

Understanding student understanding in mathematics

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Introduction

Understanding is one of the most important traits associated with the attainment of educational goals. However, Nickerson (1985) observes that although the concept of understanding is a fundamental one for education, “what it means to understand is a disarmingly simple question to ask but one that is likely to be anything but simple to answer” (p. 215). A significant concern in school mathematics is learner understanding of mathematical concepts. Kilpatrick, Swafford and Findell (2001) have described conceptual understanding as a critical component of mathematical proficiency that “is necessary for anyone to learn mathematics successfully” (p. 116). In particular, Kilpatrick et al (2001) have argued that:

Students with a conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful. They have organised their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (p. 118)

Vygotsky (1962) makes the following more theoretical observation:

concepts do not lie in the child’s mind like peas in a bag, without any bonds between them. If that were the case, no intellectual operation requiring co-ordination of thoughts would be possible, nor any general conception of the world. Not even separate concepts as such could exist; their very nature presupposes a system. (p. 110-111)

The Revised National Curriculum Statement Grade R-9 (Department of Education, 2002) observes that “mathematical ideas and concepts build on one another to create a coherent structure”. Accordingly, the teaching and learning of mathematics is supposed to enable learners to “develop deep *conceptual understandings* in order to make sense of mathematics” (p. 5, emphasis

added). It therefore suggests that finding out how learners have organised their knowledge of mathematical concepts might be a way of establishing how they understand those concepts. A strategy known as concept mapping has been associated with exploring learner understanding in terms of how they make links between concepts. There has been a growing interest in the use of concept mapping in teaching and research across various fields in education (Novak, 1998). A concept map (see, for example, Figure 1 below) is a graphical tool for representing concept relationships. In a concept map, lines are drawn between pairs of concepts to denote relationships between concepts. Linking words on the lines indicate how pairs of concepts are related (Kennedy & McNaught, 1997).

Concept mapping has frequently been used as a teaching tool to help students “learn more meaningfully” and form a “conceptual understanding of the subject” (Novak, 1990: 943). The potential of concept mapping to make a knowledge discipline more “conceptually transparent” (Novak, 1998: 162) has been particularly recommended. In a concept map, “the network of propositions interlinking a group of concepts tells us much about the meaning of the concept from the perspective of the map makers” (Roth & Roychoudhury, 1992: 357). Interrelationships between concepts are considered to be an important attribute of knowledge. Interrelationships represent an essential feature of a learner’s thinking, understanding and meaning-making in a particular learning area.

The nature of concept mapping as described above makes it a useful tool for assessing and researching learner understanding. However, while there has been widespread use of concept mapping to investigate learner understanding in science education (chemistry, biology and physics), there is a marked under-utilisation of concept mapping as a tool in mathematics education research. Raymond’s (1997) study is one of the few that have used concept mapping to explore mathematical knowledge and understanding in a qualitative way.

In the literature on concept mapping, there seems to be a taken-for-granted assumption that a concept map represents a totality of an individual's understanding (Williams, 1998). However, just as knowledge is argued to be a construction resulting from a dynamic relationship between prior and new knowledge, so is a concept map a product: a construction that is not "independent of its author" (Henderson, 1991: 160). More usefully, concept maps need to be seen as representing "contents of consciousness" that need to be "inspected, edited, and shared with others" (Eisner, 1993: 6). The potential of concept mapping as a tool in mathematics education research is explored in a broad study by Mwakapenda and Adler (2002), from which this paper emerges. The study sets out to answer the question: What do the concept maps and follow-up interviews indicate about students' understanding of specific mathematical concepts? The resulting article explores learners' understanding of specific mathematical concepts in the South African (senior phase) curriculum. The article highlights the usefulness of concept mapping in researching understanding, and shows, in particular, that students' understanding of concepts is highly related to the broader context in which they learn mathematics, an aspect that has been rarely explored in studies that have used concept mapping as a research technique. Moreover, an issue that does not seem to have had focused attention in such studies relates to the question of how one is able to determine when something has been "understood". By sharing insights into aspects of learning experiences related to specific mathematical concepts, the study reported in this article provides a stimulus for discussing questions related to the complex issue of what constitutes understanding.

