A GEOMETRIC INTERPRETATION OF THE FIGUEROA PLANES

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In [3] Grundhöfer gives a synthetic construction of a Figueroa plane of order q^3 starting from $PG(2,q^3)$ which is independent on the algebraic point of view of the action of a Singer group, as Figueroa [1] used in the original construction.

In this paper, we analyze at first the action of the collineation α of order h of $PG(2,q^h)$, $q=p^r$, p,h primes , $p,h>2$, fixing $PG(2,q)$ pointwise . If $h=5,7$ we prove that α admits three kinds of point and line orbits, as in the case $h=3$. The group of the collineations of $PG(2,q^h)$ which fix $PG(2,q)$, acts trasitively on the points and on the lines of $\mathcal{P}_3$ and $\mathcal{L}_3$ respectively, precisely when $h=3$.

In Sec.3, we analyze the Grundhöfer construction by a geometric point of view and we obtain that a Figueroa plane of order q^3 can be represented starting from $PG(2,q^3)$ leaving invariant the incidence relation and replacing the subset $\mathcal{L}_3$ of the lines by a new subset $\mathcal{L}_3^*$, any new line $r^* \in \mathcal{L}_3^*$ consisting of a subset of the old line r , union a subset of an algebraic curve defined by r .

1. PRELIMINARY RESULTS.

Let $F=GF(q^h)$ be a Galois field, $q=p^r$, p,h primes, $p,h>2$.
The field F can be regarded as an extension $K(w)$ of $K=GF(q)$, where w is a root of a polynomial $f(x)\in K[X]$, $\deg f=h$, f irreducible over K.
Let $\tilde{\alpha}: x\mapsto x^q$ be the automorphism of F fixing the subfield K elementwise. Then $\circ\tilde{\alpha}=h=|<\tilde{\alpha}>|$.
Denote by $\Pi=PG(2,q^h)$ the desarguesian plane over F and by $\Pi_o=PG(2,q)$ the subplane of Π coordinatized by K with respect to a chosen coordinate system for Π so that a point P of Π has homogeneous coordinates (x,y,z), $x,y,z\in F$, and a fixed line l_∞ has equation $z=0$; denote by α the collineation of Π defined by $(x,y,z)\alpha=(x\tilde{\alpha},\ y\tilde{\alpha},\ z\tilde{\alpha})$.
Express $\Pi=(\mathcal{P},\mathcal{L},I)$ as an incidence structure such that $\Pi_o=(\mathcal{P}_o,\mathcal{L}_o,I_o)$ where $\mathcal{P}_o\subset\mathcal{P}$, $\mathcal{L}_o\subset\mathcal{L}$, $I_o=I/_{\mathcal{P}_o\times\mathcal{L}_o}$, $I=\in$.
Let $\theta s=\{s,\ s\alpha,\ s\alpha^2,\dots,s\alpha^{h-1}\}$ be the orbit of the element $s\in\mathcal{P}\cup\mathcal{L}$ under the action of the group $<\alpha>$.

<u>LEMMA 1.1</u>- i) the group $<\alpha>$ is planar and fixes precisely the points of $\mathcal{P}_o$ (the lines of $\mathcal{L}_o$);

ii) $\forall s\in\mathcal{P}\cup\mathcal{L}$, $|\theta s|=h$ if and only if $s\notin\mathcal{P}_o\cup\mathcal{L}_o$;

iii) $r\supset\theta P$ for some $P\in r$ if and only if $r\in\mathcal{L}_o$ ($P\in\bigcap\theta r$ for some $r\ni P$ if and only if $P\in\mathcal{P}_o$).

Proof:

A point $P=(x,y,z)$ is in $\mathcal{P}_o$ if and only if
$X=xz^{-1}$, $Y=yz^{-1}$ are in K when $z\neq 0$ or
$X=xy^{-1}$ or $Y=x^{-1}y$ are in K when $z=0$.
A point P is fixed by α if and only if

$(x\tilde{\alpha},y\tilde{\alpha},z\tilde{\alpha})=(\lambda x,\lambda y,\lambda z)$; equivalently, $X\tilde{\alpha}=X$ and $Y\tilde{\alpha}=Y$,

that is, if and only if X,Y are elements of K. Hence α fixes precisely the points of Π_o. The dual argument holds for the lines.

