ON THE SUBGROUP LATTICE OF AN ABELIAN FINITE GROUP

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The aim of this paper is to give some connections between the structure of an abelian finite group and the structure of its subgroup lattice.

1 Preliminaries

Let $(G, +)$ be an abelian group. Then the set $\mathcal{L}(G)$ of subgroups of G is a modular and complete lattice.

Moreover, we suppose that G is finite of order n. If L_n is the divisors lattice of n, then the following function is well defined:

$$\text{ord} : \mathcal{L}(G) \longrightarrow L_n, \quad \text{ord}(H) = |H|, \text{ for any } H \in \mathcal{L}(G),$$

where by $|H|$ we denote the order of the subgroup H.

2 Main results

Proposition 1. *The following conditions are equivalent:*

(i) *G is a cyclic group.*

(ii) ord *is an one-to-one function.*

(iii) ord *is a homomorphism of the semilattice* $(\mathcal{L}(G), \cap)$ *into the semilattice* $(L_n, (,))$.

(iv) ord *is a homomorphism of the semilattice* $(\mathcal{L}(G), +)$ *into the semilattice* $(L_n, [,])$.

(v) ord *is an isomorphism of lattices.*

Proof. (i) $\Longrightarrow$ (ii) Obvious.

(ii) $\Longrightarrow$ (i) For any $d \in L_n$ let M_d be the set of elements $x \in G$ having the order d. Then the family $\{M_d \mid d \in L_n\}$ is a partition of G, therefore we have:

$$n = \sum_{d \in L_n} |M_d|. \tag{1}$$

A set M_d is nonempty if and only if there exists a cyclic subgroup H_d of G having the order d. In this situation H_d is the unique subgroup of G with the order d and we have:

$$M_d = \{x \in G \mid < x >= H_d\}.$$

It results that $|M_d| = \varphi(d)$, where φ is the Euler function. Using the relation (1) and the identity

$$n = \sum_{d \in L_n} \varphi(d),$$

we obtain that $|M_d| = \varphi(d)$, for any $d \in L_n$. For $d = n$ we have $|M_d| = \varphi(n) \geq 1$, so that G contains an element of order n.

(i) $\Longrightarrow$ (iii) Let G_1, G_2 be two subgroups of G, $d_i = |G_i|$, $i = 1, 2$ and $d = |G_1 \cap G_2|$. Since $G_1 \cap G_2$ is a subgroup of G_i, $i = 1, 2$, we obtain that d/d_i, $i = 1, 2$. If d' is a divisor of d_1 and d_2, then $d' \in L_n$, so that there exists $G' \in \mathcal{L}(G)$ with $|G'| = d'$. We have $G' \subseteq G_i$, $i = 1, 2$, therefore $G' \subseteq G_1 \cap G_2$. It results that d'/d, thus $d = (d_1, d_2)$.

(iii) $\Longrightarrow$ (ii) Let G_1, G_2 be two subgroups of G such that $\text{ord}(G_1) = \text{ord}(G_2)$. From (iii) we obtain that $\text{ord}(G_1 \cap G_2) = \text{ord}(G_i)$, $i = 1, 2$, therefore we have $G_1 = G_1 \cap G_2 = G_2$.

(i) $\Longrightarrow$ (iv) Similarly with (i) $\Longrightarrow$ (iii).

(iv) $\Longrightarrow$ (ii) Similarly with (iii) $\Longrightarrow$ (ii).

(i) $\Longrightarrow$ (v) Obvious.

Next aim is to find necessary and sufficient conditions for $\mathcal{L}(G)$ in order to be a distributive lattice, respectively a complemented lattice in which every element has a unique complement.

Lemma 1. *If $\mathcal{L}(G)$ is a distributive lattice or a complemented lattice in which every element has a unique complement, then, for any $H \in \mathcal{L}(G)$, the lattice $\mathcal{L}(H)$ has the same properties.*

Proof. The first part of the assertion is obvious.

