha$, for all $\alpha \in A$. If *A* is finite, $A = \{1, \dots, n\}$ then be obtained $\times_A X_\alpha = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) | x_i \in X_i, i = 1, \dots, n\}$.

Definition 2.2. [3] Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of *R*-modules. Defined the operations in $\times_\Lambda M_\lambda$, for every $(x_\lambda)_{\lambda \in \Lambda}$, $(y_\lambda)_{\lambda \in \Lambda} \in \times_\Lambda M_\lambda$ and $r \in R$ then
$(x_\lambda)_{\lambda \in \Lambda} + (y_\lambda)_{\lambda \in \Lambda} = (x_\lambda + y_\lambda)_{\lambda \in \Lambda}$ and $r(x_\lambda)_{\lambda \in \Lambda} = (rx_\lambda)_{\lambda \in \Lambda}$.
Next, the cartesian product $\times_\Lambda M_\lambda$, together with the above operations is *R*-modules. Furthermore, the module $\times_\Lambda M_\lambda$ is said to be the direct product of $\{M_\lambda\}_{\lambda \in \Lambda}$ and be written $\prod_\Lambda M_\lambda$.

Definition 2.3. [3] Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of *R*-modules. The external direct sum of $\{M_\lambda\}_{\lambda \in \Lambda}$ is defined as $\oplus_\Lambda M_\lambda = \{m \in \prod_\Lambda M_\lambda \mid \pi_\lambda(m) \neq 0 \; for \; \lambda \in \Lambda \text{ is finite}\}$.

2.2 Exact Sequences

The concept of exact sequences of *R*-modules and *R*-module homomorphisms and their relation to direct summands is a useful tool to have available in the study of modules. We start by defining exact sequences of *R*-modules.

Definition 2.4. [6] Let *R* be a ring. A sequence of *R*-modules *M* and *R*-module homomorphisms *f*

$$\dots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \dots \quad (1)$$

is said to be exact at M_i if $Im(f_i) = Ker(f_{i+1})$. The sequence is said to be exact if it is exact at each M_i.

As particular cases of Definition 2.1. note that if M, M_1, and M_2 are *R*-modules

1. $0 \to M_1 \xrightarrow{f} M$ is exact if and only if *f* is injective,
2. $M \xrightarrow{g} M_2 \to 0$ is exact if and only if g is surjective, and
3. The sequence
$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0 \quad (2)$$
is exact if and only if *f* is injective, g is surjective and $Im(f) = Ker(g)$.

Definition 2.5. [7] Given a sequence of *R*-modules

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0 \quad (3)$$

1. The sequence (3) is said to be a short exact sequence if it is exact.
2. The sequence (3) is said to be a split exact sequence (or just split) if it is exact and if $Im(f) = Ker(g)$ is a direct summand of *M*.

Next, in the following theorem will be given a characterization of split exact sequence.

Theorem 2.1. [7] If

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0 \quad (4)$$

is a short exact sequence of *R*-modules, then the following are equivalent:

1. There exists a homomorphism $\alpha: M \to M_1$ such that $\alpha \circ f = id_{M_1}$.
2. There exists a homomorphism $\beta: M_2 \to M$ such that $g \circ \beta = id_{M_2}$.
3. The sequence (4) is split exact.
If these equivalent conditions hold then
$$\begin{aligned} M &\cong Im(f) \oplus Ker(\alpha) \\ &\cong Ker(g) \oplus Im(\beta) \\ &\cong M_1 \oplus M_2 \end{aligned}$$

2.3 Semisimple Module

Next theory is needed in the next discussion is a semisimple module and some of its properties. However, it will first be defined as a simple module.

Definition 2.6. [3] A non-zero R-module M is called a simple module if M has no non-trivial submodules. In other words, the submodule of M is only $\{0\}$ and M.

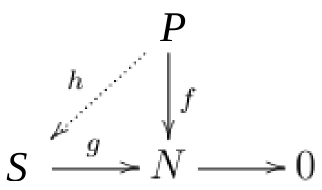
Definition 2.7. [6] An R-module M is called a semisimple module if M is a direct sum of simple modules.

A semisimple module has some characterization which will be given in the following proposition.