A point P of is fixed by $\alpha^i\in<\alpha>$ where $i=1,\dots,h$ if and only if $X\tilde{\alpha}^i=X$, $Y\tilde{\alpha}^i=Y$; the elements X,Y must belong to a field M such that $K\leq M<F$ $|M:F|=s$, s/h. Therefore $s=1$ and P is a point of Π_o. The dual arguments hold for the lines.

Let $s \in \mathscr{P} \cup \mathscr{L}$; then $|\Theta s| \leq h$. We have $|\Theta s| < h$ if and only if there exist i, j such that $1 \leq i < j < h-1$ and $s\alpha^i = s\alpha^j$, or, $s\alpha^{j-i} = s$, that is (by i)) , if and only if $s \in \mathscr{P}_0 \cup \mathscr{L}_0$. Furthermore,

$\Theta P \subset r$ if and only if $r = P\,P\alpha = P\alpha P\alpha^2 = r$; that is, if and only if r is a fixed line under $\langle \alpha \rangle$; $P \in \bigcap \Theta r$ if and only if P is a point fixed by $\langle \alpha \rangle$.

REMARK 1- From iii) it follows that $h / p^{rh} - p^r$.

LEMMA 1.2- Three points of ΘP are collinear if and only if there exist i, j such that $1 \leq i < j < h$ and $P, P\alpha^i, P\alpha^j$ are collinear.

Proof:

Let $P\alpha^k, P\alpha^m, P\alpha^n$ be three points of ΘP, $k<m<n$. Then $P\alpha^k$ $P\alpha^m, P\alpha^n \in r$ where $r \in \mathscr{L}$ if and only if $P, P\alpha^i, P\alpha^j \in r^{h-k}$ where $i=m+n-k$, $j=n+h-k$.

REMARK 2- If $P \in r$ and $\mathcal{P} \neq P$, then $r = P\,P\alpha P\alpha^2$ is equivalent to $r \in \mathscr{L}_0$.

Assume that an orbit ΘP contains three collinear points. From Lemma 1.2 it follows that this is equivalent to assuming that there exists a line $r \in \mathscr{L}$ such that $r = P\,P\alpha^i P\alpha^j$ where $1 \leq i < j < h$.

LEMMA 1.3- If a) $j=2i$ or $j+i=h$ or

b) $i=h-2s$ and $j=h-s$ for some s , then r is a line of $\mathscr{L}_0$.

Proof:

The length of the orbit Θr is less than h if and only if there exists s , $1 \leq s < h$ such that $r\alpha^s = P\alpha^s\,P\alpha^{i+s}\,P\alpha^{j+s} = r = P\,P\alpha^i\,P\alpha^j$.

If $j=2i$, then $r\alpha^i = P\alpha^i\,P\alpha^{2i}\,P\alpha^{j+i}$ and

$$r \cap r\alpha^i \supset \{P^i, P^{2i} = P^j\} \quad ;$$

if $j+i=h$, then $r\alpha^i = P\alpha^i\,P\alpha^{2i}\,P\alpha^{j+i}$ and $r \cap r\alpha^i \supset \{P\alpha^i, P\alpha^{j+i} = P\alpha^h = P\}$;

in both cases , $r = r\alpha^i$.

If $i=h-2s$ and $j=h-s$, then $r\alpha^s = P\alpha^s\,P\alpha^{i+s}\,P\alpha^{j+s} = P\alpha^s\,P\alpha^{h-s}\,P\alpha^h = P\alpha^s\,P\alpha^{h-s}\,P$ and since $r = P\,P\alpha^i\,P\alpha^j = P\,P\alpha^{h-2s}\,P\alpha^{h-s}$, we have $r \cap r\alpha^s \supset \{P, P\alpha^{h-s}\}$; hence $r = r\alpha^s$.

LEMMA 1.3'- The dual of Lemma 1.3.

PROPOSITION 1.1- If $h=5$, then the point-orbits θP are of the following three types: 1) trivial; 2) incident a line ; 3) a 5-arc;

the line-orbits θr are of the following three types: 1') trivial; 2') confluent in a point; 3') a 5-gon.

Proof:

Let P be a point of Π ; if $P \in \mathcal{P}_o$ then θP is trivial ; if $P \in r$ where $r \in \mathcal{L}_o$ and $P \notin \mathcal{P}_o$, then $\theta P \subset r$.