Study design

The study involved first-year students from the University of the Witwatersrand. There were three groups: students with at least a 60% pass on the Higher Grade Matriculation (Grade 12) mathematics examination who had enrolled for a mathematics major; students with at least a 60% pass on the Standard Grade Matriculation mathematics examination who were enrolled at the College of Science (an access college); and students who did not obtain a 60% pass on the Standard Grade Matriculation mathematics examination who had enrolled in a Foundation mathematics course. The study therefore involved three quite different groups of mathematics students in terms of previous school performance

and their current university enrolment. Participation in the research was purely voluntary. The study had intended to involve thirty students, ten from each group. A total of twenty-two students volunteered to participate in the research.

An important aspect of the study's methodological approach concerned a reflection on concept mapping itself as an activity in school mathematics. Although concept mapping has been described as a teaching tool, particularly in the sciences, none of the students involved in this study had any experience of concept mapping in their learning of science and mathematics. A critical component of the implementation of the study involved introducing these students to the nature and activity of concept mapping (Mwakapenda, 2001). Students were then asked to construct a concept map which they would use to show a friend how the following 16 concepts were related: *ratio, parallel, function, tangent, infinity, perpendicular, inverse, zero, equation, limit, absolute value, similar, gradient, angle, variable, bisector*. Why were these particular concepts selected? There is little doubt that *zero, ratio* and *angle* can be regarded as elementary and basic concepts in school mathematics. The mathematical concepts used in this investigation were drawn, in consultation with mathematics education specialists at Wits University, from a study of the curriculum, textbooks and assessment items being used for senior secondary mathematics. These concepts were selected because they were considered by mathematics education specialists to be key mathematical concepts. These were included in the task because of their prevalence and significance in the South African school curriculum. These concepts cut across the algebraic, numerical and geometric settings of secondary school mathematics. The aim was to see how students would link concepts across topics and settings that are often fragmented in the way they are presented in the curriculum. The number of concepts to be mapped in the task was relatively large. Because students would have been familiar with these key concepts, having encountered them over and over again in their five years of secondary mathematics study, it would be interesting to see the ways in which they had understood possible connections between them. Although the students' familiarity with the concepts could be assumed, the connecting of concepts in the form of a concept map was not common practice in school mathematics activity, hence the need for an extensive introduction on concept-mapping

