

INEQUALITIES FOR SOME SPECIAL FUNCTIONS (*)

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Abstract

The present paper is expository. We give a survey of some simple methods for finding bounds for some special functions. Our results are based on the recurrence relations satisfied by these functions and on some integral representations.

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1. Introduction . The purpose of the present paper is to describe some elementary methods which can be used to establish inequalities for some special functions. In particular we derive inequalities for the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$. The bounds are obtained directly from the recurrence relations satisfied by these functions.

Furthermore we describe a method for getting a simple inequality for elliptic integrals K and E of the first and the second kind , respectively.

2. Bounds for modified Bessel functions . Let $I_\nu(t)$ and $K_\nu(t)$ be the modified Bessel functions of the first and the third kind, respectively. Several authors studied inequalities for these functions. For example Bordelon [2] and Ross [8] proved the following result

$$e^{x-y} \left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)} < e^{y-x} \left(\frac{x}{y}\right)^\nu, \quad \nu > 0, 0 < x < y$$

and Ifantis and Siafarikas [3] established the upper bound

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu \left(\frac{x^2 + j_{\nu 1}^2}{y^2 + j_{\nu 1}^2}\right)^{\frac{j_{\nu 1}^2}{4(\nu+1)}}, \quad \nu > -1, 0 < x < y$$

where $j_{\nu 1}$ denotes the first positive zero of the Bessel function $J_\nu(x)$ of the first kind.

It is possible to derive inequalities for the function $I_\nu(t)$ using the recurrence relations [10, p. 79]

$$(2.1) \quad t I'_\nu(t) - \nu I_\nu(t) = t I_{\nu+1}(t)$$

and

$$(2.2) \quad t I'_\nu(t) + \nu I_\nu(t) = t I_{\nu-1}(t) .$$

Similarly for the function $K_\nu(t)$ we need the recurrence relations [10, p. 79]

$$(2.3) \quad t K'_\nu(t) + \nu K_\nu(t) = -t K_{\nu-1}(t)$$

and

$$(2.4) \quad t K'_\nu(t) - \nu K_\nu(t) = -t K_{\nu+1}(t) .$$

Theorem 2.1 . For real ν let $I_\nu(t)$ be the modified Bessel function of the first kind . Then the following inequalities

$$\frac{I_\nu(x)}{I_\nu(y)} > e^{x-y} \left(\frac{x}{y}\right)^\nu , \quad \nu \geq -\frac{1}{2} , 0 < x < y$$

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y} \left(\frac{y}{x}\right)^\nu , \quad \nu \geq \frac{1}{2} , 0 < x < y$$

hold .

Proof . We use the inequality

$$I_\nu(t) > I_{\nu+1}(t) , \quad \nu \geq -\frac{1}{2}$$

established by Nasell [7] and Soni [9] in the recurrence relation (2.1). So we find

$$\frac{I'_\nu(t)}{I_\nu(t)} < 1 + \frac{\nu}{t} \quad , \quad \nu \geq -\frac{1}{2} \quad .$$

Integrating between x and y we obtain

$$\frac{I_\nu(x)}{I_\nu(y)} > e^{x-y} \left(\frac{x}{y}\right)^\nu \quad , \quad \nu \geq -\frac{1}{2}$$

which is the desired lower bound.

The proof of the upper bound is similar. Now we refer to the recurrence relation (2.2) and we get

$$\frac{I'_\nu(t)}{I_\nu(t)} > 1 - \frac{\nu}{t} \quad , \quad \nu \geq \frac{1}{2} \quad .$$

Integrating on $[x,y]$ we find

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y} \left(\frac{y}{x}\right)^\nu \quad , \quad \nu \geq \frac{1}{2} \quad .$$

This completes the proof of the theorem.

Theorem 2.2 . For real ν let $K_\nu(t)$ be the modified Bessel function of the third kind. Then the following inequalities

$$\frac{K_\nu(x)}{K_\nu(y)} < e^{y-x} \left(\frac{y}{x}\right)^\nu \quad , \quad \nu > \frac{1}{2} \quad , \quad 0 < x < y$$

and

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x} \left(\frac{y}{x}\right)^\nu \quad , \quad 0 < \nu < \frac{1}{2} \quad , \quad 0 < x < y$$

hold.

Proof . The proof is similar to the one of theorem 2.1 . We use the inequality [9]

$$K_{\nu+1}(t) > K_{\nu}(t) \quad , \quad \nu > -\frac{1}{2}$$

in (2.3) obtaining

$$\frac{K'_{\nu}(t)}{K_{\nu}(t)} = -\frac{\nu}{t} - \frac{K_{\nu-1}(t)}{K_{\nu}(t)} > -\frac{\nu}{t} - 1 \quad , \quad \nu > \frac{1}{2} \quad ,$$

and integrate on $[x,y]$, leading to the desired upper bound. The starting point for the proof of the lower bound is the recurrence relation (2.4) and the inequality

$$K_{\nu-1}(t) > K_{\nu}(t) \quad , \quad 0 < \nu < \frac{1}{2} \quad .$$

This inequality can be checked in several ways. For example it follows immediately from the integral formula [10, p. 181]

$$K_{\nu}(t) = \int_0^{\infty} e^{-t \cosh z} \cosh(\nu z) dz \quad .$$

From (2.4) we get

$$\frac{K'_{\nu}(t)}{K_{\nu}(t)} < -\frac{\nu}{t} - 1 \quad , \quad 0 < \nu < \frac{1}{2}$$

and integrating on $[x,y]$ the desired lower bound follows.