Let P be a point of $\mathcal{P} - \mathcal{P}_o$ such that $P \notin r \ \forall r \in \mathcal{L}_o$; such a point does exist since Π_o is not a Baer subplane. The orbit θP contains three collinear points if and only if there exist i,j $1 \le i < j < 5$ such that $P, P\alpha^i, P\alpha^j$ are collinear (see Lemma 1.2).

Let $r = P\,P\alpha^i\,P\alpha^j$; by Lemma 1.3,a), we obtain $r = r\alpha^i$ $\forall (i,j) \in \{(1,2),$ $2,4), (1,4), (2,3)\}$, and by b), $r = r\alpha^s$, $s=1,2$ $\forall (i,j) \in \{(3,4),(1,3)\}$.

This means that for all the possibilities of the choices i,j $1 \le i < j < 5$ the line r is a line of $\mathcal{L}_o$, $\theta P \subset r$, a contradiction. Hence there exist no exponents i,j such that $P , P\alpha^i , P\alpha^j \in \theta P$ are collinear. Equivalently, the orbit θP is a 5-arc. The dual arguments hold for the line orbits.

Let θP be a point-orbit such that $P \notin \mathcal{P}_o$ and $\theta P \not\subset r \ \forall r \in \mathcal{L}$. Assume that θP contains three collinear points, that is, there exists a line $r = P\,P\alpha^i\,P\alpha^j$ $1 \le i < j < h$ and $|\theta r| \ne 1$.

Let $\mathcal{S} = (\theta P, \theta r, I_{/\theta P \times \theta r})$ be the incidence structure consisting of the points of the orbits of P and the lines of the orbit of r .

LEMMA 1.4- $\mathcal{S}$ is an incidence structure with parameters $b=v=h$, $r,k \geq 3$.

Proof:

Any line $m \in \theta r$ contains at least three distinct points , namely $P\alpha^s$, $P\alpha^{i+s}$, $P\alpha^{j+s}$ as $m = r\alpha^s$ for some s . For any point $Q \in \theta P$, $Q = P\alpha^t$ for some t ; hence Q is incident with the following three distinct lines: $r\alpha^t$, $r\alpha^s$ where $s=t-i$, $r\alpha^{s'}$ where $s'=t-j$ as image of P under α^t; of $P\alpha^i$ under α^s ; of $P\alpha^j$ under $\alpha^{s'}$, respectively.

PROPOSITION 1.2- If $h=7$ then the point-orbits θP are of the following three types : 1) trivial ; 2) incident a line ; 3) a 7-arc ;

the line-orbits θr are of the following three types : 1')trivial ; 2') confluent in a point ; 3') a 7-gon.

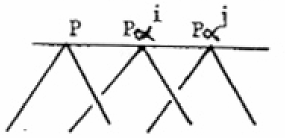
Proof:

Let $P \in \mathcal{P}$; if $P \in \mathcal{P}_0$ then θP is trivial ; if $P \notin \mathcal{P}_0$ and $P \in r$, where $r \in \mathcal{L}_0$, then $\theta P \subset r$.

Let P be a point of $\mathcal{P} - \mathcal{P}_0$ such that $P \notin r \ \forall r \in \mathcal{L}_0$. Such a point does exist since Π_0 is not a Baer subplane.

Assume that r contains three collinear points, that is, assume that there exists $r = P\,P\alpha^i\,P\alpha^j$, $1 \leq i < j < 7$.

From Lemma 1.4 it follows that each of the points P, $P\alpha^i$, $P\alpha^j$ is incident with two distinct lines of θr other than r ; that is, the lines through P $P\alpha^i$, $P\alpha^j$ are all the lines of θr since $h=7$. Hence $r \cap r\alpha^s \in \{P, P\alpha^i, P\alpha^j\}$ $\forall$ $s=1,\ldots,6$. We prove that, $\forall\, t,t'=1,\ldots,6$, $r\alpha^t \cap r\alpha^{t'} \in \theta P$:



$R = r\alpha^t \cap r\alpha^{t'}$ is equivalent to $R\alpha^{-t} = r \cap r\alpha^{t'-t}$.

From the above we obtain $R\alpha^{-t} \in \{P, P\alpha^i, P\alpha^j\}$ or $R \in \{P\alpha^t, P\alpha^{i+t}, P\alpha^{j+t}\}$. Thus any two lines of θr are incident with a point of θP.

