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On the structure of composite odd integers and prime numbers

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Abstract

With the expression "structure of an odd integer" we mean the set of properties of the integer n which specifies the odd integer $2n+1$ and brings about its behaviour. These properties of n, for both composite and prime numbers, are expounded in detail, together with their geometrical implications. In this context, a set, in a two dimensional space, where all the composite odd integers in $[2a+1, 2n+1]$ are localized, is illustrated.

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1 Introduction

Let $2n+1$ be any composite odd integer, and $2i+1$, $2j+1$ a couple of its divisors. In [1] it is pointed out that

$$2n+1=(2i+1)(2j+1) \iff n=k_{ij}, \tag{1}$$

where k_{ij} is the following symmetric form in i and j

$$k_{ij}=i+j+2ij, \quad (i,j)\in\mathbb{N}. \tag{2}$$

Let $\mathbb{K}=\{k_{ij}|(i,j)\in\mathbb{N}\}\subset\mathbb{N}$. Since any odd integer is either a composite or a prime number, it follows that

$n\in\mathbb{K}\iff 2n+1$ is a composite number, $\quad n\in\mathbb{N}\setminus\mathbb{K}\iff 2n+1$ is a prime number.

Some properties of the matrix $\{k_{ij}\}$ of finite order are reported in the appendix 1 of [1], where the relation $n=k_{ij}$ has been used to identify composite odd integers $2\nu+1$, with $4=k_{11}\le\nu\le n$, and having the same number of couples of divisors. The distributions of odd integers $\le 2n+1$ vs. the number of their couple of divisors have been computed up to $n\simeq 5\cdot 10^7$.

The motivation for this work is just the reference [1], unique reference for this paper, and very useful for the analysis of the problem under study.

In section 2 we enlighten the basic properties of n for both composite odd numbers and primes. In section 3 their main geometrical implications are illustrated. In particular, the relation $n=k_{ij}$ for composite odd integers suggested to investigate the localization of the pairs (i,j) producing k_{ij} in a given interval. Thus, a set $\Omega(a,n)$ of points $(x,y)\in\mathbb{R}^2$ containing all the points with integer coordinates (i,j) such that $a\le k_{ij}\le n$ is defined.

2 The basic properties of n specifying the behaviour of $2n+1$

For a pair (i,j), identifying any possible couple of divisors of an odd integer $2n+1$, we should have

$$n=k_{ij}, \tag{3}$$

where k_{ij} is defined in (2). Furthermore, any couple $(2i+1,2j+1)$ of divisors of $2n+1$, with $i\le j$, must satisfy the following inequality

$$2i+1 \leq \sqrt{2n+1} \leq 2j+1 \quad (equality\ holds\ iff\ i=j).$$

It follows that

$$1 \leq i \leq I_n = \frac{1}{2}(-1+\sqrt{2n+1}) \leq j. \tag{4}$$

Let $\mathbb{I}_n = \{1, 2, ..., i_n\}$, with $i_n = [I_n]$ =integer part of I_n.

The implications (1) state that the special form (2) of n is a necessary and sufficient condition in order for any odd integer $2n+1$ to be a composite number. Moreover, if $2n+1$ is a prime, then n cannot be put in the form (2). In the following theorem, given a generic n, necessary and sufficient conditions are given for either $n \in \mathbb{K}$ or $n \in \mathbb{N} \setminus \mathbb{K}$.

Theorem 2.1. *Let* $n \in \mathbb{N}$. *Then*

$$n \in \mathbb{K} \quad iff \quad \exists i \in \mathbb{I}_n : (2i+1) \mid (n-i), \tag{5}$$

$$n \in \mathbb{N} \setminus \mathbb{K} \quad iff \quad \forall i \in \mathbb{I}_n : (2i+1) \nmid (n-i). \tag{6}$$

Proof. A non implicit relation between i and j is obtained by making explicit the variable j in (3). In doing so, (3) is written in the form of a real homographic function of the integer variable i

$$y = \phi_n(i) = \frac{n-i}{2i+1}, \quad i \in \mathbb{I}_n,\ y \in \mathbb{R}. \tag{7}$$

If for some i we have $y \in \mathbb{N}$, then the pair (i, j) with $j = y = [y]$ identifies a couple of divisors of $2n+1$. Thus, in this way we can identify all the couples of divisors of $2n+1$, if they exist. The theses follow directly from (7).

