Ratio Mathematica **ISSN**: 1592-7415
Vol. 33, 2017, pp. 5-20 **eISSN**: 2282-8214

On Some Applications of the Vougiouklis Hyperstructures to Probability Theory

Antonio Maturo[1], Fabrizio Maturo[2]

[3]**doi**: 10.23755/rm.v33i0.372

Abstract

Some important concepts about algebraic hyperstructures, especially from a geometric point of view, are recalled. Many applications of the H_v structures, introduced by Vougiouklis in 1990, to the de Finetti subjective probability theory are considered. We show how the wealth of probabilistic meanings of H_v-structures confirms the importance of the theoretical results obtained by Vougiouklis. Such results can be very meaningful also in many application fields, such as decision theory, highly dependent on subjective probability.
Keywords: algebraic hyperstructures; subjective probability; H_v structures, join spaces.
2010 AMS subject classification: 20N20; 60A05; 52A10.

[1] Department of Architecture, University of Chieti-Pescara, Italy; antomato75@gmail.com
[2] Department of Business Administration, University of Chieti-Pescara, Italy; f.maturo@unich.it
[3] ©Antonio Maturo and Fabrizio Maturo. Received: 15-10-2017. Accepted: 26-12-2017. Published: 31-12-2017.

1 Introduction

The theory of the algebraic hyperstructures was born with the paper (Marty, 1934) at the VIII Congress of Scandinavian Mathematicians and it was developed in the last 40 years.

In the book "Prolegomena of hypergroup theory" (Corsini, 1993) all the fundamental results on the algebraic hyperstructures, until 1992, have been presented. A complete bibliography is given in the appendix. A review of the results until 2003 is in (Corsini, Leoreanu, 2003).

Perhaps the most important motivation for the study of algebraic hyperstructures comes from the basic text "Join Geometries" by Prenowitz and Jantosciak (1979), which in addition to giving an original and general approach to the study of Geometry, introduces an interdisciplinary vision of Geometry and Algebra, showing how the Euclidean Spaces can be drawn as Join Spaces, i.e. commutative hypergroups that satisfy an axiom called "incidence property". Moreover, various other geometries, such as the Projective Geometries (Beutelspacher, Rosembaum, 1998), are also Join Spaces.

Considering, for example, the Affine Geometries, it is seen that associative property is not satisfied in many important geometric spaces. This and other important geometric and algebraic issues have led to the study of weak associative hyperstructures. The theory of such hyperstructures, called H_v-structures, was carried out by Thomas Vougiouklis, who introduced the concept of H_v-structures in the work “The fundamental relation in hyperrings. The general hyperfield” (1991),**.** presented at the 4th AHA Conference, Xanthi, Greece, 1990. Subsequently Vougiouklis found many fundamental results on the H_v-structures in numerous works (e.g. Vougiouklis, 1991, 1992, 1994a, 1994b; Spartalis, Vougiouklis, 1994). A collection of all the results on the subject until 1994 is in the important book “Hyperstructures and their representations” (Vougiouklis, 1994c).

Subsequent insights into H_v-structures were made by Vougiouklis in many subsequent works (1996a, 1996b, 1996c, 1997, 1999a, 1999b, 2003a, 2003b, 2008, 2014), also in collaboration with other authors (Dramalidis, Vougiouklis, 2009, 2012; Vougiouklis et al., 1997; Nikolaidou, Vougiouklis, 2012).

From the H_v-structures of Vougiouklis, the idea in the Chieti-Pescara research group was conceived to interpret some important structures of subjective probability as algebraic structures. Some paper on this topic are (Doria, Maturo, 1995, 1996; Maturo, 1997a, 1997b, 1997c, 2000a, 2000b, 2001a, 2001b, 2003c, 2008, 2010).

The study of applications of hyperstructures to the treatment of uncertainty and decision-making problems in Architecture and Social Sciences begins with a series of lectures held at the Faculty of Architecture in Pescara by Giuseppe Tallini in 1993, on hyperstructures seen from a geometric point of view, and was developed at various AHA conferences (Algebraic Hyperstructures and Applications) as well as various seminars and conferences with Piergiulio Corsini from 1994 to 2014.

For example, in December 1994 and October 1995, two conferences on "Hyperstructures and their Applications in Cryptography, Geometry and Uncertainty Treatment" were organized by Corsini, Eugeni and Maturo, respectively in Chieti and Pescara, with which it was initiated the systematic study of the applications of hyperstructures to the treatment of uncertainty and Architecture.

In (Corsini, 1994), it is proved that the fuzzy sets are particular hypergroups. This fact leads us to examine properties of fuzzy partitions from a point of view of the theory of hypergroups. In particular, crisp and fuzzy partitions given by a clustering could be well represented by hypergroups. Some results on this topic and applications in Architecture are in the papers of Ferri and Maturo (1997, 1998, 1999a, 1999b, 2001a, 2001b). Applications of hyperstructures in Architecture are also in (Antampoufis et al., 2011; Maturo, Tofan, 2001). Moreover, the results on fuzzy regression by Fabrizio Maturo, Sarka Hoskova-Mayerova (2016) can be translate as results on hyperstructures.

A new research trend concerns the applications of hypergroupoid to Social Sciences. Vougiouklis, in some of his papers (e.g. 2009, 2011), propose hyperstructures as models in social sciences; Hoskova-Mayerova and Maturo analyze social relations and social group behaviors with fuzzy sets and H_v-structures (2013, 2014), and introduce some generalization of the Moreno indices.

2 Fundamental Definitions on Hyperstructures

Let us recall some of the main definitions on the hyperstructures that will be applied in this paper to represent concepts of Logic and Subjective Probability.

