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Pairwise Paracompactness

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Abstract

The purpose of this paper is to introduce and study a new paracompactness in bitopological spaces using (τ_i, τ_j)-$g^*\omega\alpha$-closed sets. Further, the properties of (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous functions and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps and (τ_i, τ_j)-$g^*\omega\alpha$-paracompact spaces are discussed in bitopological spaces.
Keywords: (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-open sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps, (τ_i, τ_j)-$g^*\omega\alpha$-paracompact spaces.
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1 Introduction

The research in topology over last two decades has reached a high level in many directions. Topological methods are widely used in many other branches of modern mathematics such as differential equation, functional analysis, classical mechanics, general theory of relativity, mathematical economics, quantum theory, biology etc.

Bitopological space is a triplet (X, τ_1, τ_2), where X is a non empty set and τ_1 and τ_2 are topologies on a space X. In 1963, J. C. Kelly [8] initiated the study of bitopological spaces. In 1985, Fututake [5] studied the concept of generalized closed (briefly g-closed) sets in bitopological spaces. After that, several authors turned their attention towards the generalizations of various concepts in topology by considering bitopological spaces.

In this paper, (τ_i, τ_j)-$g^*\omega\alpha$-closed sets, (τ_i, τ_j)-$g^*\omega\alpha$-continuous functions and (τ_i, τ_j)-$g^*\omega\alpha$-irresolute maps are defined and studied in bitopological spaces. Also, the concept of (τ_i, τ_j)-$g^*\omega\alpha$-paracompactness in bitopological spaces is introduced and studied.

2 Preliminaries

Throughout this present paper, let X, Y and Z always represents non-empty bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, γ_1, γ_2) on which no separation axioms are assumed unless explicitly mentioned and the integers $i, j, k \in \{1, 2\}$.

Definition 2.1. *[13] A space X is said to be $g^*\omega\alpha$-paracompact if every open cover of X has a $g^*\omega\alpha$-locally finite $g^*\omega\alpha$-refinement.*

Definition 2.2. *Let $A \subseteq X$. Then A is said to be a*
(a) $\omega\alpha$-closed [2] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in X.
(b) $g^*\omega\alpha$-closed[12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\omega\alpha$-open in X.

Definition 2.3. *A subset A of a bitopological space (X, τ_1, τ_2) is called a*
(a) (τ_i, τ_j)-g-closed [5] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in τ_i.
(b) (τ_i, τ_j)-rg-closed [1] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i.
(c) (τ_i, τ_j)-αg-closed [3] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in τ_i.
(d) (τ_i, τ_j)-$g\alpha$-closed [3] if τ_j-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in τ_i.
(e) (τ_i, τ_j)-gpr-closed [7] if τ_j-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i.
(f) (τ_i, τ_j)-g^*-closed [14] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in τ_i.

(g) (τ_i, τ_j)-$\omega\alpha$-closed [11] if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω-open in τ_i.
In all the above definitions $i \neq j$.

Definition 2.4. *A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is called a*
(a) τ_j-μ_k-continuous [10] if $f^{-1}(G) \in \tau_j$ for every open set G in μ_k.
(b) $D(\tau_i, \tau_j)$-μ_k-continuous [10] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-g-closed in (X, τ_1, τ_2).
(c) $D_r(\tau_i, \tau_j)$-μ_k-continuous [1] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-rg-closed in (X, τ_1, τ_2).
(d) $C(\tau_i, \tau_j)$-μ_k-continuous [6] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-ω-closed in (X, τ_1, τ_2).
(e) $D^*(\tau_i, \tau_j)$-μ_k-continuous [14] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-g^*-closed in (X, τ_1, τ_2).
(f) (τ_i, τ_j)-αg-continuous [4] if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-αg-closed in (X, τ_1, τ_2).

Definition 2.5. *[8] A bitopological space (X, τ_1, τ_2) is said to be pairwise Hausdorff if for each pair of distinct points x and y of X, there exist $U \in P_i$ and $V \in P_j$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.*

3 (τ_i, τ_j)-$g^*\omega\alpha$-Closed Sets

This section deals with the concept of $g^*\omega\alpha$-closed sets in bitopological spaces and some of their properties.

Definition 3.1. *Let $(i, j) \in \{1, 2\}$ where $i \neq j$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-$g^*\omega\alpha$-closed if τ_j-$cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$-$\omega\alpha$-open in X.*

Example 3.1. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Consider a set in the space (X, τ_1, τ_2), $A = \{n, p\}$ which is (τ_1, τ_2)-$g^*\omega\alpha$-closed.*

Remark 3.1. *If $\tau_1 = \tau_2 = \tau$ in Definition 3.1, then (τ_i, τ_j)-$g^*\omega\alpha$-closed set in (X, τ_1, τ_2) is same as $g^*\omega\alpha$-closed [12] in (X, τ).*

The family of all (τ_i, τ_j)-$g^*\omega\alpha$-closed sets in (X, τ_1, τ_2) is denoted by $P(\tau_i, \tau_j)$.

Theorem 3.1. *Every τ_j-closed (resp. (τ_i, τ_j)-regular closed) is (τ_i, τ_j)-$g^*\omega\alpha$-closed.*
However the converse need not be true in general as shown in the following example.