Proposition 2.2. [6] For an R-module M, the following properties are equivalent:
1. M is a semisimple module.
2. Every submodule of M is a direct summand.
3. Every exact sequence $0 \to K \to M \to L \to 0$ splits, for each K and L are R-modules.

3. Main Results

Based on the previous introduction, we have that p-poor module is a special case of the projective module because the projectivity domain of p-poor only consists of all semisimple modules over ring R. In other words, R-modules P is p-poor if for every semisimple R-modules S satisfies for each epimorphism $g : S \to N$ and homomorphism $f : P \to N$ there exists a homomorphism $h : P \to S$ such that $g \circ h = f$ (i.e. the following diagram commute).



Therefore, before we explain Schanuel's lemma in p-poor modules, we will first discuss Schanuel's lemma in projective modules.

3.1 Schanuel's Lemma in Projective Modules

This lemma associated with the equivalence of two modules M_1 and M_2 provided that there are two projective modules P_1 and P_2 such that $M_1 \oplus P_2$ is isomorphic to $M_2 \oplus P_1$. Furthermore, it will be discussed in the following lemma.

Lemma 3.1. [4] Given the sequences of R-modules

$$0 \to M_1 \xrightarrow{f_1} P_1 \xrightarrow{g_1} M \to 0 \tag{5}$$

$$0 \to M_2 \xrightarrow{f_2} P_2 \xrightarrow{g_2} M \to 0 \tag{6}$$

If (5) and (6) are exact with P_1 and P_2 are projective, then $M_1 \oplus P_2$ is isomorphic to $M_2 \oplus P_1$.

Proof. From R-modules P_1 and P_2 can be formed a direct sum $P_1 \oplus P_2$. Next, be formed $X = \{(p_1, p_2) \in P_1 \oplus P_2 | g_1(p_1) = g_2(p_2)\}$. Clearly, $X \subseteq P_1 \oplus P_2$ and $X \neq \emptyset$ because $(0,0) \in X$. Then, for each (x_1, x_2) and (y_1, y_2) in X and r in R, we

see that $g_1(x_1 + y_1) = g_1(x_1) + g_1(y_1) = g_2(x_2) + g_2(y_2) = g_2(x_2 + y_2)$ and $g_1(x_1 r) = g_1(x_1)r = g_2(x_2)r = g_2(x_2 r)$. So, we have $(x_1 + y_1, x_2 + y_2)$ and $(x_1 r, x_2 r)$ in X. In other words, X is submodule of $P_1 \oplus P_2$.
Next, we see that g_1 is epimorphism (surjective homomorphism) so that we have $M = g_1(P_1)$. Since g_2 is also epimorphism, then for each $g_1(P_1) \in M$ there exists $p_2 \in P_2$ such that $g_1(p_1) = g_2(p_2)$. Defined homomorphism $\pi_1 : X \to P_1$ with $\pi_1(p_1, p_2) = p_1$. Then, we have

$$\begin{aligned} Ker\,(\pi_1) &= \{(p_1, p_2) \mid \pi_1(p_1, p_2) = 0\} \\ &= \{(p_1, p_2) \mid p_1 = 0\} \\ &= \{(0, p_2) \mid g_2(p_2) = 0\} \\ &\cong Ker\,(g_2) \\ &= I'm\,(f_2) \end{aligned}$$

Furthermore, based on the particular cases of Definition 2.1, because (6) are exact, then f_2 is monomorphism (injective homomorphism), and because f_2 is injective, then we have $Im\,(f_2) \cong M_2$. As a result, we have $Ker\,(\pi_1) \cong M_2$. Next, can be formed a short exact sequence

$$0 \to M_2 \to X \xrightarrow{\pi_1} P_1 \to 0 \qquad (7)$$

Since P_1 is a projective module, there exists a homomorphism $h: P_1 \to X$ such that $\pi_1 \circ h = id_{P_1}$, then the sequence (7) is split exact, and we have $X \cong M_2 \oplus P_1$.
Furthermore in an analogous way, then can be formed a short exact sequence

$$0 \to M_1 \to X \xrightarrow{\pi_2} P_2 \to 0 \qquad (8)$$

and we have $X \cong M_1 \oplus P_2$. Therefore, we have $M_1 \oplus P_2 \cong M_2 \oplus P_1$.

