A generalisation of the Spieker circle and Nagel line

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Introduction

Many a famous mathematician and scientist have described how their first encounter with Euclidean geometry was the defining moment in their future careers. Some of the most well known are probably Isac Newton and Albert Einstein. Often these encounters in early adolescence have been poetically described as passionate love affairs. For example, the mathematician Howard Eves describes his personal experience as follows: "*...Euclid's Elements ... I leafed through the book, and found that, from a small handful of assumptions ... all the rest apparently followed by pure reasoning ... The experience had all the aspects of a romance. It was love at first sight. I soon realized I had in my hands perhaps the most seductive book ever written. I fell head over heels in love with the goddess Mathesis ... As the years have gone by I have aged, but Mathesis has remained as young and beautiful as ever*" (in Anthony, 1994: xvi-xvii)

Perhaps noteworthy is that very few famous mathematicians and scientists have ever mentioned arithmetic or school algebra as having been as influential as geometry in attracting them to mathematics. One of the reasons may be the algorithmic nature of high school algebra as pointed out by Howard Eves as follows: " *... I still think that geometry is the high school student's gateway to mathematics. It's not algebra, because high school algebra is just a collection of rules and procedures to be memorized*" (in Anthony, 1994: xvii). Moreover, the fundamental mathematical idea of proof, and that of a deductive structure and of logical reasoning, is usually introduced and developed largely within high school geometry, and hardly at all in algebra. Though elementary number theory and algebra can provide exciting opportunities for some conjecturing and proof, it is unfortunately not common practise in high school. At present, it is mainly geometry that provides a challenging, non-routine context for creative proof that requires learners to explore and discover the logical links between premises and conclusions.

The current reduction of Euclidean geometry from the new South African school curriculum at the General Education and Training (GET) and Further Education and Training (FET) levels has been largely motivated by the need to introduce some more contemporary topics. Some of these are cartesian and transformation geometry, as well as a little non-Euclidean geometry such as spherical geometry, taxi-cab geometry and fractal geometry. However, it would seem disastrous for the future development of mathematicians and scientists in our country to argue, as some do, for the complete removal of Euclidean geometry from the curriculum. Often the argument seems a purely political one: learners find geometry difficult compared to algebra; we have to improve the pass rate; so let's get rid of geometry!

Of course, the problem of geometry education is a very complex one, and is not one that I will attempt to address in this article, though some of my mathematics education research and thoughts in this regard appear in De Villiers (1997). It is also not a problem limited to our country, but is fairly international. Suffice to say that ignoring the problem will not solve it, but that it has to be faced head on, and will require the concerted, combined efforts of mathematicians, mathematics educators, teachers and researchers.

This article instead modestly aims to acquaint the reader with some results from 17th and 19th century geometry, and to combat the perspective that geometry is dead by showing that new discoveries can and are still being made. Specifically it will discuss a possibly new generalisation of the Spieker circle and the associated Nagel line, which is parallel to that of the generalisation of the nine-point circle and Euler line discussed in De Villiers (2005). Not only should these results be accessible to a fair number of undergraduate students, prospective and practising high school teachers, but also to the more mathematically talented high school learner. Unlike cutting edge research in other areas of mathematics, the results are relatively easy to understand and appreciate, even without proof, because of their visual nature.

Apart from the remarkable concurrencies of the medians, altitudes and perpendicular bisectors of a triangle mentioned in De Villiers (2005), there is a fourth concurrency theorem mentioned in a few South African textbooks, namely:

> The angle bisectors of the angles of a triangle are concurrent at the *incentre*, which is the centre of the inscribed circle of the triangle (see Figure 1).

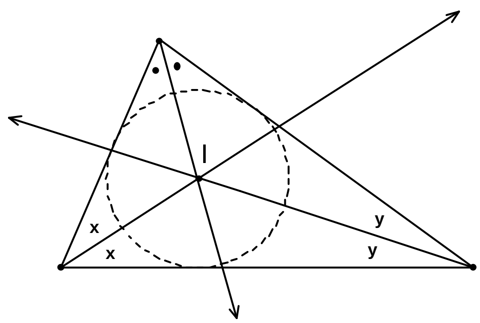


Figure 1: Incentre

Nagel point

Many mathematics teachers are not aware that there are many famous special centres associated with the triangle, and not only the four, i.e. the centroid, orthocentre, circumcentre and incentre, normally mentioned in textbooks. In fact, Clark Kimberling's two websites are worth a visit, where over 1000 special centres are associated with the triangle (see Kimberling)! Antonio Gutierrez's site also provides some beautiful, draggable dynamic geometry sketches of some of the more famous triangle centres (see Gutierrez).