For further details on hyperstructure theory, see, for example, (Corsini, 1993; Corsini, Leoreanu, 2003; Vougiouklis, 1994c).

Definition 2.1 Let H be a non-empty set and let $\wp^*(H)$ be the family of non-empty subsets of H. A *hyperoperation* on H is a function σ: $H \times H \to \wp^*(H)$, such that to every ordered pair (a, b) of elements of H associates a non-empty subset of H, noted $a\sigma b$. The pair (H, σ) is called *hypergroupoid* with *support* H and *hyperoperation* σ.

If A and B are non-empty subsets of H, we put $A\sigma B = \cup\{a\sigma b: a\in A, b\in B\}$. Moreover, $\forall a, b\in H$, we put, $a\sigma B = \{a\}\sigma B$ and $A\sigma b = A\sigma\{b\}$.

Definition 2.2 A hypergroupoid (H, σ) is said to be:

- a *semihypergroup*, if $\forall a, y, c\in H$, $a\sigma(b\sigma c) = (a\sigma b)\sigma c$ (*associativity)*;
- a *quasihypergroup*, if $\forall a\in H$, $a\sigma H = H = H\sigma a$ (*riproducibility*);
- a *hypergroup* if it is both a *semihypergroup* and a *quasihypergroup*;
- *commutative*, if $\forall a, b\in H$, $a\sigma b = b\sigma a$;
- *idempotent*, if $\forall a\in H$, $a\sigma a=\{a\}$.
- *weak associative*, if $\forall a, b, c\in H$, $a\sigma(b\sigma c) \cap (a\sigma b)\sigma c \neq \emptyset$;
- *weak commutative*, if $\forall a, b\in H$, $a\sigma b \cap b\sigma a \neq\emptyset$.

The weak associative hypergroupoid, called also H_v-semigroup by Vougiouklis (1991), appear to be particularly significant in the Theory of Subjective Probability, and all results found by Vougiouklis in later papers (e.g.1992, 1994a, 1994b), should have important logic and probabilistic meanings. Vougiouklis (1991) introduced also the notation "H_v-group" for the weak associative quasihypergroups.

A H_v-semigroup is said to be *left directed* if $\forall a, b, c\in H$, $a\sigma(b\sigma c) \subseteq (a\sigma b)\sigma c$, and *right directed* if $a\sigma(b\sigma c) \supseteq (a\sigma b)\sigma c$.

Let (H, σ) a hypergroupoid. Using a geometric language, a singleton {a}, $a\in H$, is said to be a block of order 1 (briefly 1-block) generated by a. Every hyperproduct $a\sigma b$, $a, b\in H$, is a block of order 2 (2-block), called block generated by (a, b). For every $a_1, a_2, a_3 \in H$, the hyperproducts $a_1 \sigma (a_2 \sigma a_3)$ and $(a_1 \sigma a_2) \sigma a_3$ are the 3-blocks generated by (a_1, a_2, a_3). For recurrence, for every $a_1, a_2, \ldots, a_n\in H$, $n > 2$, the blocks generated by $(a_1, a_2, \ldots, a_n)$ are the hyperproducts $A\sigma B$, with A block of order $s < n$, generated by $(a_1, \ldots, a_s)$, and B block of order n-s generated by $(a_{s+1}, \ldots, a_n)$. In general, for every $n > 1$, a block of order n (or n-block) is a hyperproduct $A\sigma B$, with A block of order $s < n$, B block of order n - s.

For every $n\in N$, let Δ_n be the set of all the blocks of order n, and let $\Delta_0 = \cup\{\Delta_n, n\in N\}$. Then for every $n\in N_0$, a geometric space (H, Δ_n) is associated to the hypergroupoid (H, σ). A polygonal with length m of (H, Δ_n) is a n-tuple $(A_1, A_2, \ldots, A_m)$ of blocks of Δ_n such that $A_i \cap A_{i+1} \neq \emptyset$. Let Π_n be the set of all the polygonals of (H, Δ_n).

The relation β_n and β_n* are defined as:

$\forall a, b\in H$, $a\ \beta_n\ b \Leftrightarrow \exists\ A\in\Delta_n: \{a, b\}\subseteq A$,

$\forall a, b\in H$, $a\ \beta_n^*\ b \Leftrightarrow \exists\ P\in\Pi_n: \{a, b\}\subseteq P$.

β_n is reflexive and symmetric, β_n^* is the transitive closure of β_n. For n = 0 we have the classical relations β and β* considered in many papers, e.g. (Freni,1991; Corsini, 1993; Gutan, 1997; Vougiouklis 1999b), A more restrictive condition than the weak associativity is the “strong weak associativity”, called also feeble associativity.

Definition 2.3 A hypergroupoid (H, σ) is said to be *feeble associative* if, for every $a_1, a_2, \ldots, a_n \in H$, the intersection of all the blocks generated by $(a_1, a_2, \ldots, a_n)$ is not empty.

If (H, σ) is a commutative quasihypergroup, the σ-division is defined in H, as the hyperoperation $/_\sigma : H\times H \to \wp^*(H)$ that to every pair $(a, b)\in H\times H$ associates the nonempty set $\{x\in H: a\in b\sigma x\}$.