□

The i_n conditions assumed to have $n \in \mathbb{N} \setminus \mathbb{K}$ are not superabundant. Consequences of theorem 1 are the following two propositions.

If $n \in \mathbb{K}$, then n has a number of representations $n = k_{ij}$ just equal to the number of couples of divisors of $2n+1$.

If $n \in \mathbb{N} \setminus \mathbb{K}$, then n can be expressed as

$$n = i + (2i+1)q_i + r_i = k_{iq_i} + r_i,$$

with $q_i, r_i \in \mathbb{N}$ and $1 \leq r_i < 2i+1,\ \forall i = 1, 2, ..., i_n$.

Remark 2.1. (r_1) $\phi_n(i)$ *is decreasing with* i *from* $\phi_n(1) = (n-1)/3$ *to* $\phi_n(i_n)$.

(r_2) *Equation* $y = \phi_n(i)$ *is a simple algorithm to compute the couples of divisors of an integer* $2n+1$*, if they exist. It can also be used as a primality test. The order of the number of operations is* $\sqrt{n/2}$.

(r_3) *By direct calculation it follows that*

$$2n + 1 = 2k_{ij} + 1 = (i + j + 1)^2 - (j - i)^2.$$

This identity shows the well known fact that any composite odd integer may be written as a difference of two squares. Possibly in more than two ways, while for a prime only holds the decomposition $2n + 1 = (n + 1)^2 - n^2$.

(r_4) *Let* m_q *be the sequence of integers defined recursively by*

$$m_q = 2m_{q-1} + 1, \quad q = 1, 2, ...,\ m_0 = 0,$$

which implies that $m_q = 2^q - 1$*. When* q *is a prime,* m_q *is a Mersenne number. Let* $q = 2r$*. Since* $m_{2r} = 2^{2r} - 1 = (2^r + 1)(2^r - 1)$ *is composite, we have that* $m_{2r-1} \in \mathbb{K}, r = 2, 3, ...$

3 Geometrical implcations of the properties of n

The relation $n = k_{ij}$ can be rewritten as

$$n = i(j + 1) + (i + 1)j, \tag{8}$$

which can be interpreted as the determinant of the matrix

$$A = \begin{pmatrix} i & -j \\ \text{i+1} & \text{j+1} \end{pmatrix}.$$

Since $det(A) = n$, n is the area of the parallelogram with the column vectors

$$u = (i, j + 1)^T,\ v = (-j, j + 1)^T,$$

as two of its sides. This property holds $\forall(i, j)$ such that $k_{ij} = n$. In addition to the equality of determinants, no other relationships have been recognized between the matrices A related to the pairs (i, j) with $k_{ij} = n$.

Moreover (8) shows that n is also the area of the union of the two rectangles of sides $i, (j+1)$ and $(i+1), j$.

Imagine that you are drawing, on a squared sheet, rectangles composed by an odd integer number of squares. The side of the squares is assumed as linear unit of measure, so that the rectangles have integer sides and area. Obviously, only integers $2n+1$ with n satisfying (8) can represent rectangles. The number $2n+1$ is the area of the rectangle of sides $(2i+1), (2j+1)$. Moreover, with the help of (8), we can show that this rectangle is the union of five rectangles: two with sides $i, j+1$, two with sides $i+1, j$, plus one unit square (Fig. 1).

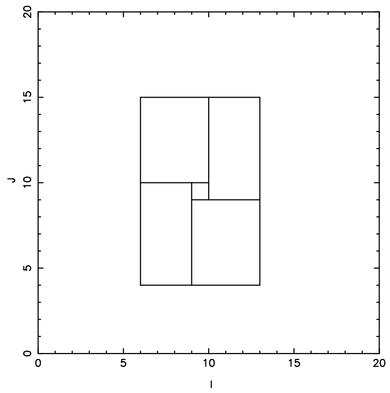

Figure 1: Decomposition of the rectangle of sides $2i+1$, $2j+1$. $i = 3$, $j = 5$, $n = 3 \cdot (5+1) + (3+1) \cdot 5 = 38$, $2n+1 = 2 \cdot 38 + 1 = 77$, $(2i+1) \cdot (2j+1) = 7 \cdot 11 = 77$

Thus we have shown that, when $n \in \mathbb{K}$, then n represents also the area of the union of two rectangles, and $2n+1$ both the area of one rectangle and the union of five rectangles.