One such notable point is the Nagel point, which is the point of concurrency of the lines from the vertices of a triangle to the points on the opposite sides where they are touched by the escribed circles (see Figure 2). This interesting point is named after its discoverer, the German mathematician Christian Heinrich von Nagel (1803-1882) and some biographical information about him can be obtained from: http://faculty.evansville.edu/ck6/bstud/nagel.html

Pascal's theorem

The French philosopher and mathematician Blaise Pascal (1623-1662) discovered and proved the following remarkable theorem at the age of sixteen: All six vertices of a hexagon lie on a conic, if and only if, the intersections of the three pairs of opposite sides are collinear (lie on a straight line) – see Figure 3 on following page. This is one of the first entirely projective theorems discovered and proved, and does not involve any measurement of sides or angles. Note that when the opposite sides of the inscribed hexagon are parallel they are assumed to meet at infinity, and all points at infinity are assumed to lie on the line at infinity. Pascal's proof has unfortunately not survived, but he probably used classical Euclidean geometry, and not modern projective methods.

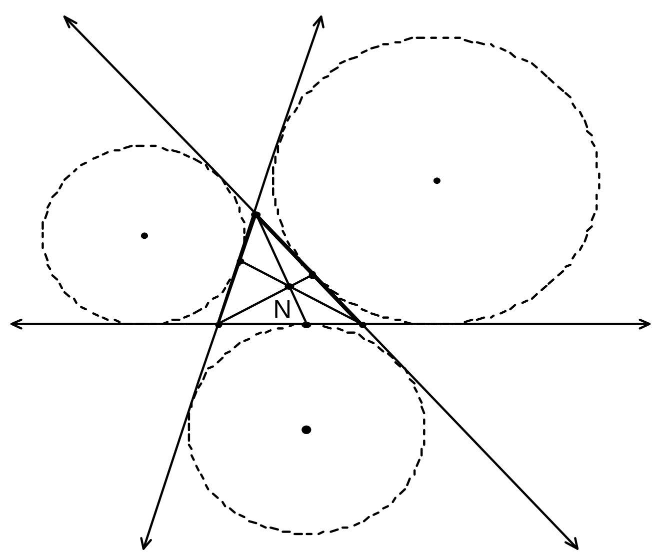


Figure 2: Nagel point

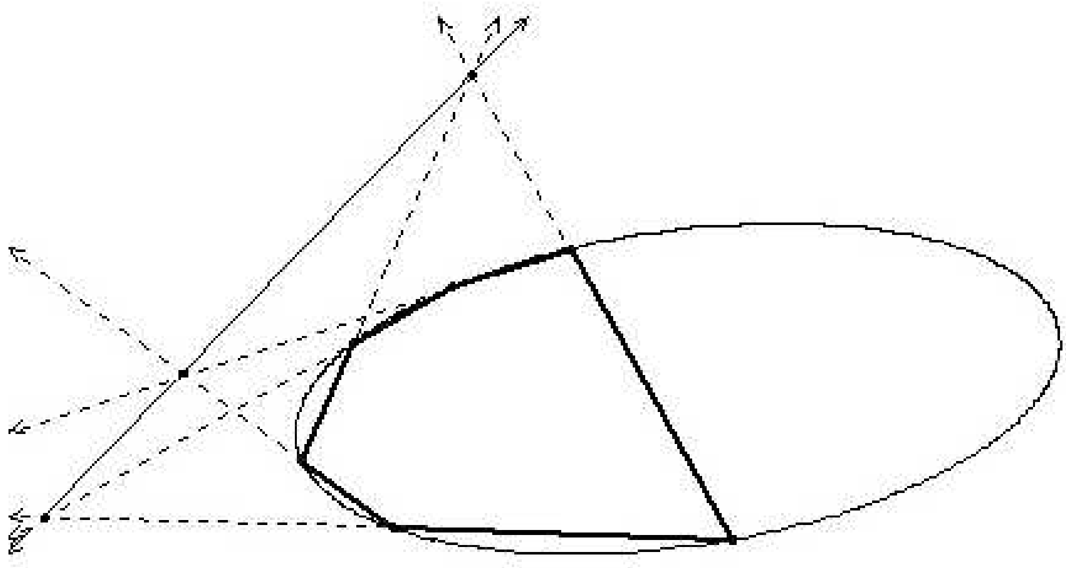
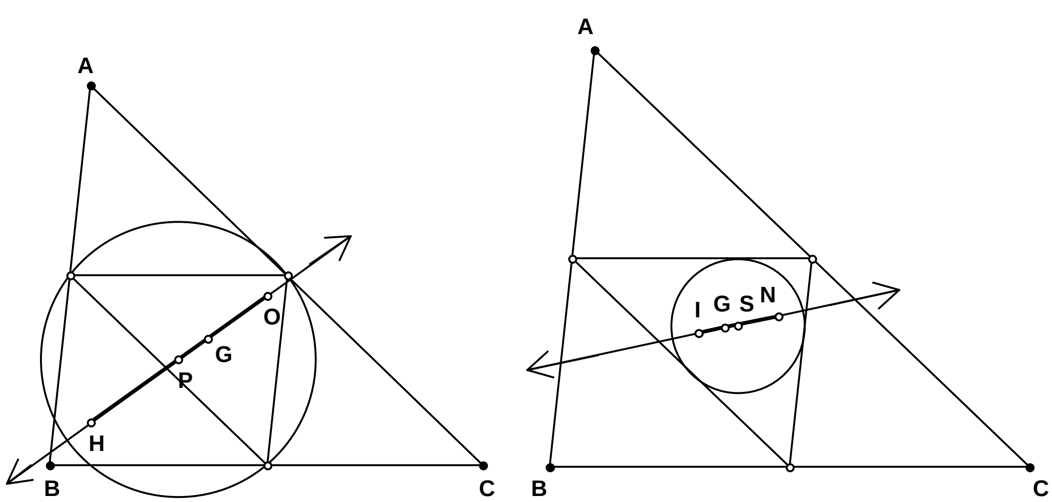


Figure 3: Pascal's theorem

Spieker circle and Nagel line

The discovery of the nine-point circle and the associated Euler-line has often been described as one of the crowning glories of post-Greek synthetic geometry (see De Villiers, 2005 for more details). However, less well known seems to be an interesting analogue or parallel result involving the Spieker circle and the Nagel line. The