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Helix-Hopes on S-Helix Matrices

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Abstract

A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. The main characteristic of the helix-hyperoperation is that all entries of the matrices are used. Such operations cannot be defined in the classical theory. Several classes of non-square matrices have results of the helix-product with small cardinality. We study the helix-hyperstructures on the representations and we extend our study up to H_v-Lie theory by using ordinary fields. We introduce and study the class of S-helix matrices.

Keywords: hyperstructures; H_v-structures; h/v-structures; hope; helix-hopes.
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1 Introduction

Our object is the largest class of hyperstructures, the H_v-structures, introduced in 1990 [10], satisfying the *weak axioms* where the non-empty intersection replaces the equality.

Definition 1.1. *In a set H equipped with a* ***hyperoperation*** *(abbreviate by* ***hope****)*

$$\cdot : H \times H \to P(H) - \{\emptyset\} : (x, y) \to x \cdot y \subset H$$

we abbreviate by
WASS the weak associativity: $(xy)z \cap x(yz) \neq \varnothing,\ \forall x, y, z \in H$ *and by*
COW the weak commutativity: $xy \cap yx \neq \varnothing,\ \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ *is called* H_v*-semigroup if it is WASS and is called* $\mathbf{H_v}-$**group** *if it is reproductive* H_v*-semigroup:* $xH = Hx = H, \forall x \in H$.

$(R, +, \cdot)$ *is called* $\mathbf{H_v}-$**ring** *if* $(+)$ *and* $(\cdot)$ *are WASS, the reproduction axiom is valid for* $(+)$ *and* $(\cdot)$ *is weak distributive with respect to* $(+)$*:*

$$x(y + z) \cap (xy + xz) \neq \varnothing,\ (x + y)z \cap (xz + yz) \neq \varnothing,\ \forall x, y, z \in R.$$

For more definitions, results and applications on H_v-structures, see [1], [2], [11], [12], [13], [17]. An interesting class is the following [8]: An H_v-structure is *very thin*, if and only if, all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, with respect to any other hopes, are singletons.

The fundamental relations $\beta*$ and $\gamma*$ are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [8], [9], [11], [12], [13], [17]. The main theorem is the following:

Theorem 1.1. *Let* $(H, \cdot)$ *be an* H_v*-group and let us denote by U the set of all finite products of elements of H. We define the relation* β *in H as follows:* $x\beta y$ *iff* $\{x, y\} \subset u$ *where* $u \in U$*. Then the fundamental relation* $\beta*$ *is the transitive closure of the relation* β*.*

An element is called *single* if its fundamental class is a *singleton*.

Motivation: The quotient of a group with respect to any partition is an H_v-group.

Definition 1.2. *Let* $(H, \cdot), (H, \otimes)$ *be* H_v*-semigroups defined on the same H.* $(\cdot)$ *is smaller than* $(\otimes)$*, and* $(\otimes)$ *greater than* $(\cdot)$*, iff there exists automorphism*

$$f \in Aut(H, \otimes) \text{ such that } xy \subset f(x \otimes y),\ \forall x, y \in H.$$

Then $(H, \otimes)$ *contains* $(H, \cdot)$ *and write* $\cdot \leq \otimes$. *If* $(H, \cdot)$ *is structure, then it is basic and* $(H, \otimes)$ *is an* H_b*-structure.*

The Little Theorem [11]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.3. *[10] The* H_v*-ring* $(R, +, \cdot)$ *is called* $\mathbf{H_v}$***-field*** *if the quotient* R/γ^* *is a field.*

This definition introduces a new class of which is the following [15]:

Definition 1.4. *The* H_v*-semigroup* $(H, \cdot)$ *is called* ***h/v-group*** *if* H/β^* *is a group.*

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The **h/v-fields** and the other related hyperstructures are defined in a similar way.

An H_v-group is called *cyclic* [8], if there is an element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator.