Definition 2.4 A commutative hypergroup (H, σ) is said to be a *join space* if the following “incidence property” hold:

$$\forall a, b, c, d \in H,\ a /_\sigma b \cap c /_\sigma d \neq \emptyset \Rightarrow a\sigma d \cap b\sigma c \neq \emptyset. \qquad (2.1)$$

A join space (H, σ) is:

- *open*, if, $\forall a, b \in H$, $a \neq b$, $a \sigma b \cap \{a, b\} = \emptyset$;
- *closed*, if, $\forall a, b \in H$, $\{a, b\} \subseteq a \sigma b$;
- *σ-idempotent*, if, $\forall a\in H$, $a \sigma a = \{a\}$;
- $/_\sigma$ *-idempotent*, if, $\forall a\in H$, $a /_\sigma a = \{a\}$.

A join space (H, σ) is said to be a *join geometry* if it is σ-idempotent and $/_\sigma$ -idempotent. We have the following theorem.

Theorem 2.1 Let $H = R^n$ and σ the hyperoperation that to every $(a, b)\in H\times H$ associates the open segment with extremes a and b if $a \neq b$, and $a \sigma a = \{a\}$. (H, σ) is a join geometry, called *Euclidean join geometry*.

Let (H, σ) be a join geometry. We can note that it is open. Using a notation like that of Euclidean join geometry, in this paper the elements of H are called points and a block aσb, with $a \neq b$, is called (open) “σ-segment” with extremes a and b or simply “segment” if only the hyperoperation σ is considered in the context.

The concept of join space leads to a unified vision of Algebra and Geometry, that can be very useful from the point of view of advanced didactics (Di Gennaro, Maturo, 2002). Also, as some of our papers show, join geometries have important applications in subjective probability. Moreover, we can introduce general uncertainty measures in join geometries such that in the Euclidean join geometries reduce to the de Finetti coherent probability (Maturo, 2003a, 2003b, 2006, 2008; Maturo et al., 2010).

3 Subjective Probability and Hyperstructures

Let us recall the concept of coherent probabilty and its geometric representation with the notation given in (Maturo, 2006).

The coherence of an assessment of probabilities $p = (p_1, p_2, \ldots, p_n)$ on a n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is defined by an hypothetical bet with a n-tuple of wins $S = (S_1, S_2, \ldots, S_n)$ (de Finetti, 1970; Coletti, Scozzafava, 2002; Maturo 2006).

For every $i \in \{1, 2, \ldots, n\}$ an individual A, called *the better*, pays the *stake* p_iS_i to an individual B, called *the bank*, and, if the event E_i occurs, A receives from B the win S_i. If $S_i < 0$ the verse of the bet on E_i is inverted, i. e. B pays the stake and A pays the win.

The total random gain G_A of A is given by the formula:

$$G_{A, p, S} = (|E_1| - p_1) S_1 + (|E_2| - p_2) S_2 + \ldots + (|E_n| - p_n) S_n. \quad (3.1)$$

where $|E_i| = 1$ if the event E_i is verified and $|E_i| = 0$ if the event E_i is not verified.

The atoms associated with the set of events $\mathbf{E} = \{E_1, E_2, \ldots, E_n\}$ are the intersections $A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_i \in \{E_i, -E_i\}$, different from the impossible event $\varnothing$. Let At($\mathbf{E}$) be the set of the atoms. Then $G_A(p, S)$ can be interpret as the function:

$$G_{A, p, S}: a = A_1 \cap A_2 \cap \ldots \cap A_n \in At(\mathbf{E}) \to (|E_1| - p_1) S_1 + (|E_2| - p_2) S_2 + \ldots + (|E_n| - p_n) S_n. \quad (3.2)$$

Definition 3.1 The probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is said to be *coherent* if, for every $S = (S_1, S_2, \ldots, S_n) \in R^n$, there are $a, b \in At(\mathbf{E})$ such that $G_{A, p, S}(a) \geq 0$ and $G_{A, p, S}(b) \leq 0$.

We note that the previous definition implies a hyperoperation. Let Λ be an algebra of events containing the set $\mathbf{E}$. Then Λ also contains At($\mathbf{E}$) and we can define the hyperoperation α on Λ:

$$\alpha: (A, B) \in \Lambda \times \Lambda \to At(A, B). \quad (3.3)$$

The above considerations show that it may be important, in a probabilistic context, to know the properties of the algebraic hyperstructure (Λ, α), introduced in (Doria, Maturo, 2006), and called *hypergroupoid of atoms*.

The coatoms associated with $\mathbf{E}$ are the nonimpossible complementary events of the atoms. Let Co($\mathbf{E}$) be the set of coatoms, and k be the number of atoms. For k = 1, At($\mathbf{E}$) = $\{\Omega\}$, where Ω is the certain event and Co($\mathbf{E}$) is empty. For k = 2, At($\mathbf{E}$) = Co($\mathbf{E}$) and for k >2 the sets At($\mathbf{E}$) and Co($\mathbf{E}$) are disjoint and with the some number of elements.

For every $A, B, C \in \Lambda$, we have (we write X Y to denote $X \cap Y$):

$(A\alpha B)\alpha C = (\{X\ C, X\ (-C), X \in At(A, B)\} \cup \{Y\ C, Y\ (-C), Y \in Co(A, B)\}) - \{\varnothing\}$,

$A\alpha(B\alpha C) = (\{A\ Z, (-A)\ Z, Z\in At(B, C)\} \cup \{A\ T, (-A)\ T, T\in Co(B, C)\}-\{\emptyset\}$,

$At\{A, B, C\} = \{X\ C, X\ (-C), X\in At(A, B)\}-\{\emptyset\} = \{A\ Z, (-A)\ Z, Z\in At(B, C)\}-\{\emptyset\}$.

Then:

$At\{A, B, C\} \subseteq (A\alpha B)\alpha C \cap A\alpha(B\alpha C)$.