Spieker circle is named after Theodor Spieker whose 1890 geometry book *Lehrbuch der ebenen Geometrie* was one of the books that greatly inspired the young Einstein (see Pyenson, 1985). The rather remarkable parallelism between the nine-point circle and Euler line on the one hand, and that of the Spieker circle and Nagel line on the other hand, is contrasted in the table below, and illustrated in Figure 4. (The reader is reminded that the median triangle is the one formed by the midpoints of the sides of a triangle.)



The nine-point circle

The Spieker circle

Figure 4: Nine-point & Spieker circles

The *nine-point* circle is the *circumcircle* of *ABC*'s median triangle and has radius half that of *circumcircle* of *ABC*.	The *Spieker* circle is the *incircle* of *ABC*'s median triangle and has radius half that of *incircle* of *ABC*.
The *circumcentre* (O), centroid (G) & *orthocentre* (H) of any triangle ABC are collinear (*Euler* line), $GH = 2GO$ and the midpoint of OH is the centre of the *nine-point* circle (P) so that $HP = 3\ PG$.	The *incentre* (I), centroid (G) & *Nagel* point (N) of any triangle are collinear (*Nagel* line), $GN = 2GI$ and the midpoint of IN is the centre of the *Spieker* circle (S) so that $NS = 3\ SG$.
The *nine-point* circle cuts the sides of *ABC* where the extensions of the altitudes through the *orthocentre* meet the sides of *ABC*.	The *Spieker* circle touches the sides of the median triangle where they meet the lines from the *Nagel* point to the vertices of *ABC*.
The *nine-point* circle passes through the midpoints of the segments from the *orthocentre* to the vertices of the triangle.	The *Spieker* circle touches the sides of the triangle whose vertices are the midpoints of the segments from the *Nagel* point to the vertices of *ABC*.