Example 3.2. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Consider the set, $A = \{m, p\}$ is (τ_1, τ_2)-$g^*\omega\alpha$-closed but not τ_2-closed (resp. (τ_i, τ_j)-regular closed).*

We have the following implification:
τ_j-closed $\rightarrow$ (τ_i, τ_j)-$g^*\omega\alpha$-closed $\rightarrow$ (τ_i, τ_j)-αg-closed

Remark 3.2. *If A and B are (τ_i, τ_j)-$g^*\omega\alpha$-closed in (X, τ_1, τ_2) then $A \cup B$ is also (τ_i, τ_j)-$g^*\omega\alpha$-closed.*

Theorem 3.2. *If a subset A of (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-closed then τ_j-$cl(A) - A$ does not contain any non empty $\omega\alpha$-closed set in τ_i.*

Proof. Let $A \subseteq (\tau_i, \tau_j)$-$g^*\omega\alpha$-closed and $F \subseteq \tau_i$-$\omega\alpha$-closed set such that $F \subseteq \tau_j$-$cl(A) - A$. Now $F \subseteq \tau_j$-$cl(A)$ and $F \subseteq X - A$. Then $A \subseteq X - F$ and by hypothesis A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $X - F$ is $\tau_i - \omega\alpha$-open. Thus from Definition 3.1, τ_j-$cl(A) \subseteq X - F$, that is $F \subseteq (X - \tau_j$-$cl(A))$. Then $F \subseteq (\tau_j$-$cl(A)) \cap (X - \tau_j$-$cl(A)) = \phi$ and so $F = \phi$ which is a contradiction. Hence τ_j-$cl(A) - A$ does not contain any non empty $\omega\alpha$-closed set. □

Remark 3.3. *A (τ_i, τ_j)-$g^*\omega\alpha$-closed set need not be τ_i-$g^*\omega\alpha$-closed or τ_j-$g^*\omega\alpha$-closed.*

Example 3.3. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}\}$ and $\tau_2 = \{X, \phi\}$. Then the set $A = \{n, p\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed but not τ_2-$g^*\omega\alpha$-closed.*
Also, if $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$ be topology on X. Then the set $A = \{m, p\}$ is(τ_i, τ_j)-$g^*\omega\alpha$-closed but not τ_1-$g^*\omega\alpha$-closed in (X, τ_1, τ_2).

Remark 3.4. *In general $P(\tau_i, \tau_j) \neq P(\tau_j, \tau_i)$.*

Example 3.4. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. Then $g^*\omega\alpha C(\tau_1, \tau_2) = \{X, \phi, \{m, n\}\}$ and $g^*\omega\alpha C(\tau_2, \tau_1) = \{X, \phi, \{n, p\}, \{p\}\}$. Hence we can observe that $g^*\omega\alpha C(\tau_1, \tau_2) \neq g^*\omega\alpha C(\tau_2, \tau_1)$.*

Remark 3.5. *If $\tau_1 \subseteq \tau_2$, then $P(\tau_2, \tau_1) \subseteq P(\tau_1, \tau_2)$ but converse is not true.*

Example 3.5. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{p\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. Then $P(\tau_2, \tau_1) = \{X\}, \phi, \{m, n\}, \{n, p\}\}$ and $P(\tau_1, \tau_2) = \{X, \phi, \{m, n\}, \{n, p\}, \{p\}\}$. Then $P(\tau_2, \tau_1) \subseteq G(\tau_1, \tau_2)$ but $\tau_1 \nsubseteq \tau_2$.*

Theorem 3.3. *A τ_i-$\omega\alpha$-open and (τ_i, τ_j)-$g^*\omega\alpha$-closed set is τ_j-closed.*

Proof. Now $A \subseteq A$. Then τ_j-$cl(A) \subseteq A$ and $A \subseteq \tau_j$-$cl(A)$. Therefore τ_j-$cl(A) = A$ and hence $A \in \tau_j$-closed. □

Theorem 3.4. *Let A be τ_i-$\omega\alpha$-open and (τ_i, τ_j)-$g^*\omega\alpha$-closed. Suppose F is τ_j-closed, then $A \cap F$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed.*

Proof. Let A be τ_i-$\omega\alpha$-open and A be (τ_i, τ_j)-$g^*\omega\alpha$-closed and F be τ_j-closed. Then from Theorem 3.3, A is τ_j-closed. So $A \cap F$ is τ_j-closed and hence (τ_i, τ_j)-$g^*\omega\alpha$-closed. □

Theorem 3.5. *If A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$, then τ_j-$cl(B)-B$ contains no non empty τ_i-$closed$ set.*

Proof. Let A be (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$. Then B is (τ_i, τ_j)-$g^*\omega\alpha$-closed follows from Theorem 3.19 [12]. Hence τ_j-$cl(B) - B$ contains no non empty τ_i-$closed$ set. □

Corolary 3.1. *If A is (τ_i, τ_j)-$g^*\omega\alpha$-closed and $A \subseteq B \subseteq \tau_j$-$cl(A)$, then τ_j-$cl(B)-B$ contains no non empty τ_i-$\omega\alpha$-closed set.*

Theorem 3.6. *Arbitrary union of (τ_i, τ_j)-$g^*\omega\alpha$-closed sets $\{A_i : i \in I\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed if the family $\{A_i : i \in I\}$ is τ_j-locally finite.*

Proof. Let $\{A_i : i \in I\}$ is τ_j-locally finite and $\{A_i : i \in I\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. Let $\cup A_i \subseteq U$ where $U \in \tau_i$-$\omega\alpha$-open. Then $A_i \subseteq U$ and $\omega\alpha$-open in τ_i. Since A is (τ_i, τ_j)-$g^*\omega\alpha$-closed the for each $i \in I$, τ_j-$cl(A_i) \subseteq U$. Consequently $\cup\tau_j$-$cl(A_i) \subseteq U$. Since the family, $\{A_i : i \in I\}$ is τ_j-locally finite τ_j-$cl(\cup A_i) = \cup(\tau_j$-$cl(A_i)) \subseteq U$. Therefore $\cup A_i$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. □

