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Fuzzy soft set connected mappings

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Abstract

In this paper, the concepts of fuzzy soft connectedness between fuzzy soft sets and fuzzy soft set connected mappings in fuzzy soft topological spaces has been introduced. It is shown that a fuzzy soft topological space is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty fuzzy soft sets. Every fuzzy soft continuous mapping is fuzzy soft set-connected a counter example is given to show the converse may not be true. Several properties of fuzzy soft set-connected mappings in fuzzy soft topological spaces have been studied.

Keywords: Fuzzy soft sets; Fuzzy soft connectedness; Fuzzy soft connectedness between fuzzy soft sets and Fuzzy soft set-connected mappings.

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1 Introduction

The notion of a fuzzy set was introduced by Zadeh [1965] as a generalization of classical set in the year 1965. Chang [1968] gave the definition of fuzzy topology and extended some topological concepts to fuzzy sets. In 1999, Molodtsov [1999] introduced the concept of soft sets to deal with uncertainties while modelling the problems with incomplete information. In 2011, Shabir and Naz [2011] initiated the study of soft topological spaces as a generalization of topological spaces. The hybrid structure of fuzzy sets and soft set called fuzzy soft set was created by Maji et al. [2001]. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [2009] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. Tanay and Kandemir [2011] introduced the fuzzy soft topology as an extension of fuzzy topology and soft topology. Fuzzy topological spaces are further investigated by Varol and Aygun [2012] and shown that a fuzzy soft topological space gives a parametrized family of fuzzy topological spaces. Roy and Samanta [2011] and Tridiv et al. [2012] are also investigated various topological concepts in fuzzy soft topological spaces. The study of connectedness in fuzzy soft topology was initiated by Karatas et al. [2015] and further studied by Kandil et al. [2017]. Connectedness between sets and set connected mappings is one of the important topic of research in Topology. In 2018, Thakur and Rajput [2018] extended and studied these concepts to soft topology. Till the date these concepts are not studied in fuzzy soft topology. Therefore to fill up this gap, the present paper introduces the concept of connectedness between fuzzy soft sets and studied some of its properties in fuzzy soft topological spaces. Further the concepts of fuzzy soft set-connected mappings are defined and established some theorems related to its characterizations and properties. The results established in this paper generalized many results which are already available in the literature.

2 Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X, I = [0,1] and I^X is the set of all fuzzy sets on X. The reader should refer Molodtsov [1999] and Zadeh [1965] for the basic concepts on fuzzy sets and soft sets.

Definition 2.1. *(Maji et al. [2001]) Let A $\subset$ E. A pair (f,A) is called a fuzzy soft set (in short* $\mathbf{FSS}$*) over X, where* $f: A \to I^X$ *defined by , (f,A)(e) =* $\mu^e_{(f,A)}$ *,where* $\mu^e_{(f,A)} = \tilde{0}$ *if* $e \notin A$ *and* $\mu^e_{(f,A)} \neq \tilde{0}$ *if* $e \in A$ *.*

The family of all $\mathbf{FSS}$s over (X,E) will be denoted by $\mathbf{FS(X,E)}$. For the notations, basic operations and properties of FSSs the reader should refer Ahmad and Kharal [2009], Maji et al. [2001].

Definition 2.2. *(Tanay and Kandemir [2011]) A subfamily τ of FS(X, E) is called a fuzzy soft topology on X if:*

(1) $\widetilde{\phi}$, $\widetilde{X}$ belong to τ.

(2) The union of any number of soft sets in τ belongs to τ.

(3) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a fuzzy soft topological space (in short FSTS). The members of τ are called fuzzy soft open sets in X and their complements called fuzzy soft closed sets in X.

The basic fuzzy soft topological concepts can be seen in Kandil et al. [2017], Karatas et al. [2015], Roy and Samanta [2011], Tridiv et al. [2012] and Varol and Aygun [2012]).

Lemma 2.1. *(Roy and Samanta [2011]) Let (f,A) and (g,B) be two FSSs. Then, $(f, A) \subseteq (g, B)^c \Leftrightarrow (f, A)\tilde{q}(g, B)$. Where $(f, A)\tilde{q}(g, B)$ means (f,A) is not quasi-coincident with (g,B).*

Lemma 2.2. *(Tridiv et al. [2012]) Let (Y, τ_Y, E) be a fuzzy soft subspace of a FSTS (X, τ, E) and (F, E) be a fuzzy soft open set in Y. If $\widetilde{Y} \in \tau$ then (F, E) $\in \tau$.*

Lemma 2.3. *(Tridiv et al. [2012]) Let (X,τ,E) be a FSTS and (Y, τ_Y,E) be a fuzzy soft subspace of (X, τ, E), then a fuzzy soft closed set (F_Y, E) of Y is fuzzy soft closed in X if and only if $\widetilde{Y}$ is fuzzy soft closed in X.*

3 Connectedness between fuzzy soft sets

Throughout this paper fuzzy soft clopen means fuzzy soft closed open.