Definition 1.5. *[11], [14], [18]. Let* $(R, +, \cdot)$ *be an* H_v*-ring,* $(M, +)$ *be COW* H_v*-group and there exists an external hope* $\cdot : R \times M \to P(M) : (a, x) \to ax$, *such that,* $\forall a, b \in R$ *and* $\forall x, y \in M$ *we have*

$$a(x + y) \cap (ax + ay) \neq \varnothing, \ (a + b)x \cap (ax + bx) \neq \varnothing, \ (ab)x \cap a(bx) \neq \varnothing,$$

then M is called an $\mathbf{H_v}$***-module*** *over R. In the case of an* H_v*-field F instead of an* H_v*-ring R, then the* $\mathbf{H_v}$***-vector space*** *is defined.*

Definition 1.6. *[16] Let* $(L, +)$ *be* H_v*-vector space on* $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$, *the canonical map and* $\omega_F = \{x \in F : \phi(x) = 0\}$, *where 0 is the zero of the fundamental field* F/γ^*. *Similarly, let* ω_L *be the core of the canonical map* $\phi' : L \to L/\epsilon^*$ *and denote again 0 the zero of* L/ϵ^*. *Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then L is an $\mathbf{H_v}$***-Lie algebra*** *over F if the following axioms are satisfied:*

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing$,
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing, \ \ \forall x \in L$

(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \varnothing, \ \ \forall x,y,z \in L$

A well known and large class of hopes is given as follows [8], [9], [11]:

Definition 1.7. *Let $(G,\cdot)$ be a groupoid, then for every subset $P \subset G, P \neq \varnothing$, we define the following hopes, called* ***P-hopes****: $\forall x,y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$
$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$, $(G,\underline{P}_l)$ are called ***P-hyperstructures****.*

The usual case is for semigroup $(G,\cdot)$, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$, and $(G,\underline{P})$ is a semihypergroup.

A new important application of H_v-structures in Nuclear Physics is in the Santilli's isotheory. In this theory a generalization of P-hopes is used, [4], [5], [22], which is defined as follows: Let (G,) be an abelian group and P a subset of G with more than one elements. We define the hyperoperation $\times_P$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e*-hope*. The hyperstructure $(G,\times_p)$ is an abelian H_v-group.

2 Small hypernumbers and H_v-matrix representations

Several constructions of H_v-fields are uses in representation theory and applications in applied sciences. We present some of them in the finite small case [18].

Construction 2.1. *On the ring $(\mathbf{Z}_4,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,*

(a) very thin minimal,

(b) COW (non-commutative),

(c) they have the elements 0 and 1, scalars.

Then, we have only the following isomorphic cases $2 \otimes 3 = \{0, 2\}$ *or* $3 \otimes 2 = \{0, 2\}$.

Fundamental classes: $[0] = \{0, 2\}, [1] = \{1, 3\}$ *and we have* $(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

Thus it is isomorphic to $(\mathbf{Z}_2 \times \mathbf{Z}_2, +)$. *In this* H_v*-group there is only one unit and every element has a unique double inverse.*

Construction 2.2. *On the ring* $(\mathbf{Z}_6, +, \cdot)$ *we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are:*

(a) very thin minimal, i.e. only one product has exactly two elements

(b) COW (non-commutative)

(c) they have the elements 0 and 1, scalars

Then we have the following cases, by giving the only one hyperproduct,

(I) $2 \otimes 3 = \{0, 3\}$ *or* $2 \otimes 4 = \{2, 5\}$ *or* $2 \otimes 5 = \{1, 4\}$
$3 \otimes 4 = \{0, 3\}$ *or* $3 \otimes 5 = \{0, 3\}$ *or* $4 \otimes 5 = \{2, 5\}$
In all 6 cases the fundamental classes are $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ *and we have* $(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot)$.

(II) $2 \otimes 3 = \{0, 2\}$ *or* $2 \otimes 3 = \{0, 4\}$ *or* $2 \otimes 4 = \{0, 2\}$ *or* $2 \otimes 4 = \{2, 4\}$ *or*
$2 \otimes 5 = \{0, 4\}$ *or* $2 \otimes 5 = \{2, 4\}$ *or* $3 \otimes 4 = \{0, 2\}$ *or* $3 \otimes 4 = \{0, 4\}$ *or*
$3 \otimes 5 = \{1, 3\}$ *or* $3 \otimes 5 = \{3, 5\}$ *or* $4 \otimes 5 = \{0, 2\}$ *or* $4 \otimes 5 = \{2, 4\}$
In all 12 cases the fundamental classes are $[0] = \{0, 2, 4\}$, $[1] = \{1, 3, 5\}$ *and we have* $(\mathbf{Z}_6, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$.

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations or by H_v-matrices [11], [14], [18].