Given P and $P\alpha^t \in \theta P$ set $\bar{r} = P\,P\alpha^t$. Since P is incident with three lines $r_1, r_2, r_3 \in \theta r$ and each line of θr contains three points of θP, on these three lines through P lie all points of θP. Hence $\bar{r} = r_i$ for some $i=1,2,3$. As $r' = P\alpha^s\,P\alpha^{s'}$ is equivalent to $r'\alpha^{-s} = P\,P\alpha^{s'-s}$, we conclude that any two of the points of θP are incident with a line of θr and $\mathcal{S} = (\theta P, \theta r, I_{/\theta P \times \theta r})$ is the projective plane $PG(2,2)$ and $2/q=p^r$, a contradiction.

Hence the orbit θP cannot have three collinear points.

2. FURTHER PROPERTIES OF THE COLLINEATION α

As in Sec. 1, let $\Pi = PG(2,q^h)$, $\Pi_o = PG(2,q)$ $\Pi_o < \Pi$.
Represent $\Pi = (\mathcal{P}, \mathcal{L}, I)$. Let α be the collineation of Π induced by the automorphism of F fixing elementwise K. Thus α fixes elementwise the points and the lines of Π_o. We can partition the sets $\mathcal{P}$ and $\mathcal{L}$ as follows:

$$\mathcal{P}_1 = \{P \in \mathcal{P} \;/\; P\alpha = P\} \quad ; \quad \mathcal{L}_1 = \{r \in \mathcal{L} \;/\; r\alpha = r\}$$

$$\mathcal{P}_2 = \{P \in \mathcal{P} \;/\; \exists r \in \mathcal{L}_1 \text{ s.t. } P\ I\ r\} \;;\; \mathcal{L}_2 = \{r \in \mathcal{L} \;/\; \exists P \in \mathcal{P}_1 \text{ s.t. } r\ I\ P\}$$

$$\mathcal{P}_3 = \mathcal{P} - (\mathcal{P}_1 \cup \mathcal{P}_2) \quad ; \quad \mathcal{L}_3 = \mathcal{L} - (\mathcal{L}_1 \cup \mathcal{L}_2) .$$

It is clear that $\Pi_o = (\mathcal{P}_1, \mathcal{L}_1, I)$.
Let A_o be the group of the automorphisms of Π which map Π_o onto itself.

<u>LEMMA 2.1</u>- For any $\sigma \in A_o$ it is :

i) $\sigma' = \alpha\sigma\alpha^{-1} \in A_o$ and σ' works as σ on Π_o ;

ii) $\sigma'(\mathcal{P}_i) = \mathcal{P}_i$ and $\sigma'(\mathcal{L}_i) = \mathcal{L}_i$ $\forall$ i=1,2,3.

Proof:

i) : for any $P \in \mathcal{P}_1$ set $P' = P\sigma$; it is $P' \in \mathcal{P}_1$ and $P\sigma' = P\,\alpha\sigma\alpha^{-1} = P\,\sigma\alpha^{-1} = P'\,\alpha^{-1} = P' = P\sigma$. The dual arguments hold for the lines of $\mathcal{L}_1$.

ii) : by i), it follows $\sigma'(\mathcal{P}_1) = \mathcal{P}_1$ and $\sigma'(\mathcal{L}_1) = \mathcal{L}_1$.
For any point $P \in \mathcal{P}_2$ there exists exactly one line $r \in \mathcal{L}_1$ such that $P\ I\ r$ and $P\sigma'$ is a point of $r\sigma' = r'$ where $r' \in \mathcal{L}_1$, by i), that is $P\sigma'$ is a point of $\mathcal{P}_2$. The dual arguments holds for the lines of $\mathcal{L}_2$. Thus $\sigma'(\mathcal{P}_2) = \mathcal{P}_2$ and $\sigma'(\mathcal{L}_2) = \mathcal{L}_2$. Therefore it must be also $\sigma'(\mathcal{P}_3) = \mathcal{P}_3$ and $\sigma'(\mathcal{L}_3) = \mathcal{L}_3$.

Let $C_o \subset A_o$ be the subset of the central collineations of Π having the center and the axis in Π_o. It is known that $\langle C_o \rangle = A_o$ (see [3]).