We suppose that $\mathcal{L}(G)$ is a complemented lattice in which every element has a unique complement and let H_1 be a subgroup of H. Then $H_1 \in \mathcal{L}(G)$, thus there exists a unique subgroup $\overline{H}_1 \in \mathcal{L}(G)$ such that $H_1 \oplus \overline{H}_1 = G$. It results that $H = G \cap H = H_1 \oplus (\overline{H}_1 \cap H)$. If $\widetilde{H}_1 \in \mathcal{L}(H)$ satisfies $H_1 \oplus \widetilde{H}_1 = H$ and K is the complement of H in G, then we have:

$$G = K \oplus H = (K \oplus H_1) \oplus (\overline{H}_1 \cap H)$$

and

$$G = K \oplus H = (K \oplus H_1) \oplus \widetilde{H}_1.$$

Since the subgroup $K \oplus H_1$ has a unique complement in G, it follows that $\widetilde{H}_1 = \overline{H}_1 \cap H$; hence, H_1 has a unique complement in H.

Proposition 2. *The following conditions are equivalent:*

(i) *G is a cyclic group.*

(ii) *$\mathcal{L}(G)$ is a distributive lattice.*

Proof. (i) $\Longrightarrow$ (ii) If G is a cyclic group, then, from Proposition 1, we have $\mathcal{L}(G) \simeq L_n$, thus $\mathcal{L}(G)$ is a distributive lattice.

(ii) $\Longrightarrow$ (i) From the fundamental theorem on finitely generated abelian groups there exist (uniquely determined by G) the numbers $d_1, d_2, ..., d_k \in \mathbb{N} \setminus \{0, 1\}$ satisfying $d_1/d_2/.../d_k$ and

$$G \simeq \bigtimes_{i=1}^{k} \mathbb{Z}_{d_i}.$$

We shall prove that $k = 1$. If we suppose that $k \geq 2$, then let p be a prime divisor of d_1. Since d_1/d_2, we obtain that G has a subgroup isomorphic to the group $\mathbb{Z}_p \times \mathbb{Z}_p$. Using Lemma 1, it is sufficiently to verify that $\mathcal{L}(\mathbb{Z}_p \times \mathbb{Z}_p)$ is not a distributive lattice.

If $p = 2$, then $\mathcal{L}(\mathbb{Z}_p \times \mathbb{Z}_p) = \mathcal{L}(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq M_3$, which is not a distributive lattice.

If $p \geq 3$, then let H_1, H_2, H_3 be the subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ generated by the elements $(\hat{1}, \hat{2})$, $(\hat{1}, \hat{0})$, respectively $(\hat{0}, \hat{1})$. It is easy to see that we have:

$$
\begin{aligned}
&H_1 \cap H_2 = H_1 \cap H_3 = \{(\hat{0}, \hat{0})\} \\
&H_1 + H_2 = H_1 + H_3 = \mathbb{Z}_p \times \mathbb{Z}_p \\
&H_2 \neq H_3,
\end{aligned}
$$

therefore $\mathcal{L}(\mathbb{Z}_p \times \mathbb{Z}_p)$ is not a distributive lattice. Hence $k = 1$, i.e. $G \simeq \mathbb{Z}_{d_1}$ is a cyclic group.

Proposition 3. *The following conditions are equivalent:*

(i) $n = |G|$ *is square–free.*

(ii) $\mathcal{L}(G)$ *is a complemented lattice in which every element has a unique complement.*

Proof. (i) $\Longrightarrow$ (ii) If n is square–free, then $\mathcal{L}(G) \simeq \mathcal{L}(\mathbb{Z}_n) \simeq L_n$. Since L_n is a complemented lattice in which every element has a unique complement (for any $d \in L_n$ there exists a unique element $\bar{d} \in L_n$, $\bar{d} = \dfrac{n}{d}$, such that $(d, \bar{d}) = 1$ and $[d, \bar{d}] = n$), $\mathcal{L}(G)$ has the same property.

(ii) $\Longrightarrow$ (i) Let G_1 be the sum of all subgroups $H \in \mathcal{L}(G)$ which are simple groups. Then there exists a unique subgroup $\overline{G}_1 \in \mathcal{L}(G)$ such that $G_1 \oplus \overline{G}_1 = G$. We shall prove that $\overline{G}_1 = \{0\}$.