Definition 3.1. *(Kandil et al. [2017], Karatas et al. [2015]) A FSTS (X,τ,E) is fuzzy soft connected if and only if there is no nonempty FSS of (X,τ,E) which is both fuzzy soft open and fuzzy soft closed in (X,τ,E).*

Definition 3.2. *A FSTS (X ,τ, E) is said to be fuzzy soft connected between FSSs (f_1,E) and (f_2,E) if and only if there is no fuzzy soft clopen set (f,E) over X such that (f_1,E) $\subset$ (f,E) and (f,E) $\tilde{q}$ (f_2,E).*

Theorem 3.1. *A* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E) *if and only if there is no fuzzy soft clopen set (f,E) over X such that* $(f_1,E) \subset (f,E) \subset (f_2,E)^c$.

Proof. *Follows from Definition 3.2. and Lemma 2.1.*□

Theorem 3.2. *If a* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E) *then* $(f_1,E) \neq \phi \neq (f_2,E)$.

Proof. *If any* $\mathbf{FSS}$ $(f_1,E) = \phi$*, then* (f_1,E) *is a fuzzy soft clopen set over X such that* $(f_1,E) \subset (f_2,E)$ *and* $(f_1,E)\ \tilde{q}\ (f_2,E)$ *and hence (X,τ,E) can not be fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E)*, which is contradiction.*□

Theorem 3.3. *If a* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E) *and if* $(f_1,E) \subset (f_3,E)$ *and* $(f_2,E) \subset (f_4,E)$ *then (X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* (f_3,E) *and* (f_4,E).

Proof. *Suppose* $\mathbf{FSTS}$ *(X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s* (f_3,E) *and* (f_4,E) *then there is a fuzzy soft clopen set (f,E) over X such that* $(f_3,E) \subset (f,E)$ *and* $(f,E)\ \tilde{q}\ (f_4,E)$ *.Clearly* $(f_1,E) \subset (f,E)$*. Now we claim that* $(f,E)\ \tilde{q}\ (f_2,E)$ *.If (f,E) q* (f_2,E) *then there exists a point* $x \in X$ *such that* $\mu^e_{(f,E)}(x) + \mu^e_{(f_2,E)}(x) \succ 1$*. Therefore* $\mu^e_{(f,E)}(x) + \mu^e_{(f_4,E)}(x) \succ \mu^e_{(f,E)}(x) + \mu^e_{(f_2,E)}(x) \succ 1$ *and (f,E) q* (f_4,E)*, a contradiction. Consequently, (X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E).□

Theorem 3.4. *A* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E) *if and only if (X,τ,E) is fuzzy soft connected between* $\mathbf{FSS}$*s* $Cl(f_1,E)$ *and* $Cl(f_2,E)$.

Proof. *Necessity : Follows from Theorem 3.3.*

Sufficiency : Suppose $\mathbf{FSTS}$ *(X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s* (f_1,E) *and* (f_2,E)*, then there exists fuzzy soft clopen set (f,E) over X such that* $(f_1,E) \subset (f,E)$ *and* $(f,E)\ \tilde{q}\ (f_2,E)$*. Since (f,E) is fuzzy soft closed,* $Cl(f_1,E) \subset Cl(f,E) = (f,E)$*. Clearly, by Lemma 2.1,* $(f,E)\ \tilde{q}\ (f_2,E) \Leftrightarrow (f,E) \subset (f_2,E)^c$*. Therefore* $(f,E) = Int(f,E) \subset Int((f_2,E)^c) = (Cl(f_2,E))^c$*. Hence,* $(f,E)\ \tilde{q}\ Cl(f_2,E)$ *and (X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s* $Cl(F_1,E)$ *and* $Cl(F_2,E)$.□

Theorem 3.5. *If* (f_1,E) *and* (f_2,E) *are two* $\mathbf{FSS}$*s in* $\mathbf{FSTS}$ *(X,τ,E) and* (f_1,E) *q* (f_2,E)*, then (X,τ,E) is fuzzy soft connected between* (f_1,E) *and* (f_2,E).

Proof. *If (f,E) is any fuzzy soft clopen set over X such that* $(f_1,E) \subset (f,E)$*, then* (f_1,E) *q* $(f_2,E) \Rightarrow (f,E)$ *q* (f_2,E)*. This proves the theorem.*□

Remark 3.1. *The converse of Theorem 3.5 need not be true .*

Example 3.1. *Let* $X = \{a,b\}$ *be universe set and* $E = \{e_1, e_2\}$ *be the set of* *parameters. The* $\mathbf{FSS}$*s Let (f,E), (*f_1*,E) and (*f_2*,E) over X are defined as follows:*

$$f_1(e_1) = \{(a, 0.3), (b, 0.4)\} \qquad f_1(e_2) = \{(a, 0.2), (b, 0.3)\}$$
$$f_2(e_1) = \{(a, 0.5), (b, 0.4)\} \qquad f_2(e_2) = \{(a, 0.6), (b, 0.5)\}$$
$$f_3(e_1) = \{(a, 0.3), (b, 0.5)\} \qquad f_3(e_2) = \{(a, 0.3), (b, 0.2)\}.$$