<u>LEMMA 2.2</u>- For any $\sigma \in C_o$ it is $\alpha\sigma = \sigma\alpha$.

Proof:

Let $\sigma \in C_o$; let $C \in \mathcal{P}_1$ and $a \in \mathcal{L}_1$ be the center and

the axis of σ, respectively, and let $P\sigma = Q$ where $P, Q \in \mathcal{P}_1$. Take $\sigma' = \alpha\sigma\alpha^{-1}$. By Lemma 2.1 we have that $\sigma' \in A_o$, $C\sigma' = C$, $\forall A \in \mathcal{P}_1$ s.t. $A\ I\ a$ then $A\sigma' = A$ and $\forall r \in \mathcal{L}_1$ s.t. $C\ I\ r$ then $r\sigma' = r$; moreover $P\sigma' = Q$. For any line r such that $C\ I\ r$, if $r \notin \mathcal{L}_1$, then $r \in \mathcal{L}_2$, $r\alpha \in \mathcal{L}_2$ and $C\ I\ r\alpha$; thus $r\sigma' = r\ \alpha\sigma\alpha^{-1} = r\alpha\alpha^{-1} = r$.
For any point $R \in \mathcal{P}_1$ such that $R\ I\ a$, if $R \notin \mathcal{P}_1$ then $R \in \mathcal{P}_2$ and $R\alpha \in \mathcal{P}_2$, $R\alpha\ I\ a$; thus $R\sigma' = R\alpha\sigma\alpha^{-1} = R\alpha\alpha^{-1} = R$.
Therefore σ' is a central collineation of Π having center C, axis a and $P\sigma' = Q$; this means that $\sigma' = \sigma$; equivalently $\alpha\sigma = \sigma\alpha$

Set $\theta s = \{s, s\alpha, \dots, s\alpha^{h-1}\}$ $\forall s \in \mathcal{P} \cup \mathcal{L}$.

PROPOSITION 2.1- For any $\sigma \in A_o$ it is $\alpha\sigma = \sigma\alpha$ and $(\theta s)\sigma = \theta(s\sigma)$.

Proof:

As $\langle C_o \rangle = A_o$, for any $\sigma \in A_o$, $\sigma = \sigma_1\sigma_2 \dots \sigma_r$ where $\sigma_i \in C_o$; it is

$$\alpha\sigma\alpha^{-1} = \alpha\sigma_1 \dots \sigma_r\alpha^{-1} = \alpha\sigma_1\alpha^{-1}\alpha\sigma_2\alpha^{-1} \dots \alpha\sigma_r\alpha^{-1} = \sigma_1\sigma_2 \dots \sigma_r = \sigma,$$

$$(\theta s)\sigma = \{s, s\alpha, \dots, s\alpha^{h-1}\}\sigma = \{s, s\alpha\sigma, \dots, s\alpha^{h-1}\sigma\} = \{s, (s\sigma)\alpha, \dots, (s\sigma)\alpha^{h-1}\} = \theta(s\sigma).$$

PROPOSITION 2.2- For any point $P \in \mathcal{P}_1$ there are q^3-q^2-1 non-identical collineations of C_o of center P : q^2-1 of them are elations, q^3-2q^2 of them are homologies.

Proof:

The non-identical elations of C_o of center P are as many are the lines of Π_o incident P, that is, $q+1$, times $q-1$, where $q-1$ is the number of the points of $\ell \cap \mathcal{P}_1$, any $\ell \in \mathcal{L}_1$ $\ell \ni P$, which are different from P and from a chosen and fixed point of $\ell \cap \mathcal{P}_1$.

The non identical homologies of C_o of center P are as many are the lines r of $\mathcal{L}_1$ incident P, that is, q^2, times $q-2$ which is the number of the points $\mathcal{G}_1$ of any line ℓ of π_o through P, different from P, from $\ell \cap r$ and from a fixed point of $\ell \cap \mathcal{G}_1$.

Choose and fix any point $P \in \mathcal{G}_3$; set
$P A_o = \{ P\sigma \ / \ \forall \sigma \in A_o \}$, $r A_o = \{ r\sigma \ / \ \forall \sigma \in A_o \}$.

THEOREM 2.1- $P A_o = \mathcal{G}_3$, $r A_o = \mathcal{L}_3$ precisely when $h=3$.