The above observations are not new, and appear together with proofs in Coolidge (1971) and Honsberger (1995). More generally, this is an example of a limited, but an interesting kind of duality not only between the incircles (and escribed circles) and circumcircles of triangles and other polygons, but also between the concepts of side and angle, equal and perpendicular diagonals (e.g. for quadrilaterals), etc. This limited duality or analogy is explored fairly extensively in De Villiers (1996) and has been useful in formulating and discovering several new results (see for example De Villiers, 2000).

Having recently rediscovered a generalisation of the nine-point circle to a nine-point conic and an associated generalisation of the Euler line (De Villiers, 2005), the author wondered how one could generalise the Spieker circle (and the Nagel line) in a similar way. The following is the result of that investigation.

Let us first prove the following useful Lemma that will come in handy a little later. The first algebraic proof is my own original one while the second geometric one was kindly sent to me by Michael Fox from Leamington Spa, Warwickshire, UK.

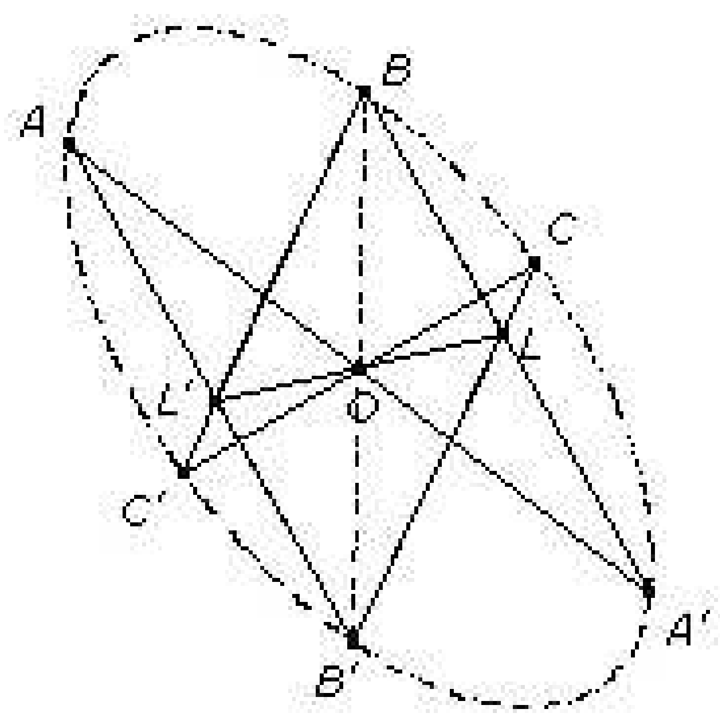


Figure 5: Lemma

Lemma

Given *A'*, *B'*, *C'* as the images of any three points *A*, *B*, *C* after a half-turn about *O*, then the six points *A'*, *B'*, *C'*, *A*, *B*, and *C* lie on a conic.

Algebraic proof

Place *O* at the origin so that the general conic equation we need to determine reduces to $ax^2 + 2hxy + by^2 + c = 0$. Divide through by *c*, so that only three unknowns now need to be determined. Due to the symmetry of the conic equation it follows that if (x_1, y_1) satisfies the equation so does its image $(-x_1, -y_1)$ under a half-turn. Therefore, only three points are needed to find the conic, one from each symmetric pair, e.g. *A* or *A'*, *B* or *B'*, *C* or *C'*.

Geometric proof

Consider the hexagon *AB'CC'BA'* shown in Figure 5. The image of *AB'* is *A'B*, therefore *AB'* // *A'B*; similarly *BC'* // *B'C*. Thus *BL'B'L* is a parallelogram, and its diagonals bisect each other. But the midpoint of *BB'* is *O*, hence *LOL'* is a straight line. These are the intersections of the opposite sides of the hexagon, so by the converse of Pascal's theorem, the vertices *A*, *B*, *C*, *A'*, *B'*, *C'* lie on a conic.

Spieker conic

Given *A'B'C'* as the median triangle of a triangle *ABC*, and *A'D*, *B'E* and *C'F* are three lines concurrent at *N*. Let *L*, *J* and *K* be the respective midpoints of *A'N*, *B'N*, and *C'N*, and *X*, *Y* and *Z* be the midpoints of the sides of *A'B'C'* as shown in Figure 6. For purposes of clarity, an enlargement of the median triangle and only the relevant points are shown in the bottom part of Figure 6 (see following page.)