If we suppose that $\overline{G}_1 \neq \{0\}$, then there exists $x \in \overline{G}_1 \backslash \{0\}$. Using the Zorn's lemma, we obtain a maximal subgroup G_2 of $\overline{G}_1$ with property that $x \notin G_2$. From Lemma 1, there exists a unique subgroup $\overline{G}_2 \in \mathcal{L}(\overline{G}_1)$ such that $G_2 \oplus \overline{G}_2 = \overline{G}_1$. It follows that $\overline{G}_2 \neq \{0\}$.

Let $G_3 \neq \{0\}$ be a subgroup of $\overline{G}_2$. Then the inclusion $G_2 \subset G_2 \oplus G_3$ implies that $x \in G_2 \oplus G_3$. From the equality $\overline{G}_1 = G_2 \oplus G_3 = G_2 \oplus \overline{G}_2$ it results that $G_3 = \overline{G}_2$, thus $\overline{G}_2$ is a simple group. We obtain $\{0\} \neq \overline{G}_2 \subset G_1 \cap \overline{G}_1$, contrary to the fact that the sum $G_1 + \overline{G}_1$ is direct.

Hence $\overline{G}_1 = \{0\}$, therefore $G = G_1$. Since G is finite, there exist $H_1, H_2, ..., H_k \in \mathcal{L}(G)$ such that H_i is a simple group, for any $i = \overline{1,k}$ and $G = \bigoplus_{i=1}^{k} H_i$. For each $i \in \{1, 2, ..., k\}$ let p_i be a prime number with the property: $H_i \simeq \mathbb{Z}_{p_i}$. Using the fact that $p_i \neq p_j$ for $i \neq j$, we obtain

$$G \simeq \bigoplus_{i=1}^{k} \mathbb{Z}_{p_i} \simeq \bigtimes_{i=1}^{k} \mathbb{Z}_{p_i} \simeq \mathbb{Z}_{p_1 p_2 \ldots p_k},$$

i.e. $n = |G|$ is square–free.

Let Ab, Lat be the categories of abelian groups, respectively of lattices. We have a functor $\mathcal{L} : \text{Ab} \longrightarrow \text{Lat}$ given by:

a) for an abelian group G, $\mathcal{L}(G)$ is the lattice of subgroups of G;

(b) for a homomorphism of groups $f : G_1 \longrightarrow G_2$, $\mathcal{L}(f) : \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)$ is the homomorphism of lattices defined by $\mathcal{L}(f)(H_1) = f(H_1)$ for any $H_1 \in \mathcal{L}(G_1)$.

Remark. $\mathcal{L}$ is an exact functor.

Proposition 4. *If $f : G_1 \longrightarrow G_2$ is an epimorphism of abelian groups and*

$$L = \{H_1 \in \mathcal{L}(G_1) / \ker f \subseteq H_1\},$$

then:

(i) *L is a sublattice of $\mathcal{L}(G_1)$;*

(ii) *the function $\widetilde{f} : L \longrightarrow \mathcal{L}(G_2)$, $\widetilde{f}(H_1) = \mathcal{L}(f)(H_1) = f(H_1)$ for any $H_1 \in L$, is an isomorphism of lattices.*

Proof. (i) Obvious.

(ii) It remains to prove only that $\widetilde{f}$ is one-to-one and onto.

Let H_1, H_1' be two elements of L such that $\widetilde{f}(H_1) = \widetilde{f}(H_1')$, i.e. $f(H_1) = f(H_1')$. For any $x \in H_1$, we have $f(x) \in f(H_1) = f(H_1')$, so that there exists $y \in H_1'$ with $f(x) = f(y)$. It results that $f(x - y) = 0$, thus $x - y \in \ker f$. Since $\ker f \subseteq H_1'$, we obtain $x \in H_1'$; hence $H_1 \subseteq H_1'$. In the same way we can check the other inclusion; therefore $\widetilde{f}$ is one–to–one.