Proof:

Let $\mathcal{G}' = \{ P' = P\sigma \ / \ \forall \sigma \in A_o \}$; it is $\mathcal{G}' \subseteq \mathcal{G}_3$. For any $\sigma \in A_o$ it is $\sigma = \sigma_1 \sigma_2$ where $\sigma_1, \sigma_2 \in C_o$.

Consider the subgroup $\Sigma_1 < C_o$ of the collineations of π_o of center $P_1 \in \mathcal{G}_1$. It is $P \Sigma_1 \subseteq P P_1 \cap \mathcal{G}_3$ where $P P_1 \in \mathcal{L}_2$ and $|P \Sigma_1| = q^3 - q^2$.

For any $P_2 \in \mathcal{G}_1$, $P_2 \neq P_1$ and $\Sigma_2 < C_o$, consider the point $P\sigma_1\sigma_2$ $\forall \sigma_i \in \Sigma_i$, $i=1,2$.

It is $P\sigma_1\sigma_2 \in P_2 P\sigma_1$; if $P\sigma_1\sigma_2 \in P_1 P P\sigma_1$, then the lines $P_2 P\sigma_1$ and $P_1 P P\sigma_1$ would have two different points $P\sigma_1$ and $P\sigma_1\sigma_2$ in common, thus $P_1 = P_2$, a contradiction. If there would be points Q in the set $P P_1 \cap \mathcal{G}_3 - P\Sigma_1$, there would exists no $\sigma \in A_o$ such that $P\sigma = Q$. Hence, $P\Sigma_1 = P P_1 \cap \mathcal{G}_3$ and $h=3$ follows.

3. THE GEOMETRIC INTERPRETATION

Let $F=GF(q^3)$ be a Galois field, $q=p^r$, p prime $p>2$, and let $K=GF(q)$ be the subfield of F of order q.
Let $\Pi=PG(2,q^3)$ be the Galois plane of order q^3 and $\Pi_o=PG(2,q)$, $\Pi_o<\Pi$.
Let α be the collineation of Π induced by the automorphism of F fixing pointwise the subfield K. The order of α is 3 and α fixes precisely the points and the lines of Π_o (see Sec. 1).
We can represent Π as an incidence structure $\Pi=(\mathcal{P},\mathcal{L},I)$ and we can partition the sets $\mathcal{P}$ and $\mathcal{L}$ into three classes $\mathcal{P}_i$ and $\mathcal{L}_i$ $i=1,2,3$ according to the three possible orbits of points, lines respectively, under the action of α (see [2]).
It is $\Pi_o=(\mathcal{P}_1,\mathcal{L}_1,I)$.
The incidence relation I can be partitionned into nine sets

$I_{ij}=I\cap(\mathcal{P}_i \times \mathcal{L}_j)$ $\quad \forall\, i,j=1,2,3$; note that
$I_{13}=\emptyset=I_{31}$ and $I_{33}\neq\emptyset$ (compare [2]).

Define a map $\mu: \mathcal{P}_3 \to \mathcal{L}_3$

(3.1) $\quad P\mu = P\alpha\, P\alpha^2$

and a map $\mu': \mathcal{L}_3 \to \mathcal{P}_3$

(3.2) $\quad r\mu' = r\alpha \cap r\alpha^2$

Take

$$I^* = (I-I_{33}) \cup I'_{33} \qquad \text{where}$$

$$(P,r)\in I'_{33} \quad \text{if and only if} \quad (r\mu',P\mu)\in I_{33}$$

The incidence structure $\Pi^*=(\mathcal{P},\mathcal{L},I^*)$ is a Figueroa plane (compare [2]).
The projective plane Π^* can be obtained by Π "redefining" the incidence relation between the points of $\mathcal{P}_3$ and the lines of $\mathcal{L}_3$ as follows:

(3.3) $\quad P\ I^*\ r$ if and only if $P\mu\ I\ r\mu'$

LEMMA 3.1- If $P \in \mathcal{P}_3$ and $r \in \mathcal{L}_3$, then

$P\ I^* r$ is equivalent to $P\ P\alpha\ I\ r \cap r\alpha'$.

Proof:

The relation (3.3) is equivalent to

(3.4) $P\alpha \cdot P\alpha^2 \quad I \quad r\alpha \cap r\alpha^2$

Applying α^3 to (3.4), we obtain $P\ P\alpha\ I\ r \cap r\alpha$.

