CALCULATION OF THE POLARIZATION PARAMETERS FOR ELECTROMAGNETIC FIELDS

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SUMMARY: A new method utilizing the "affinity", a concept of the projective geometry, is exposed in order to identify the polarization ellipse of an assigned electromagnetic field.

1. INTRODUCTION

In the study of the electromagnetic fields the problem may arise of calculating the parameters of the so-called "polarization ellipse", namely the lengths and directions of the semiaxes of such ellipse, with respect to an assigned reference frame. To clarify this problem, first we will recall how that ellipse originates.

From now on, we indicate the complex numbers by a dot typed over their symbols; because the phasors and their related operators are time-invariable, no confusion is possible with the time derivative operator, needless in the present work. For the physic vectors we assume boldface symbols. As the polarization phenomenon involves both electric and magnetic fields, is sufficient to consider only one of them, e.g. the electric one.

2. THE POLARIZATION ELLIPSE

We define "harmonic field" an electric field whose cartesian coordinates are sinusoidal function of the time, with the same angular frequency ω. The general expression of such a field is

$$\mathbf{e}(t) = \mathbf{u}_x e_x(t) + \mathbf{u}_y e_y(t) + \mathbf{u}_z e_z(t) \tag{1}$$

where $\mathbf{u}_k$ (with $k = x,y,z$) are unit vectors and

$$e_k(t) = E_k \cos(\omega t+\alpha_k) \tag{2}$$

are the components, whose peak values are indicated by E_k and the phases by α_k.

If we introduce the phasors

$$\dot{E}_k = E_k e^{j\alpha_k} \qquad (k = x,y,z)$$

corresponding to the scalar fields (2), in the frequency domain the harmonic vector (1) becomes a *complex vector*

$$\dot{\mathbf{E}} = \mathbf{u}_x \dot{E}_x + \mathbf{u}_y \dot{E}_y + \mathbf{u}_z \dot{E}_z \tag{3}$$

from which, as it is well-known, the physical real vector (1) is obtainable by the algorithm

$$\mathbf{e}(t) = Re \left(\dot{\mathbf{E}} e^{j\omega t} \right) . \tag{4}$$

It is also well-known [1] that another representation is possible for the complex vector (3): if we expand the cosines in eqn. (2) and let

$$E_{Rk} = Re (\dot{E}_k), \qquad E_{Jk} = Im (\dot{E}_k),$$

we may define two *constant and real* vectors

$$\begin{aligned} \mathbf{E}_R &= \mathbf{u}_x E_{Rx} + \mathbf{u}_y E_{Ry} + \mathbf{u}_z E_{Rz} \\ \mathbf{E}_J &= \mathbf{u}_x E_{Jx} + \mathbf{u}_y E_{Jy} + \mathbf{u}_z E_{Jz} \end{aligned} \tag{5}$$

by means, after some algebraic manipulation, the vector (1) may assume the expression

$$\mathbf{e}(t) = \mathbf{E}_R \cos \omega t + \mathbf{E}_J \cos(\omega t + \pi/2) \tag{6}$$

whose representation in the frequency domain is:

$$\dot{\mathbf{E}} = \mathbf{E}_R + j\, \mathbf{E}_J . \tag{7}$$

The relation (6) is more interesting than its equivalent (1) and (2) because it points out that the vector $\mathbf{e}(t)$ moves always lying on a stationary plane defined by the two stationary vectors (5); this important result is not easy to draw directly from eqns. (1) and (2). Besides, it is also apparent that the vector $\mathbf{e}(t)$, defined as sum of three harmonic vectors, coincides also with the sum of the following only two harmonic vectors

(8) $\mathbf{e}_R(t) = \mathbf{E}_R \cos \omega t, \qquad \mathbf{e}_J(t) = \mathbf{E}_J \cos(\omega t+\pi/2),$

whose spatial directions form the angle

$$(9) \qquad \vartheta = \text{arc} \frac{\mathbf{E}_R \cdot \mathbf{E}_J}{E_R \quad E_J} ;$$

it is noteworthy that, whatsoever their spatial angle ϑ is, their representation on the complex plane always form a right angle, as $\mathbf{e}_R(t)$ lags $\pi/2$ behind $\mathbf{e}_J(t)$.

The trajectory described by the tip of the vector $\mathbf{e}(t)$ is an ellipse, because it is the composition of the harmonic motions of the two tips of vectors (8); the field $\mathbf{e}(t)$ is said to be "elliptically polarized". This ellipse becomes a circle if $E_R=E_J$ and $\vartheta=\pi/2$ (field "circularly polarized"), or may degenerate into a segment if $\vartheta=0$, or $E_R=0$, or $E_J=0$ (field "linearly polarized").

The phenomenon of the elliptical polarization is very important because an electric (or magnetic) polarized field does stress the medium in a more complicate way than a variable field with constant direction does. In many problems, especially if the medium is anisotropic, it may be necessary to determine the maximum and the minimum stresses (and their respective directions) caused by a polarized field.