Theorem 3.7. *For an element x in X, the set $X - \{x\}$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed or $X - \{x\}$ is $\omega\alpha$-open in τ_i.*

Proof. Suppose $X - \{x\}$ is not $\omega\alpha$-open in τ_i, then X is the only $\omega\alpha$-open set containing $X - \{x\}$, that is τ_j-$cl(X - \{x\}) \subseteq \tau_j$-$cl(\{x\}) = X$. Hence τ_j-$cl(X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is $\tau_i, \tau_j)$-$g^*\omega\alpha$-closed. □

Definition 3.2. *A subset A of a bitopological space (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-open if its complement is (τ_i, τ_j)-$g^*\omega\alpha$-closed.*

Definition 3.3. *For a subset A of a bitopological space (X, τ_1, τ_2), (τ_i, τ_j)-$g^*\omega\alpha$-interior of A is denoted by (τ_i, τ_j)-$g^*\omega\alpha$-int(A) and is defined as*
(τ_i, τ_j)-$g^*\omega\alpha$-$int(A) = \cup\{F : F \in (\tau_i, \tau_j)$-$g^*\omega\alpha$-open and $F \subseteq A\}$.

Theorem 3.8. *Let A be (τ_i, τ_j)-$g^*\omega\alpha$-open. Then $P = X$ whenever G is τ_i-$\omega\alpha$-open and τ_j-$g^*\omega\alpha$-$int(A) \cup A^c \subseteq G$.*

Theorem 3.9. *A set A is (τ_i, τ_j)-$g^*\omega\alpha$-open if and only if $F \subseteq \tau_j$-$int(A)$ whenever F is τ_i-closed and $F \subseteq A$.*

Theorem 3.10. *If A and B are separated (τ_i, τ_j)-$g^*\omega\alpha$-open sets then $A \cup B$ is also (τ_i, τ_j)-$g^*\omega\alpha$-open.*

Proof. Suppose A and B are (τ_i, τ_j)-$g^*\omega\alpha$-open sets. Let F be an τ_i-closed set such that $F \subseteq A \cup B$. Since A and B are separated, τ_i-cl(A) $\cap B = A \cap \tau_i$-cl(B) = ϕ and τ_j-$cl(A) \cap B = A \cap \tau_j$-$cl(B) = \phi$. Then $F \cap \tau_j$-$cl(A) \subseteq (A \cup B) \cap \tau_j$-$cl(A) = A$. Similarly, $F \cap \tau_j$-$cl(B) \subseteq B$. Since F is τ_i-closed, we have $F \cap \tau_i$-$cl(A), F \cap \tau_i$-$cl(B)$ are also τ_i-closed and from hypothesis A and B are (τ_i, τ_j)-$g^*\omega\alpha$-open sets, $F \cap \tau_j$-$cl(A) \subseteq \tau_j$-$int(A)$ and $F \cap \tau_j$-$cl(B) \subseteq \tau_j$-$int(B)$. Now $F = F \cap (A \cup B) \subseteq (F \cap \tau_j$-$cl(A)) \cup (F \cap \tau_j$-$cl(B)) \subseteq \tau_j$-$int(A \cup B)$. Hence $A \cup B$ is (τ_i, τ_j)-$g^*\omega\alpha$-open. □

Definition 3.4. *A bitopological space (X, τ_1, τ_2) is said to be a (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space if every (τ_i, τ_j)-$g^*\omega\alpha$-closed set is τ_j-closed.*

Example 3.6. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Then (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Theorem 3.11. *If a bitopological space (X, τ_1, τ_2) is (τ_i, τ_j)-$T_{g^*\omega\alpha}$ space, then for each $x \in X$, $\{x\}$ is τ_i-$\omega\alpha$-closed or τ_j-open.*

Proof. Suppose $\{x\}$ is not (τ_i, τ_j)-$g^*\omega\alpha$-open, then $\{x\}^c$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed. As X is is (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space, $\{x\}^c$ is τ_j-closed and hence $\{x\}$ is τ_j-open. □

Remark 3.6. *Every singleton subset of (X, τ_1, τ_2) is τ_j-closed or τ_i-$\omega\alpha$-closed but (X, τ_1, τ_2) is not (τ_i, τ_j)-$T_{g^*\omega\alpha}$-space.*

Example 3.7. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Then every singleton set $\{x\}$ of X is either τ_2-open or τ_1-$\omega\alpha$-closed. However, (X, τ_1, τ_2) is not (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Remark 3.7. *If (X, τ_1) and (X, τ_2) are both $T_{g^*\omega\alpha}$-space, then it need not imply (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Example 3.8. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{n\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{m, n\}\}$. Then (X, τ_1) and (X, τ_2) are $T_{g^*\omega\alpha}$-space, but (X, τ_1, τ_2) is not (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Remark 3.8. *The space (X, τ_1) is not generally $T_{g^*\omega\alpha}$-space if (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space.*

Example 3.9. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Then (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{g^*\omega\alpha}$-space, but (X, τ_1) is not $T_{g^*\omega\alpha}$-space.*

4 (τ_i, τ_j)-$g^*\omega\alpha$-Continuous and (τ_i, τ_j)-$g^*\omega\alpha$-Irresolute Maps

Definition 4.1. *A map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is called $P(\tau_i, \tau_j)$-μ_k-continuous (pairwise $g^*\omega\alpha$-continuous) if the inverse image of every μ_k-closed set in (Y, μ_1, μ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-closed in (X, τ_1, τ_2).*

Theorem 4.1. *Every is τ_j-μ_k-continuous function is $P(\tau_i, \tau_j)$-μ_k-continuous.*

Proof. Follows from Theorem 3.1. □
The converse need not be true as seen from the following example.