3.2 Schanuel's Lemma in P-Poor Modules

Next, can be made Schanuel's lemma in *p*-poor modules, i.e. we replace sufficient conditions projective module in Lemma 3.1. with *p*-poor module which it is also a semisimple module, or we call that module as a semisimple *p*-poor. This is because the *p*-poor module is a special case of the projective module, where the projectivity domain of *p*-poor only consists of all semisimple modules. Therefore, need a certain condition is semisimple so that the concept of its projective module can be used in *the p*-poor module.

Lemma 3.2. Given the sequences of R-modules

$$0 \to M_1 \xrightarrow{f_1} P_1 \xrightarrow{g_1} M \to 0 \qquad (9)$$

$$0 \to M_2 \xrightarrow{f_2} P_2 \xrightarrow{g_2} M \to 0 \qquad (10)$$

If (9) and (10) are exact with P_1 and P_2 are semisimple *p*-poor modules, then $M_1 \oplus P_2$ is isomorphic to $M_2 \oplus P_1$.

Proof. From semisimple *p*-poor modules P_1 and P_2 , then we have $P_1 \oplus P_2$ is also semisimple *p*-poor module. Next, be formed $W = \{(p_1, p_2) \in P_1 \oplus P_2 | g_1(p_1) = g_2(p_2)\}$. Clearly, W is a submodule of $P_1 \oplus P_2$ because its proof is same with the proof of X is a submodule of $P_1 \oplus P_2$ in Lemma 3.1. Furthermore, according to [3] because every submodule of a semisimple module is semisimple, then we have W is a semisimple module.
Next, we see that g_1 is epimorphism (surjective homomorphism) so that we have $M = g_1(P_1)$. Since g_2 is also epimorphism, then for each $g_1(P_1) \in M$ there exists

$p_2 \in P_2$ such that $g_1(p_1) = g_2(p_2)$. Defined homomorphism $\pi_1: W \to P_1$ with $\pi_1(p_1, p_2) = p_1$. Then, we have

$$\begin{aligned} Ker\,(\pi_1) &= \{(p_1, p_2) \mid \pi_1(p_1, p_2) = 0\} \\ &= \{(p_1, p_2) \mid p_1 = 0\} \\ &= \{(0, p_2) \mid g_2(p_2) = 0\} \\ &\cong Ker\,(g_2) \\ &= Im\,(f_2) \end{aligned}$$

Furthermore, because f_2 is monomorphism (injective homomorphism), then we have $Im\,(f_2) \cong M_2$. As a result, we have $Ker\,(\pi_1) \cong M_2$. Next, can be formed a short exact sequence

$$0 \to M_2 \to W \overset{\pi_1}{\to} P_1 \to 0 \qquad (11)$$

Since P_1 is a *p*-poor module (i.e. projective module which its projectivity domain only consists of all semisimple modules), then for semisimple module W there exists homomorphism $h: P_1 \to W$ such that $\pi_1 \circ h = id_{P_1}$. In other words, the sequence (11) is split exact and we have $W \cong M_2 \oplus P_1$. Furthermore in an analogous way, then can be formed a short exact sequence

$$0 \to M_1 \to W \overset{\pi_2}{\to} P_2 \to 0 \qquad (12)$$

and we have $W \cong M_1 \oplus P_2$. Therefore, we have $M_1 \oplus P_2 \cong M_2 \oplus P_1$.

4. Conclusion

Some properties which have sufficient conditions of the projective module can be modified by replacing the projective module into *the p*-poor module with certain additional conditions. The result of this research only discuss how to get Schanuel's lemma in *p*-poor modules, i.e. with modify Schanuel's lemma in projective modules. Its method is to replace sufficient conditions projective module on Schanuel's lemma in projective modules with a semisimple *p*-poor module. This is because *the p*-poor module is a special case of the projective module, where the projectivity domain of *p*-poor only consists of all semisimple modules. Actually, this lemma also as an introduction of an equivalence relation in the *p*-poor module, i.e. modules M_1 and M_2 are equivalent if there exist semisimple *p*-poor modules P_1 and P_2 such that $M_1 \oplus P_2$ is isomorphic to $M_2 \oplus P_1$.

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