Inequalities similar to ones of theorems 2.1 and 2.2 can be established

for the Bessel functions.

While the proofs of theorems 2.1 and 2.2 are based on the recurrence relations, on the other hand it is possible to establish other more complicated inequalities from the differential equations satisfied by these functions. We recall here a result established by the author and M.L. Mathis [5].

Theorem 2.3 . For $\nu > -1/2$ let $I_\nu(t)$ the modified Bessel function of the first kind. Then the following inequalities

$$\frac{I_\nu(x)}{I_\nu(y)} > \left(\frac{x}{y}\right)^{\frac{1}{4} + \nu^2} \exp\left\{\frac{x^2 - y^2}{6}\right\}, \quad 0 < x < y,$$

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^{\frac{3}{16} + \frac{3}{2}\nu^2 - \nu^4} \exp\left\{\frac{x^2 - y^2}{36}\left(7 - 4\nu^2 - \frac{x^2 + y^2}{5}\right)\right\}$$

$$\text{for } |\nu^2 - 1/4|^{1/2} \leq x < y < t_1,$$

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^{\frac{1}{2}}, \quad 0 < x < \nu, \quad \text{for } \nu \geq \frac{1}{2},$$

holds, where t_1 is the first zero of

$$16t^4 - 40(7 - 4\nu^2)t^2 + 45(16\nu^4 - 24\nu^2 - 11).$$

3. **Inequalities for elliptic integrals** . We prove the following result .

Theorem 3.1 . For $0 < \kappa < 1$ let

$$(3.1) \quad E = \int_0^{\frac{\pi}{2}} (1 - \kappa^2 \sin^2 t)^{\frac{1}{2}} dt$$

and

$$(3.2) \quad K = \int_0^{\frac{\pi}{2}} (1 - \kappa^2 \sin^2 t)^{-\frac{1}{2}} dt$$

be the elliptic integrals of the second and the first kind, respectively. Then

$$2 \kappa^2 < (1 + \kappa^2) E - (1 - \kappa^2) K < \frac{3}{4} \pi \kappa^2 .$$

Proof . Let us consider the function

$$(3.3) \quad f(\kappa) = \frac{(1 + \kappa^2) E - (1 - \kappa^2) K}{\kappa^2} .$$

We have to prove that

$$2 < f(\kappa) < \frac{3}{4} \pi .$$

Taking into account (3.1) and (3.2) the function $f(\kappa)$ can be written in the form

$$f(\kappa) = \frac{1}{\kappa^2} \int_0^{\frac{\pi}{2}} \left\{ (1 + \kappa^2) (1 - \kappa^2 \sin^2 t)^{\frac{1}{2}} - (1 - \kappa^2) (1 - \kappa^2 \sin^2 t)^{-\frac{1}{2}} \right\} dt =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 t + 1 - \kappa^2 \sin^2 t}{(1 - \kappa^2 \sin^2 t)^{\frac{1}{2}}} dt$$

Let $g(\kappa)$ the integrand function. Differentiating with respect to κ we get

$$g'(\kappa) = \frac{\sin^2 t \cdot \kappa}{(1 - \kappa^2 \sin^2 t)^{\frac{3}{2}}} \left[\cos^2 t - (1 - \kappa^2 \sin^2 t) \right]$$

The term in the brackets is clearly negative, therefore $g(\kappa)$ decreases and also $f(\kappa)$ decreases. To prove the desired bounds we need only to compute $f(0)$ and $f(1)$. We find

$$f(0) = \lim_{\kappa \rightarrow 0^+} \frac{(1 + \kappa^2) E - (1 - \kappa^2) K}{\kappa^2} = \lim_{\kappa \rightarrow 0^+} \left[\frac{E - (1 - \kappa^2) K}{\kappa^2} + E(0) \right] =$$

$$\begin{aligned}
&= \lim_{\kappa \rightarrow 0^+} \int_0^{\frac{\pi}{2}} (1 - \kappa^2 \sin^2 t)^{-\frac{1}{2}} \cos^2 t \, dt + \frac{\pi}{2} \\
&= \int_0^{\frac{\pi}{2}} \cos^2 t \, dt + \frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3}{4} \pi .
\end{aligned}$$

For the value $f(1)$ we have

$$f(1) = 2 E(1) = 2 .$$

This gives

$$2 < f(\kappa) < \frac{3}{4} \pi$$

leading to the desired result . Further inequalities for elliptic integrals will be established in a forthcoming paper with S. Sismondi [6] .

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