Example 4.1. *Let $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{n\}\}$ and $\mu_2 = \{Y, \phi, \{m\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be the identity map. Then f is $P(\tau_1, \tau_2)$-μ_1-continuous but not τ_2-μ_1-continuous, since for the μ_1-closed set $A = \{m, p\}$ in Y, $f^{-1}(\{m, p\}) = \{m, p\}$ is not τ_2-closed in X.*

Remark 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be $P(\tau_i, \tau_j)$-μ_k-continuous $g : (Y, \mu_1, \mu_2) \to (Z, \gamma_1, \gamma_2)$ be $P(\mu_1, \mu_2)$-γ_m-continuous but their composition need not be $P(\tau_i, \tau_j)$-γ_m-continuous.*

Example 4.2. *Let $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{m\}, \{n, p\}\}$, $\mu_2 = \{Y, \phi, \{m\}\}$, $\gamma_1 = \{Z, \phi, \{m\}, \{m, p\}\}$ and $\gamma_2 = \{Z, \phi, \{m\}, \{m, n\}, \{m, p\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be identity map and define a map $g : (Y, \mu_1, \mu_2) \to (Z, \gamma_1, \gamma_2)$ by $g(m) = n, g(n) = m, g(p) = p$. Then f and g are pairwise $g^*\omega\alpha$-continuous maps but their composition is not pairwise $g^*\omega\alpha$-continuous, since for the γ_1-closed set $\{n, p\}$ in (Z, γ_1, γ_2), $(gof)^{-1}(\{n, p\}) = f^{-1}(g^{-1}(\{n, p\})) = f^{-1}(\{m, p\}) = \{m, p\}$ is not (τ_1, τ_2)-$g^*\omega\alpha$-closed in (X, τ_1, τ_2).*

Definition 4.2. *A map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is called pairwise $g^*\omega\alpha$-irresolute if for every $A \in P(\mu_k, \mu_e)$ in (Y, μ_1, μ_2), $f^{-1}(A) \in P(\tau_i, \tau_j)$ in (X, τ_1, τ_2).*

Theorem 4.2. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ pairwise $g^*\omega\alpha$-irresolute iff is $P(\tau_i, \tau_j)$-μ_e-continuous.*

Proof. Let F be μ_e-closed, then F is (μ_k, μ_e)-$g^*\omega\alpha$-closed in (Y, μ_1, μ_2). From Theorem 3.1, $F \in P(\mu_k, \mu_e)$. Since f is pairwise $g^*\omega\alpha$-irresolute, $f^{-1}(F) \in P(\tau_i, \tau_j)$. Therefore f is $P(\tau_i, \tau_j)$-μ_e-continuous. □
The converse of this theorem need not be true as seen from the following example.

Example 4.3. *Let* $X = Y = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$, $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$, $\mu_1 = \{Y, \phi, \{n\}, \{n, p\}\}$ *and* $\mu_2 = \{Y, \phi, \{m\}, \{m, p\}\}$. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ *be the identity map. Then f is* $P(\tau_1, \tau_2)$*-*μ_1*-continuous map but not pairwise* $g^*\omega\alpha$*-irresolute map, since for the* (μ_1, μ_2)*-*$g^*\omega\alpha$*-closed set* $\{m, p\}$ *in* (Y, μ_1, μ_2), $f^{-1}(\{m, p\}) = \{m, p\}$ *is not* (τ_i, τ_j)*-*$g^*\omega\alpha$*-closed set in* (X, τ_1, τ_2).

Theorem 4.3. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ *be a map and* (Y, μ_1, μ_2) *be* (μ_k, μ_e)*-*$T_{g^*\omega\alpha}$*-space. Then f is pairwise* $g^*\omega\alpha$*-irresolute if and only if f is* $P(\tau_i, \tau_j)$*-*μ_e*-continuous.*

Proof. Suppose f is pairwise $g^*\omega\alpha$-irresolute. From Theorem 4.2, f is $P(\tau_i, \tau_j)$-μ_e-continuous.
Conversely, let f be $P(\tau_i, \tau_j)$-μ_e-continuous map. Let F be (μ_k, μ_e)-$g^*\omega\alpha$-closed in (Y, μ_1, μ_2). By hypothesis (Y, μ_1, μ_2) is (μ_k, μ_e)-$T_{g^*\omega\alpha}$-space, F is μ_e-closed set in (Y, μ_1, μ_2). Again, since f is $P(\tau_i, \tau_j)$-μ_e-continuous, $f^{-1}(F)$ is (τ_i, τ_j)-$g^*\omega\alpha$-closed set in (X, τ_1, τ_2). Hence f is pairwise $g^*\omega\alpha$-irresolute. □

5 (τ_i, τ_j)-$g^*\omega\alpha$-Paracompact Spaces

We recall that, a collection $\xi = \{F_\lambda : \lambda \in \Gamma\}$ of subsets of a space X is called a locally finite with respect to the topology τ_i, if for each $x \in X$ there exists $U_x \in \tau_i$ containing x and U_x which intersects at most finitely many members of ξ.