Therefore, the following theorem applies:

Theorem 3.1 Let Λ be an algebra of events, and α the hyperoperation defined by (3.3). Then (Λ, α) is a commutative H_v-semigroup.

The algebra associated with the set of events $\mathbf{E} = \{E_1, E_2, \ldots, E_n\}$, denoted with Alg(**E**) is the set containing the impossible event $\emptyset$ and all the unions of the elements of At(**E**), i.e. $X\in Alg(\mathbf{E})$ iff $\exists Y\in \wp(At(\mathbf{E}))$ such that X is the union of the elements of Y. If $|At(\mathbf{E})|=s$, then $|Alg(\mathbf{E})| = 2^s$.

Let Λ be an algebra of events. We can introduce the following hyperoperation on Λ:

$$\beta: (A, B)\in \Lambda\times\Lambda\rightarrow Alg(A, B) \quad (3.4)$$

The hyperoperation β is commutative, and, since $\{A, B\} \subseteq Alg(A, B)$, (Λ, β) is a quasihypergroup. Moreover $At(A, B) \subseteq Alg(A, B)$ and so β is an extension of the operation α and we have:

$At\{A, B, C\} \subseteq (A\ \beta\ B)\ \beta\ C \cap A\ \beta\ (B\ \beta\ C)$.

Theorem 3.2 Let Λ be an algebra of events, and β the hyperoperation defined by (3.4). Then (Λ, β) is a commutative H_v-group.

Suppose A, B, C are logically independent events, then $|At(A, B| = 4$, $|Alg(A, B| = 2^4 = 16$, $|At(A, B, C)| = 8$, $Alg(A, B, C)| = 2^8 = 256$. Moreover Alg(A, B) contains $\emptyset$, Ω and other 7 elements with their complements. If X is one of these elements, then $X\ \beta\ C$ contains $\emptyset$, Ω, C, -C and other 12 elements.

Then $(A\ \beta\ B)\ \beta\ C$ has $7\times12+4= 88$ elements and 168 elements are in Alg(A, B, C) but not in $(A\ \beta\ B)\ \beta\ C$. So, in general, we can write:

$At\{A, B, C\} \cup Co\{A, B, C\} \subset (A\ \beta\ B)\ \beta\ C,\ A\ \beta\ (B\ \beta\ C) \subset Alg(A, B, C)$.

Let (H, σ) be a join geometry. From the associative and commutative properties, for every $a_1, a_2, \ldots, a_n\in H$ there is only a block $a_1\sigma\ a_2\sigma \ldots\sigma a_n$ generated by $(a_1, a_2, \ldots, a_n)$ and this block depend only by on the set $\{a_1, a_2, \ldots, a_n\}$ and not on the order of the elements. By the idempotence we can reduce to the case in which $a_1, a_2, \ldots, a_n$ are distinct.

Definition 3.2 For every $A \subseteq H$, $A \neq \emptyset$, the *convex hull* of A, in (H, σ), is the set

$[A]_\sigma = \{x\in H: \exists n\in N, \exists\ a_1, a_2, \ldots, a_n\in A : x \in a_1\sigma\ a_2\sigma \ldots\sigma a_n\}$.

If A is finite then $[A]_\sigma$ is said to be the *polytope* generated by A.

Let $E = (E_1, E_2, \ldots, E_n)$ be a n-tuple of events set and let At(E) the set of atoms associated to E. For every $a = A_1 \cap A_2 \cap \ldots \cap A_n \in$ At(E), let $x_i(a) = 1$ if $A_i = E_i$ and $x_i(a) = 0$ if $A_i = - E_i$. The atom a is identified with the point $(x_1(a), x_2(a), \ldots, x_n(a)) \in R^n$. From definition 3.1, the following theorem applies (Maturo, 2006, 2008, 2009).

Theorem 3.3 Let (R^n, ε) the Euclidean join geometry. The probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is coherent iff $p \in [At(E)]_\varepsilon$.

The theorem 3.3 opens the way to introduce measures of uncertainty that are different from the probability and coherent with respect non-Euclidean join geometries. We can introduce many possible join geometries. The following is an example.

Example 3.1 Let $H = R^n$ and κ the hyperoperation that to every $(a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n)) \in H \times H$ associated the Cartesian product of the open segments I_r with extremes a_r and b_r belonging to (R, ε). We can prove that (H, κ) is a join geometry, called the Cartesian join geometry.

Some applications of the Cartesian join geometry to problems of Architecture and Town-Planning are in (Ferri, Maturo, 2001a, 2001b).

In a general join geometry with support R^n we can introduce the following definition:

Definition 3.3 Let (R^n, σ) be a join geometry. The measure assessment $m = (m_1, m_2, \ldots, m_n)$ on the n-tuple $E = (E_1, E_2, \ldots, E_n)$ of events is said to be coherent with respect to (R^n, σ) iff $m \in [At(E)]_\sigma$.

For example, σ can be the hyperoperation that to every $(a, b) \in H \times H$ associates a particular curve with extremes a and b, and the polytope $[At(E)]_\sigma$ is a deformation of the Euclidean polytope, obtained by replacing the segments with curves. It can have important meanings in appropriate contexts of Physics or Social Sciences.

In a generic join geometry (R^n, σ) can happen that some of the most intuitive properties of the Euclidean join geometry fall. To avoid this, you should restrict yourself to join geometries where some additional properties apply. Important is the following:

Ordering condition. If a, b, c, are distinct elements of R^n, at most one of the following formulas occurs: $a \in b\sigma c$, $b \in a\sigma c$, $c \in a\sigma b$.

A join geometry (R^n, σ) with the order condition is said to be an *ordered join geometry*.

