Mathematical Problems of Computer Science 40, 31–33, 2013.

Simple Proofs of Two Dirac-type Theorems Involving Connectivity

Carlen M. Mosesyan, Mher Zh. Nikoghosyan and Zhora G. Nikoghosyan

Faculty of Mathematics and Informatics
Kh. Abovyan Armenian State University
Institute for Informatics and Automation Problems of NAS RA
e-mail: mosesyan@list.ru, zhora@ipia.sci.am

Abstract

In 1981, the third author proved that each 2-connected graph G with $\delta \geq (n+\kappa)/3$ is hamiltonian and each 3-connected graph contains a cycle of length at least $\min\{n, 3\delta - \kappa\}$, where n denotes the order, δ - the minimum degree and κ - the connectivity of G. Short proofs of these two results were given by Häggkvist and Yamashita, respectively, occupying more than three pages for actual proofs altogether. Here we give much simpler and shorter proofs actually occupying the two-thirds of a page.

Keywords: Minimum degree, Connectivity, Circumference.

Let G be a finite undirected graph without loops or multiple edges. A good reference for any undefined terms is [1]. We reserve n, δ, κ and c to denote the order (the number of vertices), minimum degree, connectivity and circumference (the length of a longest cycle) of G. The vertices and edges can be interpreted as simple cycles of lengths 1 and 2, respectively. A cycle C in G is called a Hamilton cycle if $|C| = n$ and is called a dominating if $G\backslash C$ is edgeless.

In 1952, Dirac [2] proved that each graph with $\delta \geq n/2$ is hamiltonian (i.e. has a Hamilton cycle) and in each 2-connected graph, $c \geq \min\{n, 2\delta\}$.

In 1981, the third author [5] was able to obtain two analogous Dirac-type results involving connectivity κ.

Theorem 1 [5]: *Every 2-connected graph with $\delta \geq (n + \kappa)/3$ is hamiltonian.*

Theorem 2 [5]: *In every 3-connected graph, $c \geq \min\{n, 3\delta - \kappa\}$.*

The original proofs [5] of Theorems 1 and 2 are somewhat lengthy and complicated. Short proofs of these two results were given by Häggkvist [3] and Yamashita [7], respectively, occupying more than three pages for actual proofs altogether.

In this note we present much simpler and shorter proofs of theorems 1 and 2 (actual proofs on about 2/3 page) based mainly on standard arguments and the following two theorems.

Theorem 3 [4]: *Let G be a 2-connected graph with $\delta \geq (n + 2)/3$. Then every longest cycle in G is a dominating cycle.*

Theorem 4 [6]: *Let G be a 3-connected graph. Then either $c \geq 3\delta - 3$ or every longest cycle in G is a dominating cycle.*

The set of vertices of a graph G is denoted by $V(G)$ and the set of edges - by $E(G)$. For S a subset of $V(G)$, we denote by $G\backslash S$ the maximum subgraph of G with vertex set

$V(G)\backslash S$. For a subgraph H of G we use $G\backslash H$ short for $G\backslash V(H)$. We denote by $N(x)$ the neighborhood of a vertex x in a graph G. We write a cycle C of G with a given orientation by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ the subpath of C in the chosen direction from x to y. For $x \in V(C)$, we denote the successor of x on $\overrightarrow{C}$ by x^+ and the predecessor by x^-. For $X \subset V(C)$, we define $X^+ = \{x^+|x \in X\}$.

Lemma 1: *Let G be a 2-connected graph and S - a minimum cut-set in G. If every longest cycle in G is a dominating cycle, then either $c \geq 3\delta - \kappa + 1$ or there exists a longest cycle C with $S \subseteq V(C)$.*

Proof: Choose a longest cycle C in G such that $|V(C) \cap S|$ is as great as possible. Assume the converse, that is $S \not\subseteq V(C)$ and $x \in S\backslash V(C)$. Since C is dominating, $N(x) \subseteq V(C)$. Let $\xi_1, ..., \xi_t$ be the elements of $N(x)$, occurring on $\overrightarrow{C}$ in a consecutive order. Put $M_1 = \{\xi_i | V(\xi_i^+ \overrightarrow{C} \xi_{i+1}^-) \cap S \neq \emptyset\}$ and $M_2 = N(x)\backslash M_1$. Since $x \in S$, we have $|M_1| \leq \kappa - 1$ and $|M_2| = |N(x)| - |M_1| \geq \delta - \kappa + 1$. Further, since C is extreme and $|V(C) \cap S|$ is maximum, $N(x) \cap N^+(x) \cap M_2^{++} = \emptyset$ and $c \geq |N(x)| + |N^+(x)| + |M_2^{++}| = 2|N(x)| + |M_2| \geq 3\delta - \kappa + 1$. ■