Definition 5.1. *A collection* $\xi = \{F_\lambda : \lambda \in \Gamma\}$ *of subsets of a space X is called* (τ_i, τ_j)*-P-locally finite if for each* $x \in X$ *there exist* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open* U_x *in X and* U_x *intersects at most finitely many members of* ξ.

Theorem 5.1. *Let* $\xi = \{F_\lambda : \lambda \in \Gamma\}$ *be a collection of subsets of* (X, τ_1, τ_2) *then*
(a) ξ *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-locally finite if and only if* $\{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda) : \lambda \in \Gamma\}$ *is* (τ_i, τ_j)*-P-locally finite.*
(b) if ξ *is* (τ_i, τ_j)*-P-locally finite, then* $\cup(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda) = (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(\cup F_\lambda)$.
(c) ξ *is locally finite with respect to the topology* τ_i *if and only if the collection* $\{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_\lambda : \lambda \in \Gamma)\}$ *is locally finite with respect to the topology* τ_i.

Proof. (a) Suppose ξ is (τ_i, τ_j)-P-locally finite. Then for each $x \in X$, there exists (τ_i, τ_j)-$g^*\omega\alpha$-open set U_x containing x, which meets only finitely many of the sets F_λ, say $F_{\lambda_1}, F_{\lambda_2}, ..., F_{\lambda_n}$. Since $F_{\lambda_K} \subseteq (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_{\lambda_K})$ for each k = 1,2,...,n and U_x meets (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_1}), ..., (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(F_{\lambda_n})$. Therefore $g^*\omega\alpha\text{-}cl(F_\lambda)$ where $\lambda \in \gamma$ is (τ_i, τ_j)-P-locally finite.
Conversely, let $x \in X$. Then there exists (τ_i, τ_j)-$g^*\omega\alpha$-open U_x, which meets only finitely many of the sets (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_n})$, say (τ_i, τ_j)-$g^*\omega\alpha\text{-}cl(F_{\lambda_1})$, ...,

$(\tau_i,\tau_j)g^*\omega\alpha$-$cl(F_{\lambda_n})$. Then $U_x \cap (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k}) \neq \phi$. Let $q \in U_x$ and $q \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$, implies that for every (τ_i,τ_j)-$g^*\omega\alpha$-open set V_q, we have $V_q \cap F_{\lambda_k} \neq \phi$. But, we have U_x is (τ_i,τ_j)-$g^*\omega\alpha$-open set containing q and so $U_x \cap F_{\lambda_k} \neq \phi$ for each k=1,2,...,n. Thus ξ is (τ_i,τ_j)-P locally finite.
(b) Suppose ξ is (τ_i,τ_j)-P-locally finite, then $\cup(\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda) \subseteq (\tau_i,\tau_j)g^*\omega\alpha$-$cl(\cup F_\lambda)$. On the other hand, let $q \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(\cup F_\lambda)$. Then for every (τ_i,τ_j)-$g^*\omega\alpha$-open set V_q such that $V_q \cap (\cup F_\lambda) \neq \phi$. But from the hypothesis, there exists (τ_i,τ_j)-$g^*\omega\alpha$-open set U_q such that U_q meets only finitely many of the sets F_λ, say $F_{\lambda_1}, F_{\lambda_2}, ..., F_{\lambda_n}$. Thus for each (τ_i,τ_j)-$g^*\omega\alpha$-open set V_q containing q, we have $V_q \cap (\cup F_{\lambda_k}) \neq \phi$ where k=1,2,...,n. That is, for each $q \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(\cup F_{\lambda_k})$, there exists h such that $q \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_h})$. Therefore $q \in \cup(\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda)$ and hence (τ_i,τ_j)-$g^*\omega\alpha$-$cl(\cup F_\lambda) = \cup(\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_\lambda)$.
(c) Suppose ξ is locally finite with respect to the topology τ_i, then for each $x \in X$ there exists τ_i-open set U_x which meets only finitely many set F_λ, say $F_{\lambda 1}, F_{\lambda 2}, ...$ $, F_{\lambda_n}$, but $F_{\lambda k} \subseteq (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$. Then U_x meets (τ_i,τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_1}), ...,$ (τ_i,τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_n})$. Thus (τ_i,τ_j)-$g^*\omega\alpha$-$cl(F_\lambda : \lambda \in \Gamma)$ is locally finite with respect to the topology τ_i.
Conversely, let $x \in X$. Then there exists τ_1-open set U_x which meets only finitely many of the sets (τ_i,τ_j)-$g^*\omega\alpha$-$cl(F_\lambda)$, that is (τ_i,τ_j)-$g^*\omega\alpha$-$cl(F_{\lambda_1}), ..., (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_n})$. Let $q \in U_x$ and $q \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-$cl(F_{\lambda_k})$ where k=1,2,...,n. Then for each (τ_i,τ_j)-$g^*\omega\alpha$-open set V_q containing q such that $V_q \cap F_{\lambda_k} \neq \phi$. But $q \in U_x$ and so U_x meets only finitely many of the sets F_λ. Hence ξ is locally finite with respect to the topology τ_i. □

Lemma 5.1. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a function. Then f is (τ_i,τ_j)-$g^*\omega\alpha$-closed if and only if for every $y \in Y$ and $U \in \tau_1 O(X)$ which contains $f^{-1}(y)$ there exists $V \in (\tau_i,\tau_j)$-$g^*\omega\alpha$-open set in (Y,σ_1,σ_2) such that $y \in Y$ and $f^{-1}(V) \subseteq U$.*