Let H_2 be a subgroup of G_2. Then $H_1 = f^{-1}(H_2) \in L$ and, using the fact that f is onto, we obtain:

$$\tilde{f}(H_1) = f(H_1) = (f \circ f^{-1})(H_2) = H_2;$$

therefore $\tilde{f}$ is onto.

Remark. The functor $\mathcal{L}$ reflects the isomorphisms, i.e. if $f : G_1 \longrightarrow G_2$ is a homomorphism of abelian groups such that $\mathcal{L}(f) : \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)$ is an isomorphism of lattices, then f is an isomorphism of groups.

Next aim is to study when, for two abelian groups G, G' of the same order n, the isomorphism of lattice $\mathcal{L}(G) \simeq \mathcal{L}(G')$ implies the isomorphism of groups $G \simeq G'$.

In order to solve this problem, it is necessary to minutely study the structure of the lattice $\mathcal{L}\left(\bigtimes_{i=1}^{k} \mathbb{Z}_{d_i}\right)$, where $d_i \in \mathbb{N}\backslash\{0,1\}$, $i = \overline{1,k}$ and $d_1/d_2/.../d_k$. We shall treat only the case $k = 2$, in the case $k \geq 3$ the problem remaining open.

Let $\pi : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}$ be the function defined by $\pi(x,y) = (\bar{x}, \widehat{y})$, for any $(x,y) \in \mathbb{Z}\times\mathbb{Z}$. π is an epimorphism of groups, therefore, by Proposition 4, we have the isomorphism of lattices:

$$\mathcal{L}\left(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\right) \simeq L_{d_1,d_2},$$

where $L_{d_1,d_2} = \{H \in \mathcal{L}(\mathbb{Z}\times\mathbb{Z}) \mid \ker\pi = d_1\mathbb{Z}\times d_2\mathbb{Z} \subseteq H\}$. It is a simple exercise to verify that:

Lemma 2. $\mathcal{L}(\mathbb{Z}\times\mathbb{Z}) = \{H_{p,q,r} = (p,q)\mathbb{Z} + (0,r)\mathbb{Z} \mid p,q,r \in \mathbb{N},\ q < r\}$.

From the above results, we obtain that

$$\mathcal{L}(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}) \simeq L_{d_1,d_2} =$$

$$= \left\{H_{p,q,r} = ((p,q)\mathbb{Z} + (0,r)\mathbb{Z} \mid p,q,r \in \mathbb{N},\ q < r,\ p/d_1,\ r/\left(d_2, \frac{d_1 q}{p}\right)\right\}.$$

Remark.

1) For two elements $H_{p,q,r}, H_{p',q',r'} \in L_{d_1,d_2}$ the following conditions are equivalent:

   (i) $H_{p,q,r} = H_{p',q',r'}$.

   (ii) $p = p',\ r/(q-q',r'), r'/(q-q',r)$.

   (iii) $(p,q,r) = (p',q',r')$.

2) For two elements $H_{p,q,r}, H_{p',q',r'} \in L_{d_1,d_2}$ the following conditions are equivalent:

   (i) $H_{p,q,r} \subseteq H_{p',q',r'}$.

   (ii) $p'/p,\ r'/\left(r, q - q'\dfrac{p}{p'}\right)$.

Let A_n be the set $\{(d_1,d_2) \in (\mathbb{N}\backslash\{0,1\})^2 \mid d_1/d_2, d_1d_2 = n\}$ and $g : A_n \to \mathbb{N}^*$ be the function defined by $g(d_1,d_2) = \text{card}\,\mathcal{L}(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2})$ for any $(d_1,d_2) \in A_n$. If $d_1 = p_1^{\alpha_1}p_2^{\alpha_2}...p_s^{\alpha_s}$, $d_2 = p_1^{\beta_1}p_2^{\beta_2}...p_s^{\beta_s}$ are the decompositions of d_1, respectively d_2, as a product of prime factors (i.e. p_1 = prime number, $\alpha_i, \beta_i \in \mathbb{N}$, $\alpha_i \le \beta_i$, $i = \overline{1,s}$ and $p_i \ne p_j$ for $i \ne j$), then we have the following result:

Lemma 3.

$$g(d_1,d_2) = \prod_{i=1}^{s} \frac{1}{(p_i-1)^2}[(\beta_i-\alpha_i+1)p_i^{\alpha_i+2} - (\beta_i-\alpha_i-1)p_i^{\alpha_i+1} - $$
$$-(\alpha_i+\beta_i+3)p_i + (\alpha_i+\beta_i+1)].$$

Corollary. *We have:*

(i) $\text{card}\,\mathcal{L}(\mathbb{Z}_2\times\mathbb{Z}_4) = 8$.

(i) $\text{card}\,\mathcal{L}(\mathbb{Z}_2\times\mathbb{Z}_2) = 5$.

(i) $\text{card}\,\mathcal{L}(\mathbb{Z}_p\times\mathbb{Z}_p) = p+3$, *where p is a prime number.*

Now we can prove the main result of this paper:

Proposition 5. *Let $n \geq 2$ be a natural number, s be the number of distinct prime divisors of n and G, G' be two abelian groups of order n which have (corresponding to the fundamental theorem on finitely generated abelian groups) the decompositions:*

$$G \simeq \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2},$$

respectively

$$G' \simeq \mathbb{Z}_{d'_1} \times \mathbb{Z}_{d'_2}.$$

Then, for $s \in \{1, 2\}$, the isomorphism of lattice $\mathcal{L}(G) \simeq \mathcal{L}(G')$ implies the isomorphism of groups $G \simeq G'$.

Proof. If $p_1, p_2, ..., p_s$ are the distinct prime divisors of n and $n = p_1^{h_1} p_2^{h_2} ... p_s^{h_s}$, $d_1 = p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}$, $d_2 = p_1^{\beta_1} p_2^{\beta_2} ... p_s^{\beta_s}$, $d'_1 = p_1^{\alpha'_1} p_2^{\alpha'_2} ... p_s^{\alpha'_s}$, $d'_2 = p_1^{\beta'_1} p_2^{\beta'_2} ... p_s^{\beta'_s}$ are the decompositions of n, d_1, d_2, d'_1, d'_2 as a product of prime factors (where $h_i, \alpha_i, \beta_i, \alpha'_i, \beta'_i \in \mathbb{N}$, $\alpha_i \leq \beta_i$, $\alpha'_i \leq \beta'_i$, $\alpha_i + \beta_i = \alpha'_i + \beta'_i = h_i$, $i = \overline{1, s}$), then we have:

$$g(d_1, d_2) = g(d'_1, d'_2). \tag{1}$$

Since $\mathcal{L}(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}) \simeq \mathcal{L}(\mathbb{Z}_{d'_1} \times \mathbb{Z}_{d'_2})$, there exists an isomorphism of lattice $f : L_{d_1, d_2} \longrightarrow L_{d'_1, d'_2}$. If we consider:

(i) $p_0 = 1$, $q_0 = 0$, $r_0 = d_2$,

then we have $H_{1,0,d_2} \subseteq H_{1,0,d}$ for any $d \in \mathbb{N}$, d/d_2, therefore

$$H_{p'_0, q'_0, r'_0} \overset{\text{not}}{=} f(H_{1,0,d_2}) \subseteq f(H_{1,0,d}) \overset{\text{not}}{=} H_{p'_d, q'_d, r'_d} \text{ for any } d \in \mathbb{N},\ d/d_2;$$

it results r'_d / r'_0 for any $d \in \mathbb{N}$, d/d_2, thus the number of divisors of d_2 is at most the number of divisors of r'_0, so that (because r'_0 / d'_2) at most the number of divisors of d'_2;

(ii) $p_0 = d_1$, $q_0 = 0$, $r_0 = 1$,

then we have $H_{d_1,0,1} \subseteq H_{e,0,1}$ for any $e \in \mathbb{N}$, e/d_1, therefore

$$H_{p''_0, q''_0, r''_0} \overset{\text{not}}{=} f(H_{d_1,0,1}) \subseteq f(H_{e,0,1}) \overset{\text{not}}{=} H_{p'_e, q'_e, r'_e} \text{ for any } e \in \mathbb{N},\ e/d_1;$$

it results p'_e/p''_0 for any $e \in \mathbb{N}$, e/d_1, thus the number of divisors of d_1 is at most the number of divisors of p''_0, so that (because p''_0/d'_1) at most the number of divisors of d'_1.