4 Conditional Events, Conditional Probability and Hyperstructures

The “axiomatic probability” by Kolmogorov, usually considered as the “true probability” is based on the assessment of a universal set U, whose elements are called the atoms, a σ-algebra S of subsets of U, whose elements are called the events, and a finite measure p on S, called the probability, such that p(U) = 1.

Let $S^* = S\text{-}\{\varnothing\}$. In the Kolmogorov approach to probability no consideration is given to the logical concept of conditional event E|H, with $E \in S$ and $H \in S^*$, but only the conditional probability p(E|H) is defined, only in the case in which p(H) > 0, by the formula:

$$p(E|H) = p(E\ H)/p(H). \tag{4.1}$$

On the contrary, the “subjective probability” (de Finetti,1970; Dubins, 1975; Coletti, Scozzafava 2002; Maturo, 2003b, 2006, 2008b), don’t consider the events as subsets of a given universal set U, but they are *logical propositions* that can assume only one of the truth values: *true* and *false*. A sharp separation is given among the concepts concerning the three areas of the *logic of the certain*, the *logic of the uncertain* and the *measure theory*.

The conditional event E|H is a concept belonging to the logic of the certain and it is a proposition that can assume three values: *true* if both E and H are verified, *false* if H is verified but E is not and *empty* (or *undetermined*) if H is not verified. The conditional event E|H reduces to the event E if H is the certain event Ω. In the appendix of his fundamental book (1970) de Finetti presents also some different interpretations of the logical concept of three valued proposition.

By the point of view of Reichenbach (1942) the value “empty” is replaced by the value “undetermined”. In the following we assume the notation of Reichenbach and we write T for true, F for false and U for undetermined. The set V = {F, U, T} is also ordered by putting F < U < T.

A numerical representation of the ordered set V is given by associating 0 to F, 1 to T and the number 1/2 to U. An alternative, in a fuzzy contest, we can associate to U is the fuzzy number u with support and core the interval [0, 1], then the relation $0 < u < 1$ is a consequence of the usual order relation among the trapezoidal fuzzy numbers.

In the subjective probability, the conditional probability p(E|H) of the conditional event E|H is given by an expert and no condition is given about the belonging of the events E and H to a structured set, e.g. like an algebra. The only condition of $H \neq \varnothing$ is required, because if $H = \varnothing$ we have the totally undetermined conditional event.

If C is a set of conditional events the assessment of a subjective conditional probability to the elements of C must satisfy some coherence conditions.

The coherence of an assessment of probabilities $p = (p_1, p_2, \ldots, p_n)$ on a n-tuple $K = (E_1|H_1, E_2|H_2, \ldots, E_n|H_n)$ of conditional events is defined by an hypothetical bet with a n-tuple of wins $S = (S_1, S_2, \ldots, S_n)$ (de Finetti, 1970; Coletti, Scozzafava, 2002; Maturo, 2006).

For every $i \in \{1, 2, \ldots, n\}$ an individual A, called *the better*, pays the *stake* p_iS_i to an individual B, called *the bank*, and,

- if the event E_iH_i occurs, A receives from B the win S_i;
- if the event $-H_i$ occurs, the amount paid p_iS_i is refunded to A;
- if the event $(-E_i)\, H_i$ occurs, no payment is made to A.

The total random gain G_A of A is given by the formula:

$$G_{A,\,p,\,S} = |H_1|\,(|E_1| - p_1)\, S_1 + \ldots + |H_n|(|E_n| - p_n)\, S_n. \qquad (4.2)$$

where $|E_i| = 1$ if the event E_i is verified and $|E_i| = 0$ if the event E_i is not verified, and similarly to H.

The atoms associated with the set of conditional events $\mathbf{K} = \{E_1|H_1, E_2|H_2, \ldots, E_n|H_n\}$ are the intersections $A_1 \cap A_2 \cap \ldots \cap A_n$, where $A_i \in \{E_i\, H_i, -E_i\, H_i, -H_i\}$, different from the impossible event $\varnothing$. The complement of $H = \cup\{H_i, i \in \{1, 2, \ldots, n\}\}$ is said to be the *inactive atom*.

Let $Atc(\mathbf{E})$ be the set of the atoms associated to $\mathbf{K}$. Then $G_{A,\,p,\,S}$ can be interpret as the function:

$$G_{A,\,p,\,S}: a = A_1 \cap A_2 \cap \ldots \cap A_n \in Atc(\mathbf{E}) \rightarrow (|A_1| - p_1)\, S_1 + (|A_2| - p_2)\, S_2 + \ldots + (|A_n| - p_n)\, S_n \qquad (4.3)$$

where $|A_i| = 1, 0, p_i$, if $A_i = E_i\, H_i, -E_i\, H_i, -H_i$, respectively.

Definition 4.1 The conditional probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \ldots, E_n|H_n)$ of conditional events is said to be *quasi-coherent* if, for every $S = (S_1, S_2, \ldots, S_n) \in R^n$, there are $a, b \in Atc(\mathbf{E})$ such that $G_{A,\,p,\,S}(a) \geq 0$ and $G_{A,\,p,\,S}(b) \leq 0$. Moreover, $p = (p_1, p_2, \ldots, p_n)$ is said to be *coherent* if, for any $s \leq n$ and for any $\{i1, i2, \ldots, is\} \subseteq \{1, 2, \ldots, n\}$, the conditional probability assessment $p_{i1,\, i2,\, \ldots,\, is} = (p_{i1}, p_{i2}, \ldots, p_{is})$ on $(E_{i1}|H_{i1}, E_{i2}|H_{i2}, \ldots, E_{is}|H_{is})$ is quasi-coherent.