Let $\tau = \{\tilde{0_E}, \tilde{1_E}, (f_1, E)\}$ *be a fuzzy soft topology on X, then* $\mathbf{FSTS}$ *(X,*τ*,E) is fuzzy soft connected between the* $\mathbf{FSS}$*s (*f_2*,E) and (*f_3*,E), but (*f_2*,E)* $\tilde{q}$ *(*f_3*,E).*

Theorem 3.6. *If a* $\mathbf{FSTS}$ *(X,*τ*,E) is neither fuzzy soft connected between (f,E) and (*g_0*,E) nor fuzzy soft connected between (f,E) and (*g_1*,E) then it is not fuzzy soft connected between (f,E) and (*g_0*,E)* $\cup$ *(*g_1*,E).*

Proof. *Since a* $\mathbf{FSTS}$ *(X,*τ*,E) is not fuzzy soft connected between (f ,E) and (*g_0*,E), there is a fuzzy soft clopen set (*h_0*,E) over X such that (f,E)* $\subset$ *(*h_0*,E) and (*h_0*,E)* $\tilde{q}$ *(*g_0*,E). Also since (X,*τ*,E) is not fuzzy soft connected between (f ,E) and (*g_1*,E) there exists a fuzzy soft clopen set (*h_1*,E) over X such that (f,E)* $\subset$ *(*h_1*,E) and (*g_1*,E)* $\tilde{q}$ *(*h_1*,E). Put (h,E) = (*h_0*,E)* $\cap$ *(*h_1*,E). Since any intersection of fuzzy soft closed sets is fuzzy soft closed, (h,E) is fuzzy soft closed. Again intersection of finite family of fuzzy soft open sets is fuzzy soft open, (h,E) is fuzzy soft open. Therefore (h,E) is fuzzy soft clopen set over X such that (f,E)* $\subset$ *(h,E) and (h,E)* $\tilde{q}$*((*g_0*,E)* $\cup$ (g_1, E)*). If (h,E) q ((*g_0*,E)* $\cup$ (g_1, E)*) there exists x* $\in$ *X such that* $\mu^e_{(h,E)}$ *(x) + (*$\mu^e_{(g_0,E)} \cup \mu^e_{(g_1,E)}$*)(x)* $\succ$ *1. This implies that (h,E) q (*g_0*,E) or (h,E) q (*g_1*,E) a contradiction. Hence, (X,*τ*,E) is not fuzzy soft connected between (f,E) and* $(g_0, E) \cup (g_1, E)$. □

Theorem 3.7. *A* $\mathbf{FSTS}$ *(X,*τ*,E) is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty* $\mathbf{FSS}$*s.*

Proof. *Let (f,E) and (g,E) be a pair of nonempty* $\mathbf{FSS}$*s over X. Suppose (X,*τ*,E) is not fuzzy soft connected between (f,E) and (g,E). Then there is a fuzzy soft clopen set (h,E) over X such that (f,E)* $\subset$ *(h,E) and (g,E)* $\tilde{q}$ *(h,E). Since (f,E) and (g,E) are nonempty it follows that (h,E) is a nonempty fuzzy soft proper clopen set over X. Hence, (X,*τ*,E) is not fuzzy soft connected.*

Conversely, suppose that (X,τ*,E) is not fuzzy soft connected. Then there exists a nonempty proper* $\mathbf{FSS}$ *(h,E) over X which is both fuzzy soft open and fuzzy soft closed. Consequently, (X,*τ*,E) is not fuzzy soft connected between (h,E) and* $(h, E)^c$*. Thus, (X,*τ*,E) is not fuzzy soft connected between arbitrary pair of its nonempty* $\mathbf{FSS}$*s.*□

Remark 3.2. *If a* $\mathbf{FSTS}$ *(X,*τ*,E) is fuzzy soft connected between a pair of its* $\mathbf{FSS}$*s, then it is not necessarily that it is fuzzy soft connected between each pair of its* $\mathbf{FSS}$*s and so it is not necessarily fuzzy soft connected.*

Example 3.2. *Let X = $\{a,b\}$ be an universe set, E = $\{e_1, e_2\}$ be the set of parameter and the soft sets (f_1, E) , (f_2, E) , (f_3,E), (f_4,E), (f_5,E) and (f_6,E) over X are defined as follows:*

$$
\begin{array}{ll}
f_1(e_1) = \{(a,0.3),(b,0.4)\} & f_1(e_2) = \{(a,0.2),(b,0.3)\} \\
f_2(e_1) = \{(a,0.7),(b,0.6)\} & f_2(e_2) = \{(a,0.8),(b,0.7)\} \\
f_3(e_1) = \{(a,0.3),(b,0.1)\} & f_3(e_2) = \{(a,0.4),(b,0.2)\} \\
f_4(e_1) = \{(a,0.8),(b,0.7)\} & f_4(e_2) = \{(a,0.9),(b,0.8)\} \\
f_5(e_1) = \{(a,0.2),(b,0.1)\} & f_5(e_2) = \{(a,0.1),(b,0.2)\} \\
f_6(e_1) = \{(a,0.4),(b,0.3)\} & f_6(e_2) = \{(a,0.5),(b,0.4)\}.
\end{array}
$$