Starting by the isomorphism of lattice $f^{-1} : L_{d'_1,d'_2} \longrightarrow L_{d_1,d_2}$ and making a similarly reasoning, we obtain the converses of the above inequalities. Thus, for $i \in \{1,2\}$, the number of divisors of d_i is equal with the number of divisors of d'_i, i.e. the following equalities hold:

$$\begin{cases} \prod_{i=1}^{s}(\alpha_i+1) = \prod_{i=1}^{s}(\alpha'_i+1) \\ \prod_{i=1}^{s}(\beta_i+1) = \prod_{i=1}^{s}(\beta'_i+1). \end{cases} \tag{2}$$

If $s = 1$, then, from equalities (2), we obtain $\alpha_1 = \alpha'_1$ and $\beta_1 = \beta'_1$, thus:

$$G \simeq \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} = \mathbb{Z}_{d'_1} \times \mathbb{Z}_{d'_2} \simeq G'.$$

If $s = 2$, then, from the first of the equalities (2), we obtain:

$$\alpha'_1 + 1 = u(\alpha_1+1),\ \alpha'_2+1 = \frac{1}{u}(\alpha_2+1),$$

where $u \in Q^*$.

If we consider the function $F : \mathbb{R}^*_+ \longrightarrow \mathbb{R}^*_+$ defined by

$$\begin{aligned} F(x) &= [(h_1+3-2(\alpha_1+1)x)p_1^{(\alpha_1+1)x+1} - (h_1+1-2(\alpha_1+1)x)p_1^{(\alpha_1+1)x} - \\ &\quad -(h_1+3)p_1 + (h_1+1)]\left[\left(h_2+3-2\frac{\alpha_2+1}{x}\right)p_2^{\frac{\alpha_2+1}{x}} - \right. \\ &\quad \left. -\left(h_2+1-2\frac{\alpha_2+1}{x}\right)p_2^{\frac{\alpha_2+1}{x}} - (h_2+3)p_2 + (h_2+1)\right] \end{aligned}$$

for any $x \in \mathbb{R}^*_+$, then the equality (1) becomes:

$$F(u) = F(1). \tag{3}$$

We can suppose that $u \geq 1$. On the interval $[1, \infty)$ we have $F' < 0$, so that F is an one–to–one function. Therefore the equality (3) implies $u = 1$. It follows that $\alpha_1 = \alpha'_1$ and $\alpha_2 = \alpha'_2$. Since $\alpha_1 + \beta_1 = \alpha'_1 + \beta'_1 = h_1$ and

$\alpha_2 + \beta_2 = \alpha'_2 + \beta'_2 = h_2$, it results that $\beta_1 = \beta'_1$ and $\beta_2 = \beta'_2$. Hence we have $d_1 = d'_1$ and $d_2 = d'_2$, thus:

$$G \simeq \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} = \mathbb{Z}_{d'_1} \times \mathbb{Z}_{d'_2} \simeq G'.$$

Remark. In the case when the number s of distinct prime divisors of n is arbitrary, the equalities (1) and (2) are not sufficient to obtain $d_1 = d'_1$ and $d_2 = d'_2$. However, if the following conditions are satisfied:

(i) $h_i = 1$ for any $i \in \{1, 2, ..., s\}$ (i.e. n is square–free)
or

(ii) $h_i = 1$ for any $i \in \{1, 2, ..., s\} \setminus \{i_0\}$
(i.e. n has the form $p_1 ... p_{i_0 - 1} p_{i_0}^{h_{i_0}} p_{i_0 + 1} ... p_s$),

then it is easy to see that the conclusion of Proposition 5 holds.

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