Definition 2.1. $\mathbf{H_v}$***-matrix*** *is a matrix with entries of an* H_v*-ring or* H_v*-field. The hyperproduct of two* H_v*-matrices* (a_{ij}) *and* (b_{ij})*, of type* $m \times n$ *and* $n \times r$ *respectively, is defined in the usual manner and it is a set of* $m \times r$ H_v*-matrices. The sum of products of elements of the* H_v*-ring is considered to be the n-ary circle hope on the hypersum. The hyperproduct of* H_v*-matrices is not necessarily WASS.*

The problem of the H_v-matrix representations is the following:

Definition 2.2. *Let* $(H, \cdot)$ *be* H_v*-group (or h/v-group). Find an* H_v*-ring* $(R, +, \cdot)$*, a set* $M_R = \{(a_{ij}) | a_{ij} \in R\}$ *and a map* $T : H \to M_R : h \mapsto T(h)$ *such that*

$$T(h_1 h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is H_v-matrix (or h/v-matrix) representation. If $T(h_1h_2) \subset T(h_1)(h_2)$ is called inclusion. If $T(h_1h_2) = T(h_1)(h_2) = \{T(h)|h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then T is good and then an induced representation T for the hypergroup algebra is obtained. If T is one to one and good then it is faithful.*

The main theorem on representations is [11]:

Theorem 2.1. *A necessary condition to have an inclusion representation T of an H_v-group $(H, \cdot)$ by $n \times n$, H_v-matrices over the H_v-ring $(R, +, \cdot)$ is the following:*

For all classes $\beta^*(x), x \in H$ must exist elements $a_{ij} \in H, i, j \in \{1, \ldots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij})|a_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \ldots, n\}\}$$

Inclusion $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces homomorphic representation T^* of H/β^* on R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T^* is called fundamental induced of T.

In representations, several new classes are used:

Definition 2.3. *Let $M = M_{m \times n}$ be the module of $m \times n$ matrices over R and $P = \{P_i : i \in I\} \subseteq M$. We define a P-hope $\underline{P}$ on M as follows*

$$\underline{P} : M \times M \to P(M) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is bilinear map, is strong associative and the inclusion distributive:

$$A\underline{P}(B + C) \subseteq A\underline{P}B + A\underline{P}C, \forall A, B, C \in M$$

Definition 2.4. *Let $M = M_{m\times n}$ the $m \times n$ matrices over R and let us take sets*

$$S = \{s_k : k \in K\} \subseteq R,\ \ Q = \{Q_j : j \in J\} \subseteq M,\ \ P = \{P_i : i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S} : R \times M \to P(M) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq M$$

$$\underline{Q}_+ : M \times M \to P(M) : (A, B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq M$$

$$\underline{P} : M \times M \to P(M) : (A, B) \to A\underline{P}B = \{AP_i^tB : i \in I\} \subseteq M$$

Then $(M, \underline{S}, \underline{Q}_+, \underline{P})$ is hyperalgebra on R called general matrix P-hyperalgebra.

3 Helix-hopes

Recall some definitions from [3], [4], [6], [7], [19], [20], [21]:

Definition 3.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be $m \times n$ matrix and $s, t \in N$ be naturals such that $1 \le s \le m$, $1 \le t \le n$. We define the map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst} = (a_{ij})$ where $1 \le i \le s$, $1 \le j \le t$. We call this map cut-projection of type $\underline{st}$. Thus $A\underline{cst} = (a_{ij})$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.*

We use cut-projections on all types of matrices to define sums and products.

Definition 3.2. *Let $A = (a_{ij}) \in M_{m\times n}$ be an $m \times n$ matrix and $s, t \in N$, such that $1 \le s \le m$, $1 \le t \le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ which has as entries the sets*

$$a_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \le i \le s, 1 \le j \le t \text{ and } \kappa, \lambda \in N, i + \kappa s \le m, j + \lambda t \le n\}.$$

Thus, we have the map

$$\underline{st} : M_{m\times n} \to M_{s\times t} : A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map helix-projection of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}, \forall i, j$. Obviously $A\underline{mn} = A$.

Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Then it is clear that we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-projection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ such that $1 \le s \le m$, $1 \le t \le n$. Then if A$\underline{st}$ is not a set but one single matrix then we call A *cut-helix matrix* of type $s \times t$. In other words the matrix A is a helix matrix of type $s \times t$, if A$\underline{cst}$= A$\underline{st}$.