Let τ = $\{\tilde{0}_E, \tilde{1}_E$, (f_1, E), (f_2, E)$\}$ be a fuzzy soft topology over X. Then the $\mathbf{FSTS}$ *(X, τ, E) is fuzzy soft connected between the* $\mathbf{FSS}$*s (f_3, E) and (f_4, E) but it is not fuzzy soft connected between (f_5,E) and (f_6,E). Also the* $\mathbf{FSTS}$ *(X, τ, E) is not fuzzy soft connected.*

Theorem 3.8. *Let (Y,τ_Y,E) be a fuzzy soft subspace of a* $\mathbf{FSTS}$ *(X,τ,E). If (Y,τ_Y,E) is fuzzy soft connected between the* $\mathbf{FSS}$*s (f ,E) and (g,E) over Y, then* $\mathbf{FSTS}$ *(X,τ,E) is fuzzy soft connected between (f ,E) and (g,E).*

Proof. *Suppose* $\mathbf{FSTS}$ *(X,τ,E) is not fuzzy soft connected between* $\mathbf{FSS}$*s (f,E) and (g,E),then there is fuzzy soft clopen set (h,E) over X such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Then $\tilde{Y}$ $\cap$ (h ,E) is fuzzy soft clopen over Y such that (f,E) $\subset$ (h,E) $\cap$ $\tilde{Y}$ and $\{$(h,E) $\cap$ $\tilde{Y}$ $\}$ $\tilde{q}$ (g ,E). Consequently, (Y,τ_Y,E) is not fuzzy soft connected between (f,E) and (g,E), a contradiction.*□

Theorem 3.9. *Let (Y,τ_Y,E) be a fuzzy soft clopen subspace of a* $\mathbf{FSTS}$ *(X,τ,E) and (f ,E), (g,E) $\subset$ $\tilde{Y}$. If (X,τ,E) is fuzzy soft connected between (f,E) and (g,E) then (Y,τ_Y,E) is fuzzy soft connected between(f,E) and (g,E).*

Proof. *Suppose (Y,τ_Y,E) is not fuzzy soft connected between (f,E) and (g,E). Then there is fuzzy soft clopen set (h,E) of (Y,τ_Y,E) such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Since, (Y,τ_Y,E) is fuzzy soft clopen in (X,τ,E), by Lemma 2.2 and Lemma 2.3 (h,E) is fuzzy soft clopen set of (X,τ,E) such that (f,E) $\subset$ (h,E) and (h,E) $\tilde{q}$ (g,E). Consequently, (X,τ,E) is not fuzzy soft connected between (f,E) and (g,E), a contradiction.*□

4 Fuzzy soft set-connected mappings

Definition 4.1. *A fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \rightarrow (Y, \vartheta, K)$ is said to be fuzzy soft set-connected provided, if* $\mathbf{FSTS}(X, \tau, E)$ *is fuzzy soft connected between* $\mathbf{FSS}$*s (f,E) and (g,E) then fuzzy soft subspace $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is fuzzy* *soft connected between $\varrho_{pu}(f,E)$ and $\varrho_{pu}(g,E)$ with respect to fuzzy soft relative topology.*

Theorem 4.1. *A fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\vartheta,K)$ is fuzzy soft set-connected mapping if and only if $\varrho_{pu}^{-1}(F,K)$ is a fuzzy soft clopen set over X for any fuzzy soft clopen set (h,K) of ($\varrho_{pu}(X)$,$\vartheta_{\varrho_{pu}(X)}, K)$.*

Proof. *Necessity : Let ϱ_{pu} be fuzzy soft set-connected mapping and (h,K) be fuzzy soft clopen set in $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$. Suppose $\varrho_{pu}^{-1}(h,K)$ is not fuzzy soft clopen in (X ,τ,E). Then (X ,τ,E) is fuzzy soft-connected between $\varrho_{pu}^{-1}(h,K)$ and $(\varrho_{pu}^{-1}(h,K))^c$. Therefore, $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is fuzzy soft-connected between $\varrho_{pu}(\varrho_{pu}^{-1}(h,K))$ and $\varrho_{pu}((\varrho_{pu}^{-1}(h,K))^c)$ because ϱ_{pu} is fuzzy soft set-connected. But, $\varrho_{pu}(\varrho_{pu}^{-1}(h,K)) = (h,K) \cap (\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K) = (h,K)$ and $\varrho_{pu}((\varrho_{pu}^{-1}(h,K))^c$ = $(h,K)^c$ imply that (h,K) is not fuzzy soft clopen in ($\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$, a contradiction. Hence, $\varrho_{pu}^{-1}(h,K)$ is fuzzy soft clopen in (X ,τ,E).*