Let $K = (E_1|H_1, E_2|H_2, \ldots, E_n|H_n)$ be a n-tuple of conditional events and let $Atc(\mathbf{K})$ the set of atoms associated to $\mathbf{K} = \{E_1|H_1, E_2|H_2, \ldots, E_n|H_n\}$. For every $a = A_1 \cap A_2 \cap \ldots \cap A_n \in Atc(E)$, let $x_i(a) = |A_i|$. The atom a is identified with the point $(x_1(a), x_2(a), \ldots, x_n(a)) \in R^n$. From definition 4.1, the following theorems applies:

Theorem 4.1 Let (R^n, ε) the Euclidean join geometry. The probability assessment $p = (p_1, p_2, \ldots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \ldots, E_n|H_n)$ of conditional events is quasi-coherent iff $p \in [Atc(K)]_\varepsilon$.

Theorem 4.2 The probability assessment $p = (p_1, p_2, \dots, p_n)$ on $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ is coherent iff for any $s \leq n$ and for any $\{i1, i2, \dots, is\} \subseteq \{1, 2, \dots, n\}$, the conditional probability assessment $p_{i1, i2, \dots, is} = (p_{i1}, p_{i2}, \dots, p_{is})$ on $(E_{i1}|H_{i1}, E_{i2}|H_{i2}, \dots, E_{is}|H_{is})$ belongs to $[Atc(E_{i1}|H_{i1}, E_{i2}|H_{i2}, \dots, E_{is}|H_{is})]_\varepsilon \subseteq R^s$.

Let Λ be an algebra of events. An axiomatic formalization of the coherence conditions in the case in which $\mathbf{K} = \{E|H, E\in\Lambda, H\in\Lambda - \{\varnothing\}\}$ is in Dubins (1975).

In terms of hyperstructures, conditional events can be defined by the following hyperstructure, introduced in (Doria, Maturo, 1996) and studied in (Maturo, 1997c).

Definition 4.2 Let Λ be an algebra of events. We define on Λ the hyperoperation:

$\gamma: (E, H)\in \Lambda\times\Lambda \rightarrow \{E\,H, H\}$.

We have:

$E\,\gamma\,H \cap H\,\gamma\,E = \{E\,H\}$;

$(E\,\gamma\,H)\,\gamma\,K = \{E\,H\,K, H\,K, K\}$, $\quad E\,\gamma\,(H\,\gamma\,K) = \{E\,H\,K, H\,K, E\,K, K\}$;

$E\,\gamma\,E = \{E\}$.

Then we have the following theorem.

Theorem 4.3 The hyperstructure (Λ, γ), let us call the *hyperstructure of conditional events*, is a weak commutative and idempotent H_v-semigroup. Moreover (Λ, γ) is right directed, i.e. $(E\,\gamma\,H)\,\gamma\,K \subseteq E\,\gamma\,(H\,\gamma\,K)$.

Any singleton {H} is the conditional event H|H and any set {E, H} with $E \subseteq H$ is the conditional event E|H**,** *true* if E is verified, *false* if H is verified but not E, and it is not *undetermined* if H is not verified. Many other meanings, of the finite subsets of Λ, are considered in (Maturo, 1997c).

The coherence conditions of definition 4.1 and theorems 4.1 and 4.2 lead us to associate the n-tuple $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ of conditional events with the set all the conditional events A|B with $A\in At\{E_1, E_2, \dots, E_n\}$ and B an union of elements of $\{H_1, H_2, \dots, H_n\}$. Then, if Λ is an algebra of events, and $\Theta \subseteq \Lambda$ is a set of nonempty events, closed with respect to the union, the following hyperoperation can be introduced:

$\delta: (E|H, F|K) \in (\Lambda\times\Theta)\times(\Lambda\times\Theta) \rightarrow \{A|B: A\in At\{E, F\}, B\in\{H, K, H\cup K\}\}$.

We can prove the following thorem

Theorem 4.4 The hyperstructure (Λ×Θ, δ) is a commutative H_v-semigroup, called *hypergroupoid of conditional atoms* and, for Θ = {Ω}, is isomorphic to (Λ, α).

5 Conclusions and Perspectives of Research

We have shown that all logical operations related to subjective probability can reduce to Vougiouklis hyperstructures. (Λ, α) and $(\Lambda\times\Theta, \delta)$ are commutative H_v-semigroups, and (Λ, β) is a commutative H_v-group. The hyperoperation γ isweak commutative and idempotent and (Λ, γ) is a right directed H_v-semigroup.

To verify the coherence of a subjective probability assignment $p = (p_1, p_2, \dots, p_n)$ on the n-tuple $E = (E_1, E_2, \dots, E_n)$ of events, we represent the atoms as points of the space R^n, in which the i-th axis is associated with the event E_i. The assessment p is coherent iff p belongs to the polytope of the join geometry (R^n, ε) generate from the atoms.

More complex is the coherence check of a conditional probability assessment $p = (p_1, p_2, \dots, p_n)$ on the n-tuple $K = (E_1|H_1, E_2|H_2, \dots, E_n|H_n)$ of conditional events, as in this case we must consider polytopes in all the join geometries (R^s, ε), $s \le n$ associated to subsets of $\mathbf{K} = \{E_1|H_1, E_2|H_2, \dots, E_n|H_n\}$.

A research perspective is to investigate the properties of the considered Vougiouklis structures, highlighting their meanings from the point of view of logic and subjective probability.

A further research perspective is studying the measures that can be obtained by applying the geometric coherence conditions in ordered join geometries other than the Euclidean join geometry.