Theorem 5.2. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be (τ_i,τ_j)-$g^*\omega\alpha$-irresolute. If $\xi = \{F_\lambda : \lambda \in \Gamma\}$ be a (τ_i,τ_j)-P-locally finite collection in Y, then $f^{-1}(\xi) = \{f^{-1}(F_\lambda) : \lambda \in \Gamma\}$ is (τ_i,τ_j)-P locally finite collection in X.*

Theorem 5.3. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be (τ_i,τ_j)-$g^*\omega\alpha$-continuous. If $\xi = \{F_\lambda : \lambda \in \Gamma\}$ is (τ_i,τ_j)-P locally finite collection in Y, then $f^{-1}(\xi) = \{f^{-1}(F_\lambda) : \lambda \in \Gamma\}$ is locally finite collection with respect to the topology τ_i.*

Definition 5.2. *A non empty collection $\xi = \{A_i, i \in I$, an index set$\}$ is called a (τ_i,τ_j)-$g^*\omega\alpha$-open cover of a bitopological space (X,τ_1,τ_2) if $X = \cup A_i$ and $\xi \subseteq \tau_1$-$g^*\omega\alpha O(X,\tau_1,\tau_2) \cup \tau_2$-$g^*\omega\alpha O(X,\tau_1,\tau_2)$ and ξ contains at least one member of τ_1-$g^*\omega\alpha O(X,\tau_1,\tau_2)$ and one member of τ_2-$g^*\omega\alpha O(X,\tau_1,\tau_2)$.*

Definition 5.3. *A subset A of a bitopological space (X,τ_1,τ_2) is said to be (τ_i,τ_j)-$g^*\omega\alpha$-compact if every cover of A by (τ_i,τ_j)-$g^*\omega\alpha$-open sets has a finite subcover.*

Example 5.1. *Let* $X = \{m, n, p, q\}$*,* $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ *and* $\tau_2 = \{X, \phi, \{m, n\}, \{m, n, p\}, \{m, p, q\}\}$*. Let* $\xi = \{\{m\}, \{m, n\}, \{m, n, p\}, \{m, p, q\}\}$ *be a* $g^*\omega\alpha$*-open cover of* (X, τ_1, τ_2)*. Then* (X, τ_1, τ_2) *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact.*

Definition 5.4. *A set A of a bitopological space* (X, τ_1, τ_2) *is said to be* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact relative to X if every* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open cover of A has a finite subcover as a subspace.*

Theorem 5.4. *Every* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact space is* (τ_i, τ_j) *compact.*
Proof: *Let* (X, τ_1, τ_2) *be* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i : i \in I\}$ *be* (τ_i, τ_j) *open cover of X. Then* $X = \cup A_i$ *and* $\xi \subseteq \tau_i \cup \tau_j$*, so* ξ *contains at least one member of* τ_i *and one member of* τ_j*. Since, every* τ_i*-open set is* τ_i*-*$g^*\omega\alpha$*-open, we have* $X = \cup A_i$ *and* $\xi \subseteq \tau_i$*-*$g^*\omega\alpha O(X) \cup \tau_j$*-*$g^*\omega\alpha O(X)$ *and by definition* ξ *contains at least one member of* τ_i*-*$g^*\omega\alpha O(X, \tau_1, \tau_2)$ *and one member of* τ_j*-*$g^*\omega\alpha O(X, \tau_1, \tau_2)$*. Therefore* ξ *is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open cover of X. As X is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact then* ξ *has the finite subcover and hence X is* (τ_i, τ_j) *compact.*

Theorem 5.5. *If Y is* τ_i*-*$g^*\omega\alpha$ *closed subset of a* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact space* (X, τ_1, τ_2) *then Y is* τ_j*-*$g^*\omega\alpha$ *compact.*
Proof: *Let* (X, τ_1, τ_2) *be* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i : i \in I\}$ *be a* τ_j*-*$g^*\omega\alpha$ *open cover of Y. As Y is* τ_i*-*$g^*\omega\alpha$ *closed,* Y^c *is* τ_i*-*$g^*\omega\alpha$ *open. Also,* $\xi \cup Y^c = Y^c \cup \{A_i : i \in I\}$ *be a* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open cover of X. Since X is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact, we have* $X = Y^c \cup A_1 \cup ... \cup A_n$*, so* $Y = A_1 \cup .... \cup A_n$*. Hence, Y is* τ_j*-*$g^*\omega\alpha$ *compact.*

Theorem 5.6. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* (τ_i, τ_j) *continuous, bijective and* (τ_i, τ_j)*-*$g^*\omega\alpha$*-irresolute. Then the image of a* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact space under f is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact.*
Proof: *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a* (τ_i, τ_j) *continuous surjective and* (τ_i, τ_j)*-*$g^*\omega\alpha$*-closed. Let X be* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact. Let* $\xi = \{A_i, i \in I\}$ *be a* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open cover of Y. Then* $Y = \cup A_i$ *and* $\xi \subseteq \sigma_1$*-*$g^*\omega\alpha O(Y) \cup \sigma_2$*-*$g^*\omega\alpha O(Y)$ *and* ξ *contains at least one member of* σ_1*-*$g^*\omega\alpha O(Y)$ *and one member of* σ_2*-*$g^*\omega\alpha O(Y)$*. Therefore* $X = f^{-1}(\cup(A_i)) = \cup f^{-1}(A_i)$ *and* $f^{-1}(\xi) \subseteq \tau_1$*-*$g^*\omega\alpha O(X) \cup \tau_2$*-*$g^*\omega\alpha O(X)$ *and* $f^{-1}(\xi)$ *contains at least one member of* τ_1*-*$g^*\omega\alpha O(X)$ *and one member of* τ_2*-*$g^*\omega\alpha O(X)$*. Therefore* $f^{-1}(\xi)$ *is the* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open cover of X. Since X is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact, we have* $X = \cup f^{-1}(A_i)$ *for each i = 1,...,n, that is* $Y = f(X) = \cup(A_i)$*, i=1,...,n. Hence,* ξ *has the finite subcover. Therefore Y is* (τ_i, τ_j)*-*$g^*\omega\alpha$*-compact.*