Sufficiency : Let (X ,τ,E) be fuzzy soft-connected between (f,E) and (g,E). If $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is not fuzzy soft-connected between $\varrho_{pu}(f,E)$ and $\varrho_{pu}(g,E)$ then there exists a fuzzy soft clopen set (h,K) in ($\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ such that $\varrho_{pu}(f,E) \subset (h,K) \subset (\varrho_{pu}(g,E))^c$. By hypothesis, $\varrho_{pu}^{-1}(h,K)$ is fuzzy soft clopen set over X and $(f,E) \subset \varrho_{pu}^{-1}(h,K) \subset (g,E)^c$. Therefore, (X ,$\tau$,E) is not fuzzy soft-connected between (f,E) and (g,E). This is a contradiction. Hence, ϱ_{pu} is fuzzy soft set-connected.□

Theorem 4.2. *Every fuzzy soft continuous mapping ϱ_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) is a fuzzy soft set-connected mapping.*

Proof. *It is obvious.*□

Remark 4.1. *The converse of Theorem 4.2 need not be true.*

Example 4.1. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $Y = \{y_1, y_2\}$, $K = \{k_1, k_2\}$. The soft sets (f,E) and (g,K) defined as follows:*

$$f(e_1) = \{x_1 = 0.3, x_2 = 0\}, \qquad f(e_2) = \{x_1 = 0, x_2 = 0.4\}$$
$$g(k_1) = \{y_1 = 0.6, y_2 = 0\}, \qquad g(k_2) = \{y_1 = 0, y_2 = 0.5\}$$

Let τ = $\{\tilde{0}_E, \tilde{1}_E$,(f,E) $\}$ and υ = $\{\tilde{0}_K$, $\tilde{1}_K$, (g ,K) $\}$ are fuzzy soft topologies on X and Y respectively. Then the fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\upsilon,K)$ defined by u(x_1) = y_1 , u(x_2)= y_2 and p(e_1)= k_1, p(e_2) = k_2 is fuzzy soft set-connected but it is not fuzzy soft continuous, because fuzzy soft set (g,K) is fuzzy soft open set in Y not fuzzy soft open in X.

Theorem 4.3. *Every fuzzy soft mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\vartheta,K)$ such that $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ is a fuzzy soft connected set is a fuzzy soft set-connected mapping.*

Proof. *Let* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ *be fuzzy soft connected. Then by Lemma 3.1, no nonempty proper* **FSS** *of* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ *which is fuzzy soft clopen. Hence ,* ϱ_{pu} *is fuzzy soft set-connected.*□

Theorem 4.4. *Let* $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ *be a fuzzy soft set-connected mapping. If (X ,τ,E) is fuzzy soft connected set, then* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ *is a fuzzy soft connected set of (Y,ϑ,K).*

Proof. *Suppose* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ *is not fuzzy soft connected in (Y,ϑ,K), Then by Lemma 3.1, there is a nonempty proper fuzzy soft clopen set (h,K) of* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$*. Since* ϱ_{pu} *is fuzzy soft set-connected,* $\varrho_{pu}^{-1}(h, K)$ *is a nonempty proper fuzzy soft clopen set over X. Consequently,(X ,τ,E) is not fuzzy soft connected.*□

Theorem 4.5. *Let* $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ *be a fuzzy soft set-connected mapping and (f,E) be a fuzzy soft set over X such that* $\varrho_{pu}(f, E)$ *is fuzzy soft clopen set of* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$*. Then* $\varrho_{pu}/(f, E) : (f, E) \to (Y, \vartheta, K)$ *is fuzzy soft set-connected mapping.*

Proof: Let (f,E) be fuzzy soft connected between (g,E) and (h,E). Then by Theorem 3.8, (X, τ, E) *is fuzzy soft connected between (g,E) and (h,E). Since* ϱ_{pu} *is fuzzy soft set-connected,* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$ *is fuzzy soft connected between* $\varrho_{pu}(g, E)$ *and* $\varrho_{pu}(h, E)$*. Now, since* $\varrho_{pu}(f, E)$ *is fuzzy soft clopen set of* $(\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K)$*, it follows by Theorem 3.9 that* $\varrho_{pu}(f, E)$ *is fuzzy soft connected between* $\varrho_{pu}(g, E)$ *and* $\varrho_{pu}(h, E)$*. This proves the theorem.*□

Theorem 4.6. *Let* $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ *be a fuzzy soft set-connected surjection. Then for any fuzzy soft clopen set (h,K) of* (Y, ϑ, K) *is fuzzy soft connected if* $\varrho_{pu}^{-1}(h, K)$ *is fuzzy soft connected in (X ,τ,E). In particular, if (X ,τ,E) is fuzzy soft connected then (Y,ϑ,K) is fuzzy soft connected.*

Proof. *By Theorem 4.5* $\varrho_{pu}/\varrho_{pu}^{-1}(h, K) : \varrho_{pu}^{-1}(h, K) \to (Y, \vartheta, K)$ *is fuzzy soft set-connected. And, since* $\varrho_{pu}^{-1}(h, K)$ *is fuzzy soft connected by Theorem 4.4,*
$\varrho_{pu}/\varrho_{pu}^{-1}(h, K)[\varrho_{pu}^{-1}(h, K)] = (h, K)$ *is fuzzy soft connected.*□