When $n \notin \mathbb{K}$, then n can be represented as $n = k_{iq_i} + r_i$, with $r_i \geq 1$, which leads to the following representation for $2n+1$

$$2n+1 = 2k_{iq_i} + 1 + 2r_i = (2i+1)(2q_i+1) + 2r_i, \quad 1 \leq r_i < 2i+1.$$

Since $n \notin \mathbb{K}$, this expression can not be reduced to the product of two odd integers.

Let us now consider the problem of the localization of composite odd integers $2\nu + 1$, with $4 \leq a \leq \nu \leq n$. The discrete relation $n = k_{ij}$ is viewed in the continuous form as $n = \kappa(x, y) = x + y + 2xy, \quad (x, y) \in \mathbb{R}^2$, and in explicit form it is written as

$$y = \phi_n(x) = \frac{n-x}{2x+1}, \quad 1 \leq x \leq I_n. \tag{9}$$

$\phi_n(x)$ is decreasing with x from $\phi_n(1) = (n-1)/3$ to $\phi_n(I_n) = I_n$.

Let the set $\Omega(a, n)$ be defined as

$$\Omega(a, n) = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq I_n,\ (x \leq y) \vee (a \leq \kappa(x, y) \leq n)\}. \qquad (10)$$

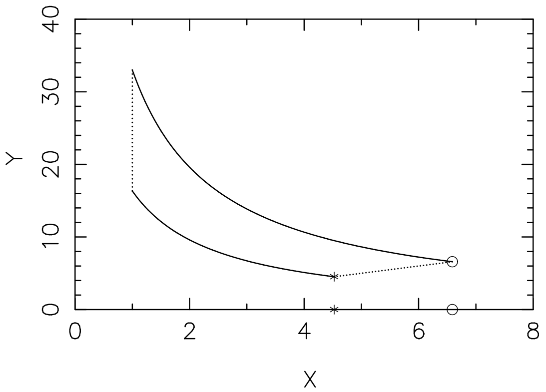

Figure 2: Set $\Omega(a, n)$: $n = 100$, $a = 50$. Continuous lines: $y = \phi_n(x)$ and $y = \phi_a(x)$; dotted lines: $x = 1$ and $y = x$; asterisk: points $(I_a, 0), (I_a, I_a)$; circle: points $(I_n, 0), (I_n, I_n)$. Different scales for x and y.

It is a bounded, closed and convex set in a plane. From its implicit definition (10), it follows that $\Omega(a, n)$ (Fig. 2) can be defined explicity by the following inequalities as the union of two sets

$$1 \leq x \leq I_a, \quad \phi_a(x) \leq y \leq \phi_n(x), \qquad (11)$$

$$I_a < x \leq I_n, \quad x \leq y \leq \phi_n(x). \qquad (12)$$

Both $\phi_a(x)$ and $\phi_n(x)$ are decreasing with x, and also their difference

$$\Delta_{n,a}(x) = \phi_n(x) - \phi_a(x) = \frac{n-a}{2x+1},\ 1 \leq x \leq I_a. \qquad (13)$$

Thus, when $n - a$ is small enough, we have that $\Delta_{n,a}(x) < 1$, and $\Omega(a, n)$ may contain at the most one integer point. Other details on this argument can be found in appendix.

4 Concluding remarks

It is worthy to draw a concluding remark. In other words, Theorem 2.1 states that any prime has just one parent: if a generic n satisfies (6) in Theorem 2.1, then it is the parent of $2n + 1$. Otherwise it is a composite number.

References

[1] G. Buffoni. On odd integers and their couples of divisors. *Ratio Mathematica*, 40: 87-111, 2021.