Definition 5.5. *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_i, τ_j)*-*$g^*\omega\alpha$*-paracompact (pairwise* $g^*\omega\alpha$ *paracompact) if every* τ_i*-open cover of X has a* (τ_i, τ_j)*-P-locally finite* (τ_i, τ_j)*-*$g^*\omega\alpha$*-open refinement.*

Example 5.2. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Let $\xi = \{\{m\}, \{m, n\}\}$. Then the space (X, τ_1, τ_2) is (τ_i, τ_j) $g^*\omega\alpha$-paracompact.*

Definition 5.6. *Let (X, τ_1, τ_2) be a bitopological space. Then X is said to be:*
(a) (τ_i, τ_j)-$g^*\omega\alpha$-regular if for each τ_i-closed set F and $x \in X$ there exist τ_i-$g^*\omega\alpha$-open set U and τ_j-$g^*\omega\alpha$-open set V such that $x \in U$ and $F \subseteq V$.
(b) (τ_i, τ_j)-$g^*\omega\alpha$-normal if there exist two disjoint τ_i-closed sets A and B , there exist disjoint (τ_i, τ_j)-$g^*\omega\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Example 5.3. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{m, n\}\}$ and $\tau_2 = \{X, \phi, \{m\}, \{p\}, \{m, p\}\}$. Then (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-regular.*

Example 5.4. *Let $X = \{m, n, p\}$, $\tau_1 = \{X, \phi, \{m\}, \{n, p\}\}$ and $\tau_2 = \{X, \phi, \{m\}\}$. Then (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-normal.*

Proposition 5.1. *A space (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-regular if and only if for each τ_i-open set U and $x \in U$ there exists $V \in (\tau_i, \tau_j)$-$g^*\omega\alpha$O(X) such that $x \in V \in (\tau_i, \tau_j)g^*\omega\alpha$-$cl(V) \subseteq U$.*

Theorem 5.7. *Every (τ_i, τ_j)-$g^*\omega\alpha$-paracompact pairwise Hausdorff bitopological space is (τ_i, τ_j)-$g^*\omega\alpha$-regular.*

Proof. Let A be τ_i-closed set with $x \notin A$. Then for each $y \in A$, choose τ_i-open sets U_y and H_x such that $y \in U_y$, $x \in H_x$ and $U_y \cap H_x = \phi$, that is $x \notin (\tau_i, \tau_j)g^*\omega\alpha$-$cl(U_y)$. Therefore the family $U = \{U_y : y \in A\} \cup \{X - A\}$ is an τ_i-open cover of X and so it has a (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement say ρ. Let $V = \{H \in \rho : H \cap A \notin \phi\}$, then V is a (τ_i, τ_j)-$g^*\omega\alpha$-open set containing A and (τ_i, τ_j)-$g^*\omega\alpha$-$cl(V) = \cup\{(\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(H) : H \in \rho$ and $H \cap A \notin \phi\}$. Therefore $U = X - (\tau_i, \tau_j)$-$g^*\omega\alpha$-$cl(V)$ is a (τ_i, τ_j)-$g^*\omega\alpha$-open set containing x such that U and V are disjoint subsets of X. Thus X is (τ_i, τ_j)-$g^*\omega\alpha$-regular. □

Corolary 5.1. *Every (τ_i, τ_j)-$g^*\omega\alpha$-paracompact pairwise Hausdorff bitopologcal space (τ_i, τ_j)-$g^*\omega\alpha$-normal.*

Theorem 5.8. *Let (X, τ_1) and (X, τ_2) be two regular spaces. Then (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact if and only if every τ_i-open cover ξ of X has a (τ_i, τ_j)-P locally finite (τ_i, τ_j)-$g^*\omega\alpha$-closed refinement say ρ.*

Proof. Necessity: Let ξ be an τ_1-open cover of X. Then for each $x \in X$ choose a member $U_x \in \xi$. Since (X, τ_1) and (X, τ_2) are τ_i-regular, there exists τ_i-open set V_x containing x such that $\tau_i \subseteq cl(V_x) \subseteq U_x$. Therefore $\Psi = \{V_x : x \in X\}$ is an τ_i-open cover of X and by hypothesis Ψ has a (τ_i, τ_j)-P-locally