References

Antampoufis N., Vougiouklis T., Dramalidis A., (2011), Geometrical and circle hyperoperations in urban applications, *Ratio Sociologica*, Vol 4, N.2, 2011,53-66.-

Beutelspacher A., Rosembaum U., (1998), *Projective Geometry*, Cambridge University Press

Coletti G., Scozzafava R., (2002), *Probabilistic logic in a coherent setting*, Kluwer Academic Publishers, London

Corsini P., (1993), *Prolegomena of hypergroup theory,* Aviani Ed. Udine, (1993).

Corsini P., (1994) Join spaces, power sets, fuzzy sets, *Proc. of the Fifth International Congress on Algebraic Hyperstructures and Applications*, Jasi, Romania,

Corsini P., Leoreanu L., (2003), *Applications of the Hyperstructure Theory*, Kluver Academic Publishers London

de Finetti B. (1970), *Teoria delle Probabilità*, vol. 1 and 2, Einaudi, Torino

Di Gennaro F., Maturo A., (2002), La teoria delle iperstrutture: un efficace strumento per una visione unitaria di algebra e geometria, *Periodico di Matematiche*, Serie VIII, Vol 2, n. 4, 5-16

Doria S., Maturo A., (1995), A hyperstructure of conditional events for artificial intelligence, in *Mathematical models for handling partial knowledge in artificial intelligence*, Plenum press, New York, pp.201-208, (1995).

Doria S., Maturo A., (1996), Hyperstructures and geometric spaces associated to a family of events, in *Rivista di Matematica Pura ed Applicata*, N 19, 1996, pp.125-137.

Dramalidis A., Vougiouklis T., (2003), On a class of geometric fuzzy H_v-structures, *8^th^ AHA, Samothraki*, 2002, Spanidis Press, Xanthi, 137-145.

Dramalidis A., Vougiouklis T., (2009), Fuzzy H_v-substructures in a two-dimensional Euclidean vector space, 10^{th} AHA, Brno, Czech Republic 2008, (2009), 151-159, and *Iranian J. Fuzzy Systems*, 6(4), 2009, 1-9.

Dramalidis A., Vougiouklis T., (2012), H_v-semigroups as noise pollution models in urban areas, *Ratio Mathematica* 23, 2012, 39-50.

Dubins L. E. (1975), Finitely additive conditional probabilities, conglomerability and disintegrations, *The Annals of Probability*, 3, 89-99

Ferri B., Maturo A., (1997), Hyperstructures as tool to compare urban projects, in *Ratio Mathematica*, 12, 1997, 79-89.

Ferri B., Maturo A., (1998), An application of the fuzzy set theory to evaluation of urban projects, in *New trends in Fuzzy Systems*, pp.82-91, Scientific Word, Singapore.

Ferri B., Maturo A., (1999a), Fuzzy Classification and Hyperstructures: An Application to Evaluation of Urban Project, in: *Classification and Data Analysis*, 55-62, Springer-Verlag, Berlin.

Ferri B., Maturo A., (1999b), On Some Applications of Fuzzy Sets and Commutative Hypergroups To Evaluation In Architecture And Town-Planning, *Ratio Mathematica*, 13, pp. 51-60.

Ferri B., Maturo A., (2001a), Mathematical models based on fuzzy classification and commutative hypergroups to solve present problems of evaluation in town planning, in *Fuzzy Systems & A. I.*, Vol. VII, Nos. 1-3, 2001, pp. 7-15.

Ferri B., Maturo A., (2001b), Classifications and Hyperstructures in problems of Architecture and Town-Planning, Journal of Interdisciplinary Mathematics, Vol 4(2001), No 1, 25-34.

Freni D., (1991), Une note sur le cœur d'un hypergroup et sur la closure transitive β^* de β, in *Rivista di Matematica pura e applicata*, 8, 153-156

Gutan M, (1997), Properties of hyperproducts and the relation β in quasihypergroups, in *Ratio Mathematica*, 12, 19-34

Hoskova-Mayerova, S., Maturo, A. (2013). Hyperstructures in Social Sciences, AWER Procedia Information Technology & Computer Science, 3(2013), 547—552, Barcelona, Spain.

Hoskova-Mayerova, S., Maturo, A. (2014). An analysis of Social Relations and Social Group behaviors with fuzzy sets and hyperstructures, Proceeding of AHA 2014, accepted in International Journal of Algebraic Hyperstructures and its Applications.

Marty F., (1934), Sur une généralization de la notion de group, *IV Congres des Mathematiciens Scandinave*, Stockholm.

Maturo A., (1997a), Probability assessments on hyperstructures of events, in *Proceedings of WUPES '97*, Prague, pp. 111-121, (1997).

Maturo A., (1997b), Conditional events, conditional probabilities and hyperstructures, in *Proceedings EUFIT '97*, September 8-11, pp. 1509/1513, Aachen, Germany.

Maturo A., (1997c), On some hyperstructures of conditional events, *Proc. of 6th AHA Congress, Prague*, 1996, Democritus Un. Press, 115-132

Maturo A. (2000a), Fuzzy events and their probability assessments, *Journal of Discrete Mathematical Sciences & Cryptography*, Vol. 3, Nos 1-3, 83-94.

Maturo A. (2000b), On a non-standard algebraic hyperstructure and its application to the coherent probability assessments, IJPAM, N 7(2000) 33-50.

Maturo A., (2001a), Hypergroups and Generalized probabilities, in *Advances in Generalized Structures, Approssimate Reasoning and Applications*, pp. 13-29, Performantica Press, Iasi, Romania.