LEMMA 3.2- a) $r \in \mathcal{L}_3$ is equivalent to $r \cap r\alpha \in \mathcal{P}_3$;

b) $|r \cap \mathcal{P}_2| = q^2+q+1$, $|r \cap \mathcal{P}_3| = q^h - q^2 - q$.

Proof:

a): Let $Q = r \cap r\alpha$; it must be $Q \in \mathcal{P}_2 \cup \mathcal{P}_3$ (otherwise, $r \in \mathcal{P}_1$); $Q \in \mathcal{P}_2$ is equivalent to "$\exists\, a \in \mathcal{L}_1$ s.t. $Q = a \cap r$", or $Q\alpha = a \cap r\alpha$ and $Q = Q\alpha$, a contradiction.

b): let P_o be a point of $r \cap \mathcal{P}_1$, then $P_o = r \cap r\alpha$ contradicts a); thus $r \cap \mathcal{P}_1 = \emptyset$. For any $r_1 \in \mathcal{L}_1$, $Q_1 = r_1 \cap r$ is a point of $r \cap \mathcal{P}_2$. Let $r_2 \in \mathcal{L}_1$, $r_2 \neq r_1$, $Q_2 = r_2 \cap r$; if $Q_1 = Q_2$ then $Q_1 = r_1 \cap r_2$ is a point of $r \cap \mathcal{P}_1$, a contradiction to a). Therefore $|r \cap \mathcal{P}_2| = |\mathcal{L}_1|$. The remaining points of r are in $\mathcal{P}_3$.

Let $r \in \mathcal{L}_3$; take $Q = r \cap r\alpha$.

LEMMA 3.3- It is :

$$\{P \in \mathcal{P} \,/\, P\ I\ r\} \cap \{P \in \mathcal{P} \,/\, P\ I^*\ r\} = \{P \in \mathcal{P}_2 \,/\, P\ I\ r\} \cup \{Q,\ Q\alpha^{h-1}\}.$$

Proof:

By Lemma 3.2, we can write $r = r_2 \cup r_3$ where $r_2 = \{P \in \mathcal{P}_2 \,/\, P\ I\ r\}$, $r_3 = \{P \in \mathcal{P}_3 \,/\, P\ I\ r\}$.

For any point $P \in r_2$ it is equivalent "$P\ I\ r$" and "$P\ I^*\ r$".

For any point $P \in r_3$, $P\ I^*\ r$ if and only if $P\ P\alpha\ I\ Q$; if $P \neq Q$ and $P\alpha \neq Q$ then $P\ P\alpha\ \not{I}\ Q$. Thus there are only two possibility : $P = Q$ or $P\alpha = Q$.

Let us introduce a coordinate system in Π so that a point P of $\Pi - r_\infty$, r_∞ a distinguished line, has non-homogeneaous coordinates (x,y), $x,y \in F$, $(x,y) \equiv (x,y,1)$, and for any point P' of r_∞ it is $P' = (x,y,0)$. Moreover, $P\alpha = (x^q, y^q)$ and $P'\alpha = (x^q, y^q, 0)$.

Any line r is represented by an equation $x=c$ or $y=xm+b$ and $r\alpha$ by $x=c^q$ or $y=xm^q+b^q$, respectively.

Set $\mathcal{C}'=\{P\in\mathcal{P}_3 / P\ I^* r\}$; by Lemma 1.1 it follows that
$\mathcal{C}'=\{P\in\mathcal{P}_3 / P\ P\alpha\ I\ Q\}$.

PROPOSITION 3.1- $\mathcal{C}'$ is a subset of an algebraic curve $\mathcal{C}$ of π of order $q+1$.

Proof:

Let $P=(x',y')$ be a point of $\mathcal{C}'$; since $P\in\mathcal{P}_3$, then $P\alpha=(x'^q,y'^q)$ where $x'^q\neq x'$ and $y'^q\neq y'$.

Let $y=xm+b$ be the equation of r; $y=xm^q+b^q$ is the equation of $r\alpha$ and $m^q\neq m$, $b^q\neq b$, as $r\in\mathcal{L}_3$ (see Sec. 1).