Since both *XK* and *LZ* are parallel and equal to half *B'N*, it follows that *XKLZ* is a parallelogram. Similarly *JXYL* and *ZJKY* are parallelograms. Let *S* be the common midpoint of the respective diagonals *XL*, *YJ*, and *ZK* of these parallelograms. Further let the intersections of *AN*, *BN* and *CN* with the sides of the median triangle be *P*, *Q*, and *R*, and their respective reflections through *S* be *P'*, *Q'*, and *R'*. If a conic is now drawn through any five of *P*, *Q*, *R*, *P'*, *Q'*, and *R'*, then the conic cuts through the sixth point, and is inscribed in the median triangle (as well as the triangle obtained from the median triangle through a half-turn around *S*).

Proof

Since *P*, *Q*, *R*, *P'*, *Q'*, and *R'* are symmetrically placed around *S* by construction, it immediately follows from the preceding lemma that all six points lie on the same conic. Furthermore, it is obvious that projecting the lines *A'D*, *B'E* and *C'F* onto the altitudes of the median triangle, reduces the conic to the Spieker circle. Since the Spieker circle is inscribed in the median triangle (as well as its half-turn around *S*), and since any conic and tangents remain a conic and tangents under projection, it therefore follows that the general Spieker conic is also inscribed in both triangles.

Nagel line generalisation

Given the above configuration for any triangle *ABC*, then the centre of the Spieker conic (*S*), the centroid of *ABC* (*G*) and *N* are collinear, and *NS* = 3 *SG*.

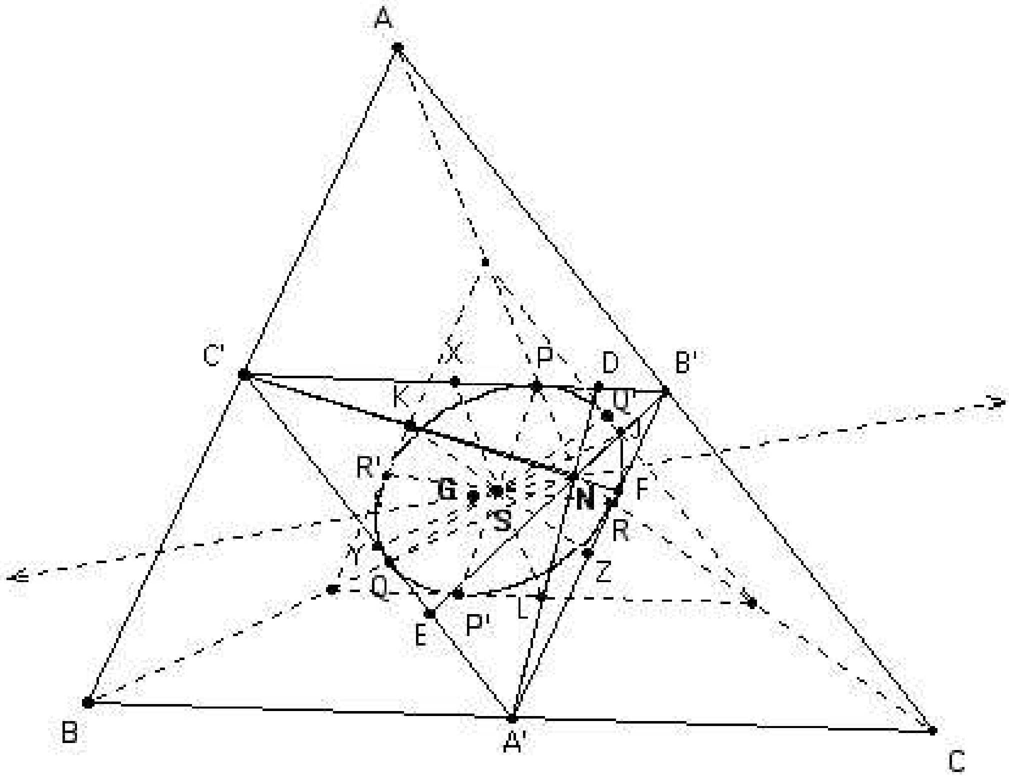
Proof

The projection of the Spieker conic onto the Spieker circle, also projects *S* onto the the centre of the Spieker circle, and the point *N* onto the Nagel point, and since collinearity is preserved under projection, *S*, *G* and *N* are collinear. However, since ratios of segments are not necessarily invariant under projection, this is not sufficient to prove *NS* = 3 *SG*.