Maturo A., (2001b), Finitely additive conditional probability with values on a hyperstructure, *Journal of Information&Optimization Sciences*, Vol22 (2001), 227-240

Maturo A., (2003a), Cooperative games, Finite Geometries and Hyperstructures, *Ratio Matematica* 14, 2003, 57-70

Maturo A. (2003b), Sull'assiomatica di Bruno de Finetti per la previsione e la probabilità coerenti: analisi critica e nuove prospettive, *Periodico di Matematiche*, Serie VIII, Vol 3, n. 1, 41-54.

Maturo A. (2003c), Probability, utility and hyperstructures, *8th AHA, Samothraki, 2002*, Spanidis Press, Xanthi, 203-214.

Maturo A., (2006), A Geometrical Approach to the Coherent Conditional Probability and its extensions, *Scientific Annals of the University Ion Ionescu de la Brad*, Tom XLIX, Vol 2., 2006, 243-255.

Maturo A., (2008a), Join coherent previsions, *Set-valued Mathematics and Applications*, Vol 1 No 2 (2008), 135-144.

Maturo, A., (2008b), La moderna visione interdisciplinare di Geometria, Logica e Probabilità in Bruno de Finetti, *Ratio Sociologica*, Vol 1, No2, 39-62.

Maturo, A., (2009). Coherent conditional previsions and geometric hypergroupoids, *Fuzzy sets, rough sets and multivalued operations and applications*, 1, N.1, 51–62.

Maturo A., Squillante M., Ventre A.G.S., (2010), Decision making, fuzzy measures an hyperstructures, *Advances and Applications in Statistical Sciences*, Vol 2, Issue 2, 233-253.

Maturo A., Tofan I., (2001), *Iperstrutture, strutture fuzzy ed applicazioni*, dierre edizioni, San Salvo.

Maturo F., Hoskova-Mayerova S., (2016), Fuzzy Regression Models and Alternative Operations for Economic and Social Sciences. In: "*Recent Trends in Social Systems: Quantitative Theories and Quantitative Models*", series: "Studies in Systems, Decision and Control". Springer International Publishing (Verlag). (2016), 235–248.

Nikolaidou P., Vougiouklis T., (2012), H_v-structures and the Bar in questionnaires, *Italian J. Pure and Applied Math*. N.29, 2012, 341-350.

Prenowitz W., Jantosciak J., (1979), *Join Geometries*, Springer-Verlag UTM, New York, (1979).

Reichenbach H., (1942), *Philosophic foundation of quantum mechanics*, General Publish Compaby, Toronto.

Ross T. J., (1997), *Fuzzy Logic with engineering applications*, MacGraw Hill.

Spartalis S., Vougiouklis T., (1994), The fundamental relations on Hv-rings, *Rivista Mat. Pura ed Appl.*, N.14 (1994), 7-20.

Vougiouklis T., (1991), The fundamental relation in hyperrings. The general hyperfield, *4th AHA, Xanthi, Greece* (1990), World Scientific (1991), 203-212

Vougiouklis T., (1993), Representations of H_v-structures, *Proc. of the Int.Conf. on Group Theory* 1992, Timisoara (1993), 159 -184**.**

Vougiouklis T., (1994a), H_v-modulus with external P-hyperoperations, *5th AHA, Iasi*, (1993), Hadronic Press (1994), 191-197, by T.Vougiouklis, A. Dramalidis

Vougiouklis T., (1994b), H_v-vector spaces, *5th Int. Congress on A.H.A., Iasi*, (1993), HadronicPress (1994), 181-190.

Vougiouklis T., (1994c), *Hyperstructures and their representations*, Hadronic Press, U.S.A.

Vougiouklis T., (1996a), Construction of H_v-structures with desired fundamental structures, *Pr. Int. Workshop Monteroduni: New Frontiers in Hyperstructures and Related Algebras*, Hadronic Press (1996),177-188.

Vougiouklis T., (1996b), H_v-groups defined on the same set, *Discrete Mathematics*, Elsevier, V.155, N 1-3, (1996), 259-265.

Vougiouklis T., (1996c), On H_v-fields, *Proc. of 6th AHA Congress*, Prague, 1996, Democritus Un. Press (1997) 151-159

Vougiouklis T., Spartalis S., Kessoglides M., (1997), Weak hyperstructures on small sets, *Ratio Matematica* N.12(1997), 90-96.

Vougiouklis T., (1997), Convolutions on WASS hyperstructures, *Discrete Mathematics*, Elsevier, V.174, (1997), 347-355

Vougiouklis T., (1999a), On H_v-rings and H_v-representations, *Discrete Mathematics*, Elsevier, 208/209 (1999), 615-620

Vougiouklis T., (1999b), Fundamental Relations in hyperstructures, *Bulletin of the Greek Mathematical Society*, V.42 (1999), 113-118

Vougiouklis T., (2003a), Special elements in H_v-structures, *Bulletin of the Greek Math. Society* V.48 (2003), 103-112.

Vougiouklis T., (2003b), Finite H_v-structures and their representations, *Rendiconti Seminario Matematico di Messina* Serie II, T.XXV, V.9(2003), 245-265.

Vougiouklis T. (2008), H_v-structures: Birth and … childhood, *J. Basic Science* 4, N.1 (2008),119-133

Vougiouklis T. (2009), Hyperstructures as models in social sciences, *Ratio Sociologica*, V.2, N.2, 2009,21-34.

Vougiouklis T. (2011), Bar and Theta Hyperoperations, *Ratio Mathematica*, 21, 2011, 27-42.

Vougiouklis T. (2014), From H_v-rings to H_v-fields, *Int. J. Algebraic Hyperstructures Appl*. Vol.1, No.1, 2014, 1-13.