Theorem 4.7. *Let* $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ *be a fuzzy soft set connected fuzzy soft open surjection and* $\varrho_{pu}^{-1}((y_k^\alpha)_K)$ *is fuzzy soft connected for each soft point* $(y_k^\alpha)_K$ *of Y. Then for every fuzzy soft clopen set (h,K) of Y is fuzzy soft connected if and only if* $\varrho_{pu}^{-1}((h, K))$ *is fuzzy soft connected.*

Proof. *Necessity: Let (h,K) be a fuzzy soft clopen fuzzy soft connected set of Y. Suppose* $\varrho_{pu}^{-1}(h, K)$ *is not fuzzy soft connected in X. Then there are fuzzy soft open sets (f,E) and (g,E) of X such that* $\varrho_{pu}^{-1}(h, K) \cap ((f, E) \cap (g, E)) = \phi$*,* $\varrho_{pu}^{-1}(h, K) = ((f, E) \cup (g, E))$ *and* $\varrho_{pu}^{-1}(h, K) \cap (f, E) \neq \phi \neq \varrho_{pu}^{-1}(h, K) \cap (g, E)$*. Since,* $\varrho_{pu}^{-1}((y_k^\alpha)_K)$ *is fuzzy soft connected either* $\varrho_{pu}^{-1}((y_k^\alpha)_K) \subset (f, E)$ *or* $\varrho_{pu}^{-1}((y_k^\alpha)_K) \subset (g, E)$ *for every fuzzy soft point* $(y_k^\alpha)_K \in (h, K)$*. Therefore* $(h, K) \cap \varrho_{pu}(f, E) \cap \varrho_{pu}(g, E) = \phi$*.* $(h, K) \subset \varrho_{pu}(f, E) \cup \varrho_{pu}(g, E)$ *and* $(h, K) \cap$

$\varrho_{pu}(f,E) \neq \phi \neq (h,K) \cap \varrho_{pu}(g,E)$. *Since,* ϱ_{pu} *is fuzzy soft open mapping* $\varrho_{pu}(f,E)$ *and* $\varrho_{pu}(g,E)$ *are fuzzy soft open set over Y. Hence, (h,K) is not fuzzy soft connected.*

Sufficiency : Follows from Theorem 4.6.□

Theorem 4.8. *Let* $\varrho_{p_1u_1}$ *: (X ,*τ*,E)* $\rightarrow$ *(Y,*ϑ*,K) be a surjective fuzzy soft set-connected and* $\sigma_{p_2u_2}$ *:(Y,*ϑ*,K)* $\rightarrow$ *(Z,*η*,T) a fuzzy soft set-connected mapping. Then* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$ *: (X ,*τ*,E)* $\rightarrow$ *(Z,*η*,T) is fuzzy soft set-connected.*

Proof. *Let (h,T) be a fuzzy soft clopen set in* $\sigma_{p_2u_2}(Y)$*. Then* $\sigma^{-1}_{p_2u_2}$*(h,T) is fuzzy soft clopen over Y =* $\varrho_{p_1u_1}$ *(X) and so* $\varrho^{-1}_{p_1u_1}$ *(*$\sigma^{-1}_{p_2u_2}$*(h,T)) is fuzzy soft clopen in (X ,*τ*,E). Now* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$*(X) =* $\sigma_{p_2u_2}$ *(Y) and* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})^{-1}$ *(h,T) =* $\varrho^{-1}_{p_1u_1}$ *(*$\sigma^{-1}_{p_2u_2}$*(h,T)) is fuzzy soft clopen in (X ,*τ*,E). Hence,* $(\sigma_{p_2u_2} o \varrho_{p_1u_1})$ *is fuzzy soft set connected.*□

Definition 4.2. *A fuzzy soft mapping* $\varrho_{pu} : (X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is said to be fuzzy soft weakly continuous if for each fuzzy soft point* $(x^{\alpha}_e)_E \in X$ *and each fuzzy soft open set (g,K) over Y containing* $\varrho_{pu}((x^{\alpha}_e)_E)$*, there exists a fuzzy soft open set (f,E) over X containing* $(x^{\alpha}_e)_E$ *such that* ϱ_{pu}*(f,E)* $\subset$ *Cl(g,K).*

Theorem 4.9. *A soft mapping* $\varrho_{pu} : (X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is fuzzy soft weakly continuous if and only if for each fuzzy soft open set (h,K) over Y,* $\varrho^{-1}_{pu}(h,K) \subset Int(\varrho^{-1}_{pu}(Cl(h,K)))$.