Proof of Theorem 2: Let G be a 3-connected graph, S be a minimum cut-set in G and let $H_1, ..., H_h$ be the connected components of $G\backslash S$. The result holds immediately if $c \geq 3\delta - 3$, since $3\delta - 3 \geq 3\delta - \kappa$. Otherwise, by Theorem 4, every longest cycle in G is a dominating cycle. If $S \not\subseteq V(C)$ for each longest cycle C then by Lemma 1, $c \geq 3\delta - \kappa + 1$. Let C be a longest cycle with $S \subseteq V(C)$. If $V(G\backslash C) = \emptyset$ then $|C| = n$ and we are done. Let $x \in V(G\backslash C)$. Assume w.l.o.g. that $x \in V(H_1)$. Put $Y_1 = N(x) \cup N^+(x)$. Clearly $|Y_1| \geq 2\delta$ and it remains to find a subset Y_2 in $V(C)$ such that $Y_1 \cap Y_2 = \emptyset$ and $|Y_2| \geq \delta - \kappa$. Abbreviate, $V_1 = V(H_1) \cup S$. Suppose first that $Y_1 \subseteq V_1$. If $V(H_2) \subseteq V(C)$, then take $Y_2 = V(H_2)$ since $|V(H_2)| \geq \delta - \kappa + 1$. Otherwise, there exist $y \in V(H_2\backslash C)$ with $N(y) \subseteq V(C)$ (since C is dominating) and we can take $Y_2 = N(y)\backslash S$. Now let $Y_1 \not\subseteq V_1$. Assume w.l.o.g. that $Y_1 \cap V(H_2) \neq \emptyset$. Since $N(x) \subseteq V_1$, we can choose $z \in N^+(x) \cap V(H_2)$. If $N(z) \subseteq V(C)$, then take $Y_2 = N(z)\backslash S$, since $N^+(x)$ is an independent set of vertices (by standard arguments) and therefore, $N(z) \cap N^+(x) = \emptyset$. Otherwise, choose $w \in N(z)\backslash V(C)$. Clearly $N(w) \subseteq V(C)$, $w \in V(H_2)$ and $N(w) \cap N^+(x) = \{z\}$. Then by taking $Y_2 = (N(w)\backslash\{z\})\backslash(S\backslash\{z^-\})$ we complete the proof. ■

Proof of Theorem 1: Assume the converse, that is G is a non-hamiltonian 2-connected graph with $\delta \geq (n+\kappa)/3$. Let S be a minimum cut-set in G. Since $\delta \geq (n+\kappa)/3 \geq (n+2)/3$, by Theorem 3, every longest cycle in G is a dominating cycle. Further, since $c \leq n \leq 3\delta - \kappa$, by Lemma 1, G contains a longest cycle C with $S \subseteq V(C)$. As in the proof of Theorem 2, $c \geq 3\delta - \kappa$, contradicting the fact that $\delta \geq (n + \kappa)/3$. ■

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[2] G. A. Dirac, "Some theorems on abstract graphs", *Proc. London, Math. Soc.*, vol. 2, pp. 69–81, 1952.

[3] R. Häggkvist and G.G. Nicoghossian, "A remark on hamiltonian cycles", *Journal Combin. Theory*, Ser. B 30, pp. 118-120, 1981.

[4] C. St. J. A. Nash-Williams, *Studies in Pure Mathematics*, (Edge-disjoint hamiltonian cycles in graphs with vertices of large valency, pp. 157–183), in: L. Mirsky (Ed), Academic Press, San Diego, London, 1971.

[5] Zh. G. Nikoghosyan, “On maximal cycle of a graph”,*DAN Arm.SSR* , (in Russian), vol. LXXII, no. 2, pp. 82-87, 1981.

[6] H.-J. Voss and C. Zuluaga, “Maximale gerade und ungerade Kreise in Graphen I”, *Wiss. Z. Tech. Hochschule Ilmenau*, vol. 23, pp. 57-70, 1977. (*Math. Soc.*, vol. 2 pp. 69–81, 1952.)

[7] T. Yamashita, “A degree sum condition for longest cycles in 3-connected graphs”, *Journal of Graph Theory*, vol. 54, no. 4, pp. 277-283, 2007.

Submitted 05.09.2013, accepted 17.10.2013.

Կապակցվածության պարամետրով Դիրակյան տեսքի երկու թեորեմների պարզ ապացույցներ

Կ. Մոսեսյան, Մ. Նիկողոսյան և Ժ. Նիկողոսյան

Ամփոփում

Երրորդ հեղինակը 1981-ին ապացուցել է, որ կամայական 2-կապակցված գրաֆ $\delta \geq (n+\kappa)/3$ պայմանի դեպքում համիլտոնյան է, իսկ կամայական 3-կապակցված գրաֆ ունի առնվազն $\min\{n, 3\delta-\kappa\}$ երկարության ցիկլ, որտեղ n-ը գրաֆի գագաթների քանակն է, δ -ն՝ նվազագույն աստիճանը, իսկ κ-ն՝ կապակցվածությունը: Հագվիսթը և Յամաշիտան, համապատասխանաբար, զգալիորեն կրճատել են այս երկու թեորեմների նախնական ապացույցները՝ երեք էջի սահմաններում: Ներկա աշխատանքում ընդհանուրը 2/3 էջի սահմաններում ներկայացվում են շատ ավելի կարճ և պարզ ապացույցներ:

Простые доказательства двух теорем Диракского типа с параметром связности

К. Мосесян, М. Никогосян и Ж. Никогосян

Аннотация

Третий автор в 1981 году доказал, что каждый 2-связный граф при условии $\delta \geq (n+\kappa)/3$ гамильтонов, а каждый 3-связный граф либо гамильтонов либо содержит цикл длины по меньшей мере $3\delta - k$, где n обозначает число вершин графа, δ- минимальная степень и k - связность. Хагвист и Ямашита, соответственно, значительно сократили оригинальные доказательства этих теорем в рамках трех страниц. В настояшей статье в рамках 2/3 страниц предлагаются новые более короткие и простые доказательства.