3. THE GEOMETRICAL PROBLEM

Generally speaking, a couple of stationary vectors like (5) completely defines a polarized field by means of the eqn. (6) and identifies the corresponding ellipse. The maximum and minimum values of the vector magnitude e(t) coincide with the semiaxes of this ellipse, whose equation is readily obtainable in an "affine plane" eliminating the variable ωt from the moduli of eqns. (8):

$$(10) \qquad \frac{e_R^2}{E_R^2} + \frac{e_J^2}{E_J^2} = 1 ;$$

from this equation it is easy to recognize that: the ellipse is centered on the origin of the frame, whose x and y axes, not necessarily orthogonal, form the angle ϑ of eqn. (9) and contain, respectively, the vectors $\mathbf{E}_R$ and $\mathbf{E}_J$ of eqns. (5). These vectors are a couple of conjugated semidiameters, that is, the polus at infinity of a diameter identifies the direction of the other: in other words, if we consider a parallelogram circumscribing the ellipse, each couple of opposite sides is parallel to the diameter passing through the tangential points of the other couple of sides.

After all, the physical problem of calculating the maximum and minimum values (which we will name e_{max} and e_{min}) of modulus of $\mathbf{e}(t)$ coincides with the geometrical problem of calculating the semiaxes of an ellipse whose only a generic couple of conjugated semidiameters is known. This problem is classically solved by resorting to the "metric" property of the ellipse [2,3]; this approach, however, leads to intricate mathematical developments, very tedious to follow.

4. THE PROJECTIVE GEOMETRY APPROACH

The flexibility of projective geometry enables us to choose as reference frame a cartesian plane, which is an affinity plane with the x and y axes forming a more conventional angle of $\pi/2$. Let us suppose we know the vectors (5) whose origin is the same of the reference frame; in order to simplify the notation, let us indicate the vectors (5) by $\mathbf{M}$ and $\mathbf{N}$ and the cartesian coordinates by x and y, rather than e_x and e_y suggested by the physical problem.

If the aforesaid vectors are known, also the implicit equations of their lines are known, that is:

$$(11) \qquad f_1(x,y) = y - mx = 0, \qquad f_2(x,y) = y - nx = 0.$$

whose angular coefficients are, respectively, m and n. The ellipse we are looking for belongs to the sheaf of conics identified by the double lines of the conjugated diameters involution [4]; then their parametric equation has the form

$$(12) \qquad f_1^2 + h\, f_2^2 + k = 0$$

where h and k are two constants to determine by imposing to the ellipse to have $\mathbf{M}$ and $\mathbf{N}$ as semidiameters. Afterwards, let us write eqn. (12) in the canonical form

(13) $$a_{11}x^2 + 2\,a_{12}x\,y + a_{22}y^2 + a_{33} = 0\ ;$$

then the following bilinear transformation holds for the angular coefficients of conjugated diameters:

(14) $$n = -\frac{a_{11} + a_{12}\,m}{a_{12} + a_{22}\,m}\ .$$

Now we impose the orthogonality of the two diameters:

(15) $$m\,n = -1\ ,$$

which means that these two generic diameters must become the two axes; combining (14) with (15) it easy to obtain the solutions:

(16) $$\mu,\nu = \frac{a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4\,a_{12}^2}}{2\,a_{12}}\ .$$

The two lines having equations

(17) $$y = \mu\,x \qquad \text{and} \qquad y = \nu\,x$$

allow us to obtain the values $e_{max}=|\mathbf{e}_{max}|$ and $e_{min}=|\mathbf{e}_{min}|$ as distances of their intersections with the ellipse (13) from the origin of coordinates.

5. A NUMERICAL EXAMPLE

In order to make clear the plainness of the proposed approach, we will apply them to a practical case.

Let us calculate the maximum and minimum values of a polarized field identified by two vectors **M** and **N**, having their origin O placed in the origin of a cartesian frame and their tips placed in the points C and D whose coordinates are, respectively, (2.5; 1) and (-1; 1.5); as consequence, the lines (11) on which the vectors lay have, respectively, equations f_1=y-0.4x and f_2=y+1.5x, so the eqn. (12) becomes

$$(y - 0.4\,x)^2 + h\,(y + 1.5\,x)^2 + k = 0\ ;$$

by imposing the passage for the points C and D we obtain h=0.16 and k=-3.61, so that the parametric equation (13) of the ellipse becomes

$$0.52\ x^2 - 0.32\ xy + 1.16\ y^2 - 3.61 = 0 \tag{18}$$

with $a_{11}=0.52$, $a_{12}=-0.16$ and $a_{22}=1.16$. Then, resorting to (16), we get $\mu=-4.236$ and $\nu=0.236$, that is, the lines on which the semiaxes lay have equations:

$$y = 0.236\ x \tag{19}$$

$$y = -4.236\ x\ . \tag{20}$$

Let AO and BO the semiaxes; introducing in eqn.(18), respectively, the eqns.(19) and (20) we obtain the coordinates $x_A= 2.663$, $y_A= 0.629$ of the point A and $x_B= -0.399$, $y_B= 1.690$ of the point B; as consequence, the semiaxes lengths are:

$$e_{max}= \sqrt{x_A^2+y_A^2} = 2.736; \qquad e_{min}= \sqrt{x_B^2+y_B^2} = 1.736\ .$$

Finally, we have obtained the moduli of the maximum and minimum stresses acting, respectively, along the directions (19) and (20); the corresponding angles they form with the positive versus of the x axis are:

$$\text{arc tg } 0.236 = 0.232 \text{ rad} \cong 13.3 \text{ deg};$$

$$\text{arc tg } (-4.236) + \pi = 1.803 \text{ rad} \cong 103.3 \text{ deg}.$$

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