Proof. *Necessity : Let (h,K) be a fuzzy soft open set over Y and let* $(x^{\alpha}_e)_E \in \varrho^{-1}_{pu}(h,K)$ *then* $\varrho_{pu}((x^{\alpha}_e)_E) \in (h,K)$*. Then, there exists a fuzzy soft open set (f,E) over X such that* $(x^{\alpha}_e)_E \in (f,E)$ *and* $\varrho_{pu}(f,E) \subset Cl(h,K)$*. Hence,* $(x^{\alpha}_e)_E \in (f,E) \subset \varrho^{-1}_{pu}(Cl(h,K))$ *and* $(x^{\alpha}_e)_E \in Int(\varrho^{-1}_{pu}(Cl(h,K)))$ *since (f,E) is fuzzy soft open.*

Sufficiency : Let $(x^{\alpha}_e)_E \in X$ *and* $\varrho_{pu}((x_e)_E) \in (h,K)$*. Then* $(x_e)_E \in \varrho^{-1}_{pu}(h,K) \subset Int(\varrho^{-1}_{pu}(Cl(h,K)))$*. Let* $(f,E) = Int(\varrho^{-1}_{pu}(Cl(h,K)))$ *then (f,E) is fuzzy soft open set containing* $(x^{\alpha}_e)_E$ *and* $\varrho_{pu}(f,E) = \varrho_{pu}(Int(\varrho^{-1}_{pu}(Cl(h,K)))) \subset \varrho_{pu}(\varrho^{-1}_{pu}(Cl(h,K))) \subset Cl(h,K)$*. Hence,* ϱ_{pu} *is fuzzy soft weakly continuous.*□

Theorem 4.10. *If a* FSTS *space (X ,*τ*,E) is fuzzy soft connected and* ϱ_{pu} : $(X,\tau,E) \rightarrow (Y,\vartheta,K)$ *is a fuzzy soft weakly continuous surjection, then (Y,*ϑ*,K) is fuzzy soft connected.*

Proof . *Suppose* (Y,ϑ,K) *is not fuzzy soft connected. Then, there exist nonempty fuzzy soft open sets* (h_1,K) *and* (h_2,K) *in Y such that* $(h_1,K) \cap (h_2,K) = \phi$ *and* $(h_1,K) \cup (h_2,K) = \tilde{X}$*. Hence we have* $\varrho^{-1}_{pu}(h_1,K) \cap \varrho^{-1}_{pu}(h_2,K) = \phi$ *and* ϱ^{-1}_{pu}*(*h_1*,K)* $\cup$ ϱ^{-1}_{pu}*(*V_2*,K) =* $\tilde{X}$*. Since* ϱ_{pu} *is surjective,* ϱ^{-1}_{pu}*(*V_j*,K)* $\neq \phi$ *for j = 1, 2. By Theorem 4.9, we have* ϱ^{-1}_{pu}*(*h_j*,K)* $\subset$*Int (*ϱ^{-1}_{pu}*(Cl(*h_j*,K))) because* ϱ_{pu} *is fuzzy soft weakly continuous. Since (*h_j*,K) is fuzzy soft open and also fuzzy soft closed, we have* ϱ^{-1}_{pu}*(*h_j*,K)* $\subset$ *Int(* ϱ^{-1}_{pu}*(*h_j*,K. Hence,* ϱ^{-1}_{pu}*(*h_j*,K) is fuzzy soft open in X for* *j = 1,2. This implies that X is not fuzzy soft connected. This is contrary to the hypothesis that X is fuzzy soft connected. Hence, (Y,ϑ,K) is fuzzy soft connected.*□

Theorem 4.11. *A fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ is fuzzy soft weakly continuous, then $Cl(\varrho_{pu}^{-1}(h, K)) \subset (\varrho_{pu}^{-1}(Cl(h, K))$ for each fuzzy soft open set (h,K) over Y.*

Proof. *Suppose there exists a fuzzy soft point $(x_e^{\alpha})_E \in Cl(\varrho_{pu}^{-1}(h, K)) - \varrho_{pu}^{-1}(Cl(h, K))$. Then $\varrho_{pu}((x_e)_E) \notin Cl(h, K)$. Hence there exists a fuzzy soft open set (g ,K) containing $\varrho_{pu}((x_e)_E)$ such that $(g, K) \cap (V, K) = \phi$. Since (h,K) is fuzzy soft open set over Y, we have $(h, K) \cap Cl(g, K) = \phi$. Since ϱ_{pu} is fuzzy soft weakly continuous ,there exists a fuzzy soft open set (f,E) over X containing $(x_e^{\alpha})_E$ such that $\varrho_{pu}(f, E) \subset Cl(g, K)$. Thus, we obtain $\varrho_{pu}(f, E) \cap (h, K) = \phi$. On the other hand, since $(x_e^{\alpha})_E \in Cl(\varrho_{pu}^{-1}(h, K))$, we have $(f, E) \cap \varrho_{pu}^{-1}(h, K) \neq \phi$ and hence, $\varrho_{pu}(f, E) \cap (h, K) \neq \phi$. Thus we have a contradiction. Hence $Cl(\varrho_{pu}^{-1}(h, K)) \subset (\varrho_{pu}^{-1}(Cl(h, K))$.*□