Appendix

Cosider the limit case $a = 4$ (Fig. 3, left), so that $I_a = 1$, $\phi_4(1) = 1$. Thus, the first set (11) consists of only one point $x = I_a = 1$. Since $\phi_a(I_a) = I_a$, (11) and (12) reduce to

$$1 \le x \le I_n, \quad x \le y \le \phi_n(x).$$

Consider the case a large, i.e. $n - a$ small (Fig. 3, right). Assume $n - a << a$. The difference $I_n - I_a$ may be written as

$$I_n - I_a = 0.5\,\sqrt{2n+1}\,(1 - \sqrt{\frac{2a+1}{2n+1}}) = 0.5\,\sqrt{2n+1}\,(1 - \sqrt{1 - \frac{2(n-a)}{2n+1}}).$$

From the assumption $n - a << a$ we have $2(n-a)/(2n+1) << 1$, so that

$$I_n - I_a \simeq \frac{n-a}{2\sqrt{2n+1}} < 1. \tag{14}$$

Thus, the second set (12) reduces to a very small interval.

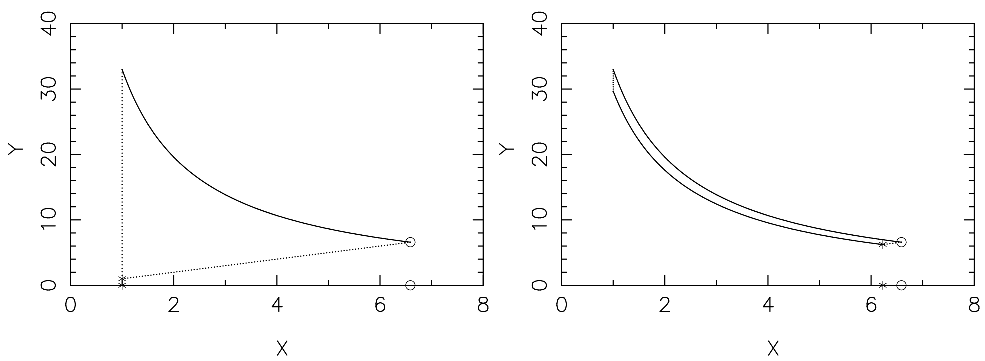

Figure 3: Set $\Omega(a,n)$: $n = 100$. Left: limit case $a = 4$; right: case $a = 90$, so that $I_n - I_a < 1$. Notes as in caption of Fig. 2

Consider now the set of points with integer coordinates. The set $\Omega^*(a,n)$ of the pairs (i,j), such that $1 \le i \le j,\ a \le k_{ij} \le n$, defined by

$$\Omega^*(a,n) = \{(i,j) \in \mathbb{N} \mid i = 1,2,...,I_n,\ (i \le j) \vee (a \le k_{ij} \le n)\}, \qquad (15)$$

is the set of the points $(x, y) \in \Omega(a, n)$ with x and y integer coordinates. Afterwards, it is straightforward to formulate the explicit definition of the pairs $(i, j) \in \Omega^*(a, n)$. Then, $\Omega^*(a, n)$ is again defined as the union of two sets

$$i = 1,2,...,[I_a], \quad j = J(i), J(i)+1,...,[\phi_n(i)], \qquad (16)$$

where either $J(i) = [\phi_a(i)] + 1$ when $\phi_a(i) \notin \mathbb{N}$ or $J(i) = \phi_a(i)$ when $\phi_a(i) \in \mathbb{N}$, and

$$i = [I_a]+1,...,[I_n], \quad j = i, i+1,...,[\phi_n(i)]. \qquad (17)$$

When $b-a$ is small enough, it follows that $I_n - I_a < 1$, so that either $[I_n] = [I_a]$ or $[I_n] = [I_a] + 1$. The second set (17) is missing when $[I_n] = [I_a]$, and consists of only one point when $[I_n] = [I_a] + 1$; thus (16) and (17) reduce to

$$i = 1,2,...,[I_n], \quad j = J(i), J(i)+1,...,[\phi_n(i)]. \qquad (18)$$

If $\nu \in \{a, a+1,...,n\} \setminus \{k_{ij} \mid (i,j) \in \Omega^*(a,n)\}$, then $2\nu + 1$ is a prime.