Definition 3.3. **a.** *Let $A = (a_{ij}) \in M_{m\times n}, B = (b_{ij}) \in M_{u\times v}$, be matrices and s=min(m,u), t=min(n,u). We define a hope, called helix-addition or* ***helix-sum****, as follows:*

$$\oplus : M_{m\times n} \times M_{u\times v} \to P(M_{s\times t}) : (A, B) \to$$

$$A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where

$$(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij} = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$$

b. *Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$, be matrices and s=min(m,u). We define a hope, called helix-multiplication or **helix-product**, as follows:*

$$\otimes : M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}) : (A,B)\to$$
$$A\otimes B = A\underline{ms}\cdot B\underline{sv} = (\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij}) = \{(c_{ij}=(\sum a_{it}b_{tj})|a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$$

The helix-sum is an external hope and the commutativity is valid. For the helix-product we remark that we have $A\otimes B = A\underline{ms}\cdot B\underline{sv}$ so we have either $A\underline{ms}=A$ or $B\underline{sv}=B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the **helix-Lie Algebra** is defined [22], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ϵ^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where $m<n$. We have analogous results if $m>n$ and for $m=n$ we have the classical theory.

Notation. For given $\kappa\in\mathbb{N}-\{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa}=m$ if the remainder is zero. Thus a matrix $A=(a_{\kappa\lambda})\in M_{m\times n}, m<n$ is a cut-helix matrix if we have $a_{\kappa\lambda}=a_{\kappa\underline{\lambda}}, \forall\kappa\lambda\in\mathbb{N}-\{0\}$.

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the **cut-helix unit matrix** which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$.

Proposition 3.1. *For $m<n$ in $(M_{m\times n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.*

Proof. Let $A,B\in M_{m\times n}$ then in the helix-multiplication, since $m<n$, we take helix projection of the matrix A, therefore, the result $A\otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m\times m$. Moreover, in order to have $A\otimes B = A\underline{mm}\cdot B = B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c=(c_\kappa\lambda)$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}, \forall\kappa,\lambda\in\mathbb{N}-\{0\}$, is necessarily the left scalar unit.

Let $A=(a_{uv})\in M_{m\times n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\le\kappa\le m, 1\le\lambda\le n$, of the form

$$\sum \underline{a}_{\kappa s}c_{s\lambda} = \sum \underline{a}_{\kappa s}\delta_{s\underline{\lambda}} = \underline{a}_{\kappa\underline{\lambda}} \ni a_{\kappa\lambda}.$$

Therefore $A \otimes I_c \ni A, \forall A \in M_{m\times n}$. $\square$

4 The S-helix matrices

Definition 4.1. *Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ such that $1 \leq s \leq m$, $1 \leq t \leq n$. Then if $A\underline{st}$ is a set of upper triangular matrices with the same diagonal, then we call A an **S-helix matrix of type** $\mathrm{s} \times \mathrm{t}$. Therefore, in an S-helix matrix A of type $s \times t$, the $A\underline{st}$ has on the diagonal entries which are not sets but elements.*

In the following, we restrict our study on the case of $A = (a_{ij}) \in M_{m\times n}$ with $m < n$.

Remark. According to the cut-helix notation, we have,

$$a_{\kappa\lambda} = a_{\kappa\underline{\lambda}} = 0, \text{ for all } \kappa > \lambda \text{ and } a_{\kappa\lambda} = a_{\kappa\underline{\lambda}}, \text{ for } \kappa = \underline{\lambda}.$$

Proposition 4.1. *The set of S-helix matrices $A = (a_{ij}) \in M_{m\times n}$ with $m < n$, is closed under the helix product. Moreover, it has a unit the cut-helix unit matrix I_c, which is left scalar.*

Proof. It is clear that the helix product of two S-helix matrices, $X = (x_{ij}), Y = (a_{ij}) \in M_{m\times n}, X \otimes Y$, contain matrices $Z = (z_{ij})$, which are upper diagonals. Moreover, for every z_{ii}, the entry ii is singleton since it is product of only $z_{(i+km),(i+km)} = z_{ii}$, entries.

The unit is, from Proposition 3.1, the matrix $I_c = I_{m\times n}$, where we have $I_{m\times n} = I_{mm} = I_m$. $\square$

An example of hyper-matrix representation, seven dimensional, with helix-hope is the following:

Example 4.1. *Consider the special case of the matrices of the type 3×5 on the field of real or complex. Then we have*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{11} & x_{15} \\ 0 & x_{22} & x_{23} & 0 & x_{22} \\ 0 & 0 & x_{33} & 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{15}\} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{11} & y_{15} \\ 0 & y_{22} & y_{23} & 0 & y_{22} \\ 0 & 0 & y_{33} & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} & x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} & x_{11}y_{11} & x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22} \\ 0 & x_{22}y_{22} & x_{22}y_{23} + x_{23}y_{33} & 0 & x_{22}y_{22} \\ 0 & 0 & x_{33}y_{33} & 0 & 0 \end{pmatrix}$$

Therefore the helix product is a set with cardinality up to 8.