The equation of the line $P\ P\alpha$ is $y=xn+c$ where :

$$n=(y'-y'^q)(x'-x'^q)^{-1} \quad , \quad c=(x'y'^q-x'^qy')(x'-x'^q)^{-1} \quad .$$

The line $P\ P\alpha$ is incident to the point Q if and only if $P\ P\alpha$ belongs to the bundle (Q) of the lines of π with center Q, equivalently if and only if there exists $\lambda,\mu\in F$ such that

(3.5) $\lambda+\mu = 1$, $\lambda m+\mu m^q = n$, $\lambda b+\mu b^q = c$.

The relations (3.5) are equivalent to

$$(m - m^q)(b - b^q)^{-1} = (n - m^q)(c - b^q)^{-1}$$

or ,

$$(3.6)\qquad (m-m^q)(x'y'^q-x'^qy')-(b-b^q)(y'-y'^q)+(bm^q-b^qm)(x'-x'^q)=0 \quad .$$

The equation (3.6) in x',y' represents an algebraic curve $\mathcal{C}$ of π of order $q+1$.

As π^* is a projective plane and a point $P=(x,y)$ belongs to $\mathcal{P}_1$ if and only if $P\alpha=(x^q,y^q)=(x,y)=P$ (see Sec. 1) then we can easily prove the following:

PROPOSITION 3.2- For any line r of $\mathcal{L}_3$, the curve $\mathcal{C}$ contains all the points of π_0 and $\mathcal{C}'=\mathcal{C}-(\mathcal{C}\cap\mathcal{P}_1)$ consists of q^3-q^2-q points of $\mathcal{P}_3$.

For any line $r \in \mathcal{L}_3$ of equation $y=xm+b$ set :

$r_2 = \{(x,y) \in \mathcal{P}_2 \ / \ y=xm+b\}$, $r_3 = \{(x,y) \in \mathcal{P}_3 \ / \ y=xm+b\}$ and

$r_3^* = \{P \in \mathcal{P}_3 \ / \ P \ I^* \ r\} = \{P=(x,y) \in \mathcal{P}_3 \ / \ (x,y) \text{ satisfy } (3.6)\}$

It is $r = r_2 \cup r_3$.

Take

$$(3.7) \quad r^* = r_2 \cup r_3^* \quad ; \quad \mathcal{L}_3^* = \{r^* \ / \ \forall r \in \mathcal{L}_3\} \ ; \ \mathcal{L}^* = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3^*$$

As an easy consequence of Lemmas 3.1, 3.3 and of the Propositions 3.1, 3.2 we can state the following

<u>THEOREM 3.1</u>- The Figueroa plane $\Pi^* = (\mathcal{P}, \mathcal{L}, I^*)$ of order q^3 is represented by $(\mathcal{P}, \mathcal{L}^*, I)$.

<u>REMARK 1</u> - The representation of Π^* by $(\mathcal{P}, \mathcal{L}^*, I)$ starts again from Π ; the subset $\mathcal{L}_3$ of the lines is replaced by $\mathcal{L}_3^*$, any new line $r^* \in \mathcal{L}_3^*$ consisting of two subsets : the subset r_2 of the "old" line r and the subset $\mathcal{C}'$ of the curve $\mathcal{C}$ defined by r , which replaces the points of $r \cap \mathcal{P}_3$.

<u>REMARK 2</u> - Let α be the collineation of Π described by the diagonal matrix A 3x3 over F , $A=\mathrm{diag}(1, r^i, r^{i+1})$ where $2i+1=h$, r is an element of F of order h . It is $A^h = I$. Define a map $\mu : \mathcal{P}' \longrightarrow \mathcal{L}$ where $\mathcal{P}' \subset \mathcal{P}$ is the set of the point of Π not fixed by α , $P\mu = P\alpha^i \ P\alpha^{i+1}$ and a map $\mu' : \mathcal{L}' \rightarrow \mathcal{P}$ where $\mathcal{L}' \subset \mathcal{L}$ is the set of the lines of Π not fixed by α , $s\mu' = s\alpha^i \cap s\alpha^{i+1}$ (compare [2] , for i=1). We can easily prove that the mappings μ and μ' are involutorial birational reciprocities as $(x,y,z)\mu = [\theta yz, xz, xy]$ in homogeneous coordinates of points, resp., lines in Π ,where $\theta = -r^{i^2}(1+r^i)$; the analogous holds for μ'.

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