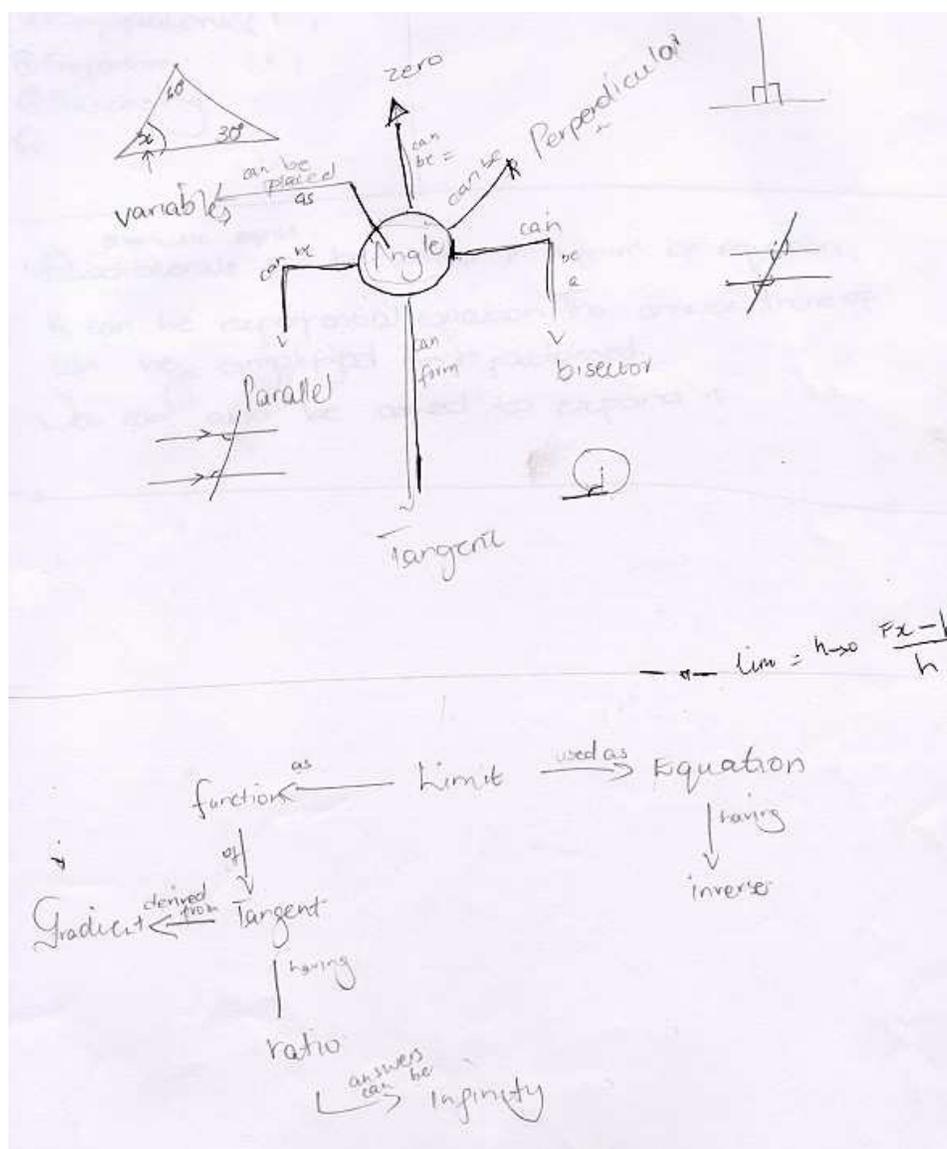


Figure 1: Angie's concept map

activity. This is described fully in Mwakapenda (2001).

Reflective interviews were then conducted with students on their completed concept maps. The aim was to probe students' understanding. The interviews presented an opportunity for students to explain and elaborate on the meanings represented in the links, and to allow them to provide appropriate examples to illustrate these links. This made it possible for the researcher to gain qualitative insights into the various dimensions of understanding which the students may have developed in their learning of school mathematics,

and what these might mean for mathematics education and practice. The dimensions of understanding being explored here are closely linked to the notion of the "functional access" students may have to mathematical knowledge (Lawson & Chinnappan, 2000: 34). This level of access involves "not only having knowledge but also doing something with it" (Nickerson, 1985: 234).

Data analysis

In many concept mapping studies, the analysis of concept maps is predominantly quantitative and proceeds by scoring various aspects of student

maps such as the presence and accuracy of hierarchy levels, propositions, links and cross-links, and specific examples provided to illustrate links (Ruiz-Primo & Shavelson, 1996). In this study, I identified the number of concept links and examples related to these links. Novak and Musonda (1991: 127) have argued that “any map scoring procedure reduces some of the richness and detail of information contained in a concept map”. Therefore, in the greater part of this study, students’ maps, and their elaboration of these in the interviews, were analysed in terms of the organisational principles (Prawat, 1989) which students seem to have used in constructing the maps. For example, the maps were examined to determine whether students considered certain concepts as central in developing links between concepts. The central concepts (e.g. angle or gradient) that students used were identified. The meanings students associated with these concepts were examined. Students’ descriptions of their maps were then analysed to examine the “completeness” (Nickerson, 1985) of connections made. As well as providing insights into the meanings and nature of links between concepts, the analysis also raised questions about students’ understanding of specific concepts.

An analysis of one student’s concept map

Detailed analyses and findings related to students’ concept maps from the broad study have been presented elsewhere (see Mwakapenda, 2004; Mwakapenda & Adler, 2003). In this article, the analysis focuses on the map drawn by Angie (pseudonym), a student from the Foundation Mathematics group. In the discussion that will follow, some implications arising from the analysis of the data from Angie are described and discussed in the context of the broad study involving other students. Figure 1 shows Angie’s concept map.

In relation to the given task, and in quantitative terms, we can see from Figure 1 that Angie used 14 out of 16 given concepts. We can also see that the map has two parts: the top part consisting of seven (mainly geometric) concepts with angle as a central concept, and the bottom part consisting of eight (largely algebraic) concepts. Seen in this way, the top part can be described as a more centralised assemblage of concepts. Qualitatively, in comparison with the top part, the bottom part of the concept map is not organised around any central concept. It can be described as a linear assemblage of concepts. However, the fact that Angie drew a concept map shows that she *saw* that there were links between various concepts and she

was able to display these links as can be seen in Figure 1. It is also observed that the concept of tangent was used twice while two concepts: “similar” and “absolute value”, were not used at all in the map. This suggests that Angie did not immediately see how these two concepts were linked to other concepts presented in the task. The appearance of the concept “tangent” in both parts of the concept map suggests a link between the geometric and algebraic concepts, an aspect that Angie did not seem to have noticed.

In a follow-up interview, Angie said the following in an attempt to describe and clarify the links she had displayed in her concept map (WM = Interviewer).

Angie: I looked at these words. Okay.

Then I said, what can I do with these?

Then I said I am going to concentrate on one word, which is angles.

WM: You said you should concentrate on angles. How did you decide?

Angie: I thought there are so many things that can be linked to angle from these given words. So I said angles can be perpendicular [draws a sketch], can be a bisector here cut into two equal parts [draws a sketch]. We can form a tangent [draws a sketch]. Angle can form a tangent, can be parallel [draws a sketch], Angles can be placed as variables [points at angle x in a triangle]. And angles can be zero.

As can be seen, Angie was able to see that a number of concepts, six in this case, “could” be linked to angles. However, stating and *describing* the conceptual linkages that she saw appears to have presented problems for Angie. For example, what does it mean when she says: “Angles can be perpendicular” and “Angle[s] can be parallel”? In the context of school geometry, it may be more adequate to say that lines (rather than angles) can be perpendicular or parallel. Nevertheless, the links Angie describes are revealing when compared with the corresponding “sketches” (Henderson, 1991) shown on her concept map in the top part of Figure 1. The sketches (drawn as a consequence of probing) in the top part shows that Angie associated angle with the following contexts: parallel lines and a transversal, an unknown angle (i.e. variables) in a triangle, perpendicular lines, and a tangent to a circle. What is revealed here is that Angie is able to draw sketches to describe links between angle and other concepts. What can also be seen in the interview excerpt above is Angie’s apparent struggle to appropriately

verbalise these links. Also, the sketches Angie drew did not seem to have helped her express these links more adequately. There is some inconsistency between the sketches she drew and the statements she makes about the sketches in Figure 1. On their own, the sketches describe a representation of mathematical knowledge that may be said to be more identifiable with the context of school geometry.

Angie said the following when she was asked to explain the links she had made between the concepts in the bottom part of the concept map.

Angie: Limit can be used as equations having an inverse.

WM: Can you give an example? What you mean by “limit can be used as equations”?

Angie: Having an inverse?

WM: Yah.

Angie: [long pause] ... I can't ...

WM: Would you like to think about it more?

Angie: ... limit ... [struggles to give examples] ... well I just can't. I don't remember exactly... Like this is, it's a long time I have done this.

Angie seems to be able to see some links between the more “algebraic” concepts in the bottom part of Figure 1. For example, she sees “limit” and “equations” as concepts that are connected in some way, although it is not clear what she means by “Limit can be used as equations”. Describing links between algebraic concepts and providing examples to illustrate and explain these links was not easy for Angie. The inability to give appropriate examples appears to be due to her not having been able to “remember exactly” what these could be. Given that these concepts were taught while in high school, it can be suggested that Angie found it easier to remember the “contexts” or mathematical situations in which the geometry concepts were learnt and what some of the organising features of the geometry were. Angle appears to be an organising feature for the geometry concepts in Angie’s understanding in relation to the given task. A question that arises here and one that needs further exploration is: what could be the organising features of “algebra”? To what extent can we say that Angie was unable to remember or think about the organising features of the algebraic concepts given in the concept mapping task?

In the above analysis, one is able to see how Angie explained her difficulty in describing links she was able to see between concepts. Her

explanation concerned the fact that she could not remember what these links were, given that “it was a long time” since she had learnt these concepts. The point is that being able to *remember* concepts and how they may have been used in school seems to have played an important role in Angie’s ability to see links between concepts.

Discussion

The analysis has indicated a lack of “fluency” (Williams, 1998: 414) in Angie’s articulation of the perceived connections between given concepts. This lack of fluency suggests that there are specific ways of expressing mathematics that Angie (as well as other students in her Foundation mathematics group) seems not to have adequately developed while in school. This is confirmed by the following remark made by a student in Angie’s group:

You know, in maths, we are taught to do maths. You know, to discuss maths, maths is not expressed in that way. At school we are taught to work maths on a paper. Sometimes it’s even difficult to understand a teacher ... when he talks. But it would be far better if you write something down. I cannot be with you in maths but when you write something there I will understand. When you write it down rather than expressing it... We cannot express maths like some other subjects... You can talk about psychology, what you are discussing, unlike maths.

We can see from above that lack of expertise in expressing mathematics and inadequate understanding of mathematical language and mathematics itself are likely to constrain students’ abilities to describe conceptual links between mathematical concepts. Expertise in expressing mathematics and how this is enabled by concept mapping as a pedagogical tool needs further exploration.

The above analysis underscores the central issue that concepts are not seen as entities on their own. The analysis suggests that, apart from being linked to other concepts, concepts are linked to contexts (represented by sketches or diagrams) associated with the learning and experiences of school mathematics. Angie found it easier to remember contexts in which particular concepts (e.g. angle) were learned than to describe ways in which such concepts are related. The link between concepts and contexts is important since it supports the widely acknowledged view that knowledge

cannot be separated from the situations in which it is learned and used. The individual's participation in the production of this knowledge is also critical (Lave, 1991).

What then can be said about the use of concept mapping as a tool for exploring students' understanding of mathematics? In particular, what does the concept map and follow-up interview indicate about students' understanding of specific mathematical concepts? Proponents of concept mapping assume that knowledge within a content domain is organised around central concepts and that to be knowledgeable in that domain, students need to be able to display a highly integrated structure of concepts (McKeown & Beck, 1990). Based on this assumption, the analysis presented here suggests that Angie displayed a partial integration of knowledge of concepts. Describing the links in this knowledge was, however, problematic. For Angie as well as other students involved in the study, it may be more accurate to characterise her mathematical connections as "representations" (Novak & Gowin, 1984: 40) of what they know and how they came to know about specific concepts. The above analysis indicates that it is possible to access some insight into students' understanding of mathematical concepts by examining the connections they make in a concept map. The ability to make connections between concepts is an important aspect of understanding the specific concepts concerned.

What can be said about the completeness of the connections students made? It is possible that, although students participating in this study may have been taught specific concepts, they may not have understood well enough to be able to communicate the knowledge gained. This result is not unusual given the culture of many classrooms where mathematics is typically taught as a disconnected set of facts and rules, unrelated to each other and to other knowledge disciplines. It is widely acknowledged that mathematics instruction often prevents meaningful learning and does not give students an opportunity to understand mathematical concepts and to critically and freely reflect on relationships between ideas (Boaler, 1997). Roth and Roychoudhury (1992) have pointed out that even though instruction may attempt to show connections,

textbooks and teachers can never provide all possible connections. Besides, no matter how many formulations there are and how explicit they are, students will always have to construct their own ways

of expressing the relationship between pairs of concepts. (p. 547)

Concept mapping is therefore a useful tool for exploring and documenting these connections and for promoting understanding of their conceptual meanings in mathematics. However, as indicated in the research design, interviews need to be set up to augment information from students' concept maps. The interview provides space for obtaining a more informed account of students' understanding of mathematics and why they connect mathematical concepts in the way they do. This is to emphasise the point that knowledge cannot be divorced from the individuals involved in the production of knowledge (Lerman, 1998).

A critical aspect concerning these connections relates to the issue about *what* we can learn by looking at the concept map. It is noted here that depending on the perspective one takes, not much can be learnt about student knowledge by focusing on the concept map only. Concept mapping originated from practitioners in the cognitive science field, a field that does not seem to openly problematise the claim that a concept map provides a totality of an individual's knowledge. Allchin (2002) argues that concept maps are "inherently selective. They can only represent *selectively*, based on the mapmaker's purpose" (p. 146, emphasis in original). While a map is a model of reality, one needs to understand the map's context in order to appropriately interpret how it represents that reality. The map externalises only a part of an individual's thoughts (Roth & Roychoudhury, 1992). It is possible to get a fuller picture of a student's learning if we consider a concept map as a product that depends highly on the contexts of its production: the individuals who produced it, and who they are and where they come from: their learning histories, and the mathematical opportunities and resources to which they have access. A concept map is therefore not considered as an end product that represents a totality of an individual's knowledge. Rather, it is perceived as a working representation of what students seem to currently know and have experienced. In this regard, Nickerson (1985) suggests that "one's understanding must depend on the amount of knowledge one has about the concepts involved", and that "the degree to which one understands [the concepts] must depend on the richness of the conceptual context in which the [concepts] can be interpreted" (p. 217). In acknowledging the context-dependent nature of understanding, Nickerson then makes the key point that "one's understanding of something should probably not be

thought of as right or wrong, but rather as ... more or less complete" (p. 220). Concept mapping is therefore seen as a vehicle for entering into a *dialogue* with students about mathematical knowledge and key concepts in school mathematics.

The findings illustrate that to know something (e.g. a mathematical concept) is not to know it as an entity having a life of its own, but it is to know it in relation to something else: its context. The concept map provides a partial representation of this knowledge. The ability to establish meaning from this knowledge is likely to be hindered by students' inability to talk clearly about concepts. Concept mapping gives students the space to talk about concepts. It creates an opportunity for students to clarify what they have learnt about mathematical concepts, and, in the process, it identifies for students and educators what further learning and relearning needs to be sought. Concept mapping is emerging as a reflective tool in mathematics learning in South African classrooms (see, for example, van Rensburg et. al, 2001: 21, 34, 49). However, there is a need to articulate and reflect on the epistemological assumptions guiding the appropriation of concept mapping as a tool in the developing context of mathematics education in South Africa. For mathematics education research generally, the activity of concept mapping opens up possibilities for gaining insights into what a learner knows and understands and the form that this understanding takes.

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"If you want mathematics to be meaningful,
you must resign of certainty.
If you want certainty, get rid of meaning.
You cannot have both."
Imre Lakatos, "Proof and Refutations"

(quoted by Amy J. Hackenberg,
one of Pythagoras' reviewers. from the University of Georgia)