The unit of this type is $I_c = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

Definition 4.2. *We call a matrix* $A = (a_{ij}) \in M_{m\times n}$ *an* $\mathbf{S_0}$***-helix matrix*** *if it is an S-helix matrix where the condition* $a_{11}a_{22}\dots a_{mm} \neq 0$*, is valid. Therefore, an* S_0*-helix matrix has no zero elements on the diagonal and the set* S_0 *is a subset of the set S of all S-helix matrices. We notice that this set is closed under the helix product not in addition. Therefore it is interesting only when the product is used not the addition.*

Proposition 4.2. *The set of* S_0*-helix matrices* $A = (a_{ij}) \in M_{m\times n}$ *with* $m < n$*, is closed under the helix product, it has a unit the cut-helix unit matrix* I_c*, which is left scalar and* S_0*-helix matrices X have inverses* X^{-1}*, i.e.* $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.

Proof. First it is clear that on the helix product of two S_0-helix matrices, the diagonal has not any zero since there is no zero on each of them. Therefore, the helix product is closed. The entries in the diagonal are inverses in the H_v-field. In the rest entries we have to collect equations from those which correspond to each element of the entry-set. □

Example 4.2. *Consider the special case of the above Example 4.1, of the matrices of the type* 3×5*. Suppose we want to find the inverse matrix* $Y = X^{-1}$*, of the matrix X. Then we have* $I_c \in X \otimes Y \cap Y \otimes X$*. Therefore, we obtain*

$$x_{11}y_{11} = x_{22}y_{22} = x_{33}y_{33} = 1$$

$$x_{11}y_{12} + \{x_{12}, x_{15}\}y_{22} \ni 0, x_{11}y_{13} + \{x_{12}, x_{15}\}y_{23} + x_{13}y_{33} \ni 0,$$
$$x_{11}y_{15} + \{x_{12}, x_{15}\}y_{22} \ni 0, x_{23}y_{22} + x_{33}y_{23} \ni 0,$$

Therefore a solution is

$$y_{11} = \frac{1}{x_{11}}, y_{22} = \frac{1}{x_{22}}, y_{33} = \frac{1}{x_{33}}$$

$$y_{23} = \frac{-x_{23}}{x_{22}x_{33}}, y_{12} = \frac{-x_{12}}{x_{11}x_{22}}, y_{15} = \frac{-x_{15}}{x_{11}x_{22}}, \text{ and}$$
$$y_{13} = \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{12}}{x_{11}x_{22}x_{33}} \text{ or } y_{13} = \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{14}}{x_{11}x_{22}x_{33}}$$

Thus, a left and right inverse matrix of X is

$$X^{-1} = \begin{pmatrix} \frac{1}{x_{11}} & \frac{-x_{12}}{x_{11}x_{22}} & \frac{-x_{13}}{x_{11}x_{33}} + \frac{x_{23}x_{12}}{x_{11}x_{22}x_{33}} & \frac{1}{x_{11}} & \frac{-x_{15}}{x_{11}} \\ 0 & \frac{1}{x_{22}} & \frac{-x_{23}}{x_{22}x_{33}} & 0 & \frac{1}{x_{22}} \\ 0 & 0 & \frac{1}{x_{33}} & 0 & 0 \end{pmatrix}$$

An interesting research field is the finite case on small finite H_v-fields. Important cases appear taking the generating sets by any S_0-helix matrix.

Example 4.3. *On the type* 3×5 *of matrices using the Construction 2.1, on* $(\mathbf{Z}_4, +, \cdot)$ *we take the small* H_v*-field* $(\mathbf{Z}_4, +, \otimes)$*, where only* $2 \otimes 3 = \{0, 2\}$ *and fundamental classes* $\{0, 2\}, \{1, 3\}$*. We consider the set of all* S_0*-helix matrices and we take the* S_0*-helix matrix:*

$$X = \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Then the powers of X are:

$$X^2 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \textit{ and so on}$$

We obtain that the generating set is the following

$$\begin{pmatrix} 1 & \{0,2\} & \{0,2\} & 1 & \{0,2\} \\ 0 & \{1,3\} & \{0,1\} & 0 & \{1,3\} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

where in the 22 and 25 entries appears simultaneously 1 or 3.

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