Theorem 4.12. *If a fuzzy soft surjection ϱ_{pu} : (X ,τ,E) $\to$ (Y,ϑ,K) is fuzzy soft weakly continuous ,then ϱ_{pu} is fuzzy soft set-connected.*

Proof. *Let (h,K) be any fuzzy soft clopen set over Y. Since (h,K) is fuzzy soft closed, We have Cl(h,K) = (h,K). Thus, by Theorem 4.9, we obtain $\varrho_{pu}^{-1}(h, K) \subset Int(\varrho_{pu}^{-1}(h, K))$. This shows that $\varrho_{pu}^{-1}(h, K)$ is fuzzy soft open set over X. Moreover , by Theorem 4.11, we obtain $Cl(\varrho_{pu}^{-1}(h, K)) \subset \varrho_{pu}^{-1}(h, K)$. This shows that $\varrho_{pu}^{-1}(h, K)$ is a fuzzy soft closed set over X. Since ϱ_{pu} is fuzzy soft surjection, by Theorem 4.1, we observe that ϱ_{pu} is a fuzzy soft set-connected mapping.*□

Remark 4.2. *The converse of Theorem 4.12 is not true.*

Example 4.2. *Let$X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and $Y = \{y_1, y_2\}$, $K = \{k_1, k_2\}$. The fuzzy soft sets (f,E)and (g,K) are defined as follows :*

$$f(e_1) = \{x_1 = 0.3, x_2 = 0\}, \qquad f(e_2) = \{x_1 = 0, x_2 = 0.4\}$$
$$g(k_1) = \{y_1 = 0.4, y_2 = 0\}, \qquad g(k_2) = \{y_1 = 0, y_2 = 0.5\}$$

Let τ = $\{\tilde{0}_E, \tilde{1}_E$,(h,E) $\}$ and υ = $\{\tilde{0}_K$, $\tilde{1}_K$, (G ,K) $\}$ are fuzzy soft topologies on X and Y respectively. Then fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \upsilon, K)$ defined by u(x_1) = u(x_2)=y_1 and p(e_1)= k_1 , p(e_2) = k_2 is fuzzy soft set-connected but it is not fuzzy soft weakly continuous.

Theorem 4.13. *Let (Y,ϑ,K) be an fuzzy soft extremally disconnected space .If a fuzzy soft mapping $\varrho_{pu} : (X, \tau, E) \to (Y, \vartheta, K)$ is fuzzy soft set-connected, then ϱ_{pu} is fuzzy soft weakly continuous.*

Proof. *Let $(x_e^{\alpha})_E$ be a fuzzy soft point of X and (g,K) any fuzzy soft open set over Y containing $\varrho_{pu}((x_e^{\alpha})_E)$. Since (Y,$\vartheta$,K) is fuzzy soft extremally disconnected,* *Cl(g,K) is fuzzy soft clopen set over Y. Thus $Cl(g,K) \cap \varrho_{pu}(\tilde{X})$ is fuzzy soft clopen set in the fuzzy soft subspace ($\varrho_{pu}(X), \vartheta_{\varrho_{pu}(X)}, K$). Put $\varrho^{-1}(Cl(g,K) \cap \varrho_{pu}(\tilde{X})) = (f,E)$. Then, since ϱ_{pu} is fuzzy soft set-connected, it follows from Theorem 4.1 that (f,E) is fuzzy soft clopen set over X. Therefore, (f,E) is a fuzzy soft open set containing $(x_e^\alpha)_E$ over X such that $\varrho_{pu}(f,E) \subset Cl(g,K)$. This implies that ϱ_{pu} is fuzzy soft weakly continuous.*□

Theorem 4.14. *Let (Y,ϑ,K) be a fuzzy soft extremally disconnected space. A fuzzy soft surjection mapping $\varrho_{pu} : (X,\tau,E) \to (Y,\vartheta,K)$ is fuzzy soft set-connected if and only if ϱ_{pu} is fuzzy soft weakly continuous.*

Proof. *It follows from Theorem 4.12 and Theorem 4.13.*□

5 Conclusions

Connectedness is an important and major area of topology and it can give many relationships between other scientific areas and mathematical models. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by assigning a strength of connectedness to every possible path between every possible pair of image elements. This paper, introduces the notion of fuzzy soft connectedness between fuzzy soft sets in fuzzy soft topological spaces. It is shown that a fuzzy soft topological space is fuzzy soft connected if and only if it is fuzzy soft connected between every pair of its nonempty fuzzy soft sets. Further two new classes of fuzzy soft mappings called fuzzy soft set connected and soft weakly continuous have been introduced. It is shown that the class of fuzzy soft set connected (respt. fuzzy soft weakly continuous) mappings properly contains the class of all fuzzy soft continuous mappings. Several properties and characterizations of fuzzy soft set connected and fuzzy soft weakly continuous mappings have been studied. Hope that the concepts and results established in this paper will help researcher to enhance and promote the further study on fuzzy soft topology to carry out a general framework for the development of information systems.

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