However, this follows directly from the nine-point conic result and associated Euler generalisation discussed in De Villiers (2005). In Figure 6, the nine-point conic result implies that *X*, *K*, *Y*, *E*, *L*, *Z*, *F*, *J*, and *D* also lie on a conic, and that it has the same centre *S* as the Spieker conic. Hence, the Euler line corollary of this inscribed nine-point Spieker conic, directly proves the Nagel generalisation above, so that the centre *S* of this nine-point conic, the centroid *G* of *ABC* and the point of concurrency *N*, are collinear, and *NS* = 3 *SG*.

Concluding comments

It is hoped that this article has to some extent expelled the myth that the ancient Greeks already discovered and proved everything there is to find and prove in geometry. Apart from these results being easily accessible to undergraduate students, they are probably also within reach of talented high school students, particularly those at the level of the Third Round of the Harmony SA Mathematics Olympiad.



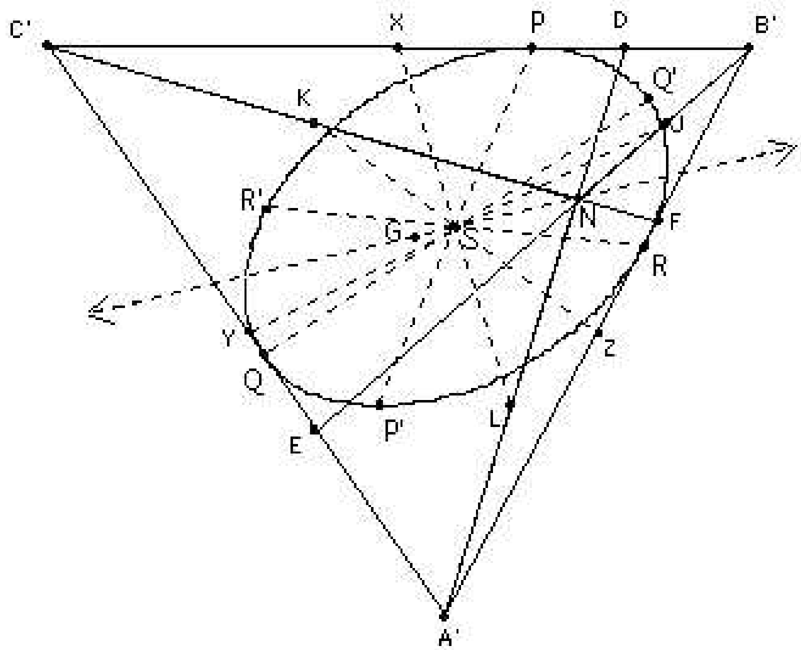


Figure 6: Spieker conic

Moreover, this article has hopefully also demonstrated that possible new geometric discoveries such as the nine-point and Spieker conics discussed here can still be made. In fact, it is quite likely that using dynamic geometry software in teaching geometry at high school or tertiary level may enable learners and students to more easily make their own discoveries, as the author has found on several occasions when working with prospective and in-service mathematics teachers. In particular, dynamic geometry software encourages an experimental approach that enables students to make and test geometric conjectures very efficiently.

In recent years there has been a general increase in geometry research on many fronts. We've seen the development and expansion of Knot Theory and its increased application to biology, the use of Projective Geometry in the design of virtual reality programs, the application of Coding Theory to the design of CD players, an investigation of the geometry involved in robotics, use of Search Theory in locating oil or mineral deposits, the application of geometry to voting systems, the application of String Theory to the origin, nature and shape of the cosmos, etc. Even Soap Bubble Geometry is receiving new attention as illustrated by the special session given to it at the Burlington MathsFest in 1995.

Even Euclidean geometry is experiencing an exciting revival, in no small part due to the recent development of dynamic geometry software such as *Cabri*, *Sketchpad* and *Cinderella*. Indeed, Philip Davies (1995) already ten years ago predicted a possibly rosy, new future for research in triangle geometry. Just a brief perusal of some recent issues of mathematical journals like the *Mathematical Intelligencer*, *American Mathematical Monthly*, *The Mathematical Gazette*, *Mathematics Magazine*, *Mathematics & Informatics Quarterly*, *Forum Geometricorum*, etc. easily testify to the greatly increased activity and interest in traditional Euclidean geometry involving triangles, quadrilaterals and circles. Of note too is a specific Yahoo discussion group which is specifically dedicated to current research in triangle geometry and traditional Euclidean geometry. Readers are invited to visit:
http://groups.yahoo.com/group/Hyacinthos/

It is therefore unfortunate that, with the exception of a handful of South African universities, hardly any courses are offered in advanced Euclidean, affine, projective or other geometries. In this respect, we seem to be lagging behind some leading overseas universities where there is a resurgence of interest in geometry not only at the undergraduate, but also at the postgraduate and research level. Not only does this tendency in South Africa narrow the potential field of research for a young mathematical researcher, but especially impacts negatively on the training of future mathematics teachers, who then return to teach matric geometry, having studied no further than matric geometry themselves.

In contrast to the present South African tertiary scene, the Mathematics Department at Cornell University, for example, is currently running more geometry courses at the graduate level (our postgraduate level) than any other courses (according to a personal communication to the author from David Henderson about four or five years ago). Moreover, Peter Hilton, one of the leading algebraic topologists (now retired from Binghamton University), is well known for frequently publicly stating that geometry is a marvelous and indispensable source of challenging problems, though algebra is often needed to solve them. It is also significant that the recent proof of Fermat's Last Theorem by Wiles relied heavily on many diverse fields in mathematics, including fundamental geometric ideas (see Singh, 1997).

Note: A Dynamic Geometry (*Sketchpad 4*) sketch in zipped format (Winzip) of the results discussed here can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/spieker.zip

(This sketch can also be viewed with a free demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/sketchpad/sketchdemo.html)

References

Anthony, J.M. (1994). *In Eve's Circles*. Notes of the Mathematical Association of America, 34. Washington: MAA.

Coolidge, J.L. (1971). *A Treatise on the Circle and the Sphere* (pp 53-57). Bronx, NY: Chelsea Publishing Company (original 1916).

Honsberger, R. (1995). *Episodes in Nineteenth & Twentieth Century Euclidean Geometry* (pp 7-13). The Mathematical Association of America. Washington: MAA.

Davies, P.J. (1995). The rise, fall, and possible transfiguration of triangle geometry. *American Mathematical Monthly*, *102(3)*, 204-214.

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*, University of Durban-Westville (now University of KwaZulu-Natal).

De Villiers, M. (1997). The future of secondary school geometry. *Pythagoras*, *44*, 37-54. (A pdf copy can be downloaded from: http://mysite.mweb.co.za/residents/profmd/future.pdf)

De Villiers, M. (2000). Generalizing Van Aubel using duality. *Mathematics Magazine*, *73(4)*, 303-306. (A pdf copy can be downloaded from: http://mysite.mweb.co.za/residents/profmd/aubel.pdf)

De Villiers, M. (2005). A generalisation of the nine-point circle and Euler line. *Pythagoras*, *62*, 31-35.

Gutierrez, A. *Triangle Centers* at: http://agutie.homestead.com/files/Trianglecenter_B.htm

Kimberling, C. *Triangle Centers* at: http://faculty.evansville.edu/ck6/tcenters/index.html

Kimberling, C. *Encyclopedia of Triangle Centers* at: http://faculty.evansville.edu/ck6/encyclopedia/

Pyenson, L. (1985). *The Young Einstein: The Advent of Relativity*. Boston: Adam Hilger.

Singh, S. (1997). *Fermat's Last Theorem*. London: Fourth Estate Publishers.

At the age of 12, I experienced a second wonder of a totally different nature: in a little book dealing with Euclidean plane geometry, which came into my hands at the beginning of the school year. Here were assertions, as for example the intersection of the three altitudes of a triangle in one point, which -- though by no means evident -- could nevertheless be proved with such certainty that any doubt appeared to be out of the question. This lucidity and certainty made an indescribable impression on me.

– Albert Einstein (Autobiographical Notes)