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A note on $\alpha-$irresolute topological rings

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Abstract

In [4], we introduced the notion of $\alpha-$irresolute topological rings in Mathematics. This notion is independent of topological rings. In this note, we point out that under certain conditions an $\alpha-$irresolute topological ring is topological ring and vice versa. We prove that the Minkowski sum $\mathfrak{A} + \mathfrak{B}$ of an $\alpha-$compact subset $\mathfrak{A} \subseteq \mathcal{R}$ and an $\alpha-$closed subset $\mathfrak{B} \subseteq \mathcal{R}$ of an $\alpha-$irresolute topological ring $(\mathcal{R}, \Im)$ is actually a closed subset of $\mathcal{R}$. In the twilight of this note, we pose several natural questions which are noteworthy.

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1 Introduction

Let's first introduce some notations. We denote a topological space by $(X,\ \Im)$ (or simply X) where $\Im$ is a non-trivial topology on X. For every topological space $(X,\ \Im)$, there is a finer topology on X which is called the $\alpha-$topology on X and is denoted by $\Im^\alpha$. In fact, $\Im^\alpha$ is the family of all $\alpha-$open sets in X (with respect to $\Im$). Njastad [3] showed that $\Im^\alpha$ is a topology on X.

In 2019 ([4]), we introduced a category of $\alpha-$regular spaces called $\alpha-$irresolute topological rings. An $\alpha-$irresolute topological ring, denoted by $(\mathcal{R},\ \Im)$, is a ring $\mathcal{R}$ that is endowed with a topology $\Im$ such that the following mappings

(1) $$(\mathcal{R},\ \Im^\alpha) \times (\mathcal{R},\ \Im^\alpha) \longmapsto (\mathcal{R},\ \Im^\alpha)$$

$$(\varsigma,\ \xi) \longrightarrow \varsigma - \xi, \text{ for all } \varsigma,\ \xi \in \mathcal{R}$$

and

(2) $$(\mathcal{R},\ \Im^\alpha) \times (\mathcal{R},\ \Im^\alpha) \longmapsto (\mathcal{R},\ \Im^\alpha)$$

$$(\varsigma,\ \xi) \longrightarrow \varsigma.\xi \text{ for all } \varsigma,\ \xi \in \mathcal{R}$$

are continuous.

The notion of $\alpha-$irresolute topological rings has structural nuances and conceptual niceties with the notion of topological rings. Structurally, it sounds that there may be a strong relationship between these concepts. In this note, we point out some conditions with which an $\alpha-$irresolute topological ring is topological ring and vice versa. For subsets $\mathfrak{A}$, $\mathfrak{B}$ of $\mathcal{R}$, we can define the so-called Minkowski addition of two sets as

$$\mathfrak{A} + \mathfrak{B} = \{a + b : a \in \mathfrak{A},\ b \in \mathfrak{B}\}$$

We describe that the Minkowski sum $\mathfrak{A} + \mathfrak{B}$ of $\mathfrak{A} \subseteq \mathcal{R}$, an $\alpha-$compact subset, and $\mathfrak{A} \subseteq \mathcal{R}$, an $\alpha-$closed subset, is closed subset of $\mathcal{R}$.

Definition 1.1. *A subset U of a topological space $(X,\ \Im)$ is said to be*

(1) $\alpha-$open [3] if $U \subseteq Int(Cl(Int(U)))$.
(2) semi-open [2] if $U \subseteq Cl(Int(U))$.

The complement of an $\alpha-$open (resp. semi-open) set is called $\alpha-$closed (resp. semi-closed).

2 Main results

For an $\alpha-$irresolute topological ring $(\mathcal{R},\ \Im)$, let Γ denotes the collection of all $\alpha-$open subsets of $\mathcal{R}$ containing the additive identity 0 of $\mathcal{R}$. In the sequel, we use the following lemma:

Lemma 2.1. *([4, Corollary 3.9.1]) Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Then for every $\mathfrak{V} \in \Gamma$, there exists a symmetric $\sigma \in \Gamma$ such that $\sigma + \sigma \subseteq \mathfrak{V}$.*

Definition 2.1. *A subset U of a topological space $(X,\ \Im)$ is said to be $\alpha-$compact [1] if every cover of U by $\alpha-$open subsets of X has a finite subcover.*

Theorem 2.1. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any subsets of $\mathcal{R}$ such that $\mathfrak{A}$ is $\alpha-$compact and $\mathfrak{B}$ is $\alpha-$closed satisfying $\mathfrak{A} \cap \mathfrak{B} = \emptyset$. Then there exists a set $\sigma \in \Gamma$ with the property $(\mathfrak{A}+\sigma) \cap (\mathfrak{B}+\sigma) = \emptyset$.*

Proof. Let ς be an element of $\mathfrak{A}$. By Lemma 2.1, there exists a symmetric $\sigma_\varsigma \in \Gamma$ such that