finite (τ_i, τ_j)-$g^*\omega\alpha$-refinement say $\Omega = \{W_\lambda : \lambda \in \Gamma\}$. Consider the collection $(\tau_i, \tau_j)g^*\omega\alpha - \Omega = \{(\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(W_\lambda) : \lambda \in \Gamma\}$ is a (τ_i, τ_j)-P-locally finite of (τ_i, τ_j)-$g^*\omega\alpha$-closed subsets of (X, τ_1, τ_2). Since for every $\lambda \in \Gamma$, (τ_i, τ_j)-$g^*\omega\alpha$-$cl(W_\lambda) \subseteq (\tau_i, \tau_j)g^*\omega\alpha\text{-}cl(V_x) \subseteq \tau_1\text{-}cl(V_x) \subseteq U_x$ for some $U_x \in \xi$, therefore (τ_i, τ_j)-$g^*\omega\alpha$-$cl(\Omega)$ is a refinement of ξ.
Sufficiency: Let ξ be an τ_i-open cover of X and Ψ be a (τ_i, τ_j)-P locally finite (τ_i, τ_j)-$g^*\omega\alpha$-closed refinement of ξ. Then for each $x \in X$ choose $W_x \in (\tau_i, \tau_j)$-$g^*\omega\alpha$O(X) such that $x \in W_x$ and W_x intersects at most finitely many member of Ψ. Let Σ be (τ_i, τ_j)-$g^*\omega\alpha$-closed (τ_i, τ_j)-P-locally finite refinement of $\Omega = \{W_x : x \in X\}$. Then for each $V \in \Psi$, $V^1 = X - H$, where $H \in \Sigma$ and $H \cap V = \phi$. Then $\{V^1 : V \in \Psi\}$ is a (τ_i, τ_j)-$g^*\omega\alpha$-open cover of X. Now for each $V \in \Psi$, let us choose $U_v \in \Xi$ such that $V \subseteq U_v$. Hence the collection $\{U_v \cap V^1 : V \in \Psi\}$ is a (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement of ξ. Thus (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact. □

Theorem 5.9. *Let A be a $(i, j)regular$ closed subset of a bitopological space (X, τ_1, τ_2). Then $(A, \tau_{i_A}, \tau_{j_A})$ is (i, j)-$g^*\omega\alpha$-paracompact.*

Proof. Let $\Sigma = \{V_\lambda : \lambda \in \Gamma\}$ is an τ_i-open cover of A in $(A, \tau_{i_A}, \tau_{j_A})$. Then for each $\lambda \in \Gamma$, choose an $U_\lambda \in \tau_i$ such that $V_\lambda = A \cap U_\lambda$. Then the collection $\xi = \{U_\lambda : \lambda \in \Gamma\} \cup \{X - A\}$ which is an τ_i-open cover of the (i, j)-$g^*\omega\alpha$-paracompact space X and so it has a (i, j)-P-locally finite (i, j)-$g^*\omega\alpha$-open refinement say $\Sigma = \{W_\delta : \delta \in \Delta\}$. But we have $(i, j) - RO(X) \subseteq (i, j) - O(X)$, then the collection $\{A \cap W_\delta : \delta \in \Delta\}$ is a (i, j)-P-locally finite (i, j)-$g^*\omega\alpha$-open refinement of Σ in $(A, \tau_{i_A}, \tau_{j_A})$. □

Theorem 5.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be (τ_1, σ_1) and (τ_2, σ_2) closed (τ_i, τ_j)-$g^*\omega\alpha$-irresolute surjective function such that $f^{-1}(y)$ is τ_i-compact in (X, τ_1) for each $y \in Y$. If (Y, σ_1, σ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact, then (X, τ_1, τ_2) is also (τ_i, τ_j)-$g^*\omega\alpha$-paracompact.*

Proof. Let $\xi = \{U_\lambda : \lambda \in \Gamma\}$ be an τ_i-open cover of a bitopological space (X, τ_1, τ_2). Then for each $y \in Y$, ξ is an τ_i-open cover of the τ_i-compact subspace $f^{-1}(y)$. So there exist a finite subcover Γ_y of Γ such that $f^{-1}(y) \subseteq \cup U_\lambda$ for each $\lambda \in \Xi_\lambda$. Let $U_\lambda = \cup U_\lambda$ which is an τ_i-open in (X, τ_1). As f is (τ_1, σ_1)-closed, then for each $y \in Y$ there exists σ_1-open set V_y in Y such that $y \in V_y$ and $f^{-1}(V_y) \subseteq U_\lambda$. Then the collection $\Psi = \{V_y : y \in Y\}$ is an σ_1-open cover of the (τ_i, τ_j)-$g^*\omega\alpha$-paracompact space (Y, σ_1, σ_2) and so it has a (τ_i, τ_j)-P locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement say $\Omega = \{W_\gamma : \gamma \in \Delta\}$. As f is (τ_i, τ_j)-$g^*\omega\alpha$-irresolute, the collection $f^{-1}(\Omega) = \{f^{-1}(W_\gamma) : \gamma \in \Delta\}$ which is an (τ_i, τ_j)-$g^*\omega\alpha$-open (τ_i, τ_j)-P-locally finite cover of (X, τ_1, τ_2) such that for each $\gamma \in \delta$, $f^{-1}(W_\gamma) \subseteq U_y$ for some $y \in Y$. Then the collection $\{f^{-1}(W_\gamma) \cap U_\gamma : \gamma \in$

$\delta, \lambda \in \Gamma_y\}$ is an (τ_i, τ_j)-P-locally finite (τ_i, τ_j)-$g^*\omega\alpha$-open refinement of ξ. Thus (X, τ_1, τ_2) is (τ_i, τ_j)-$g^*\omega\alpha$-paracompact. □

Conclusion

The notions of sets and functions in topological spaces are extensively developed and used in many fields such as particle physics, computational topology, quantum physics. By researching generalizations of closed sets, some new Paracompact spaces have been founded and they turn out to be useful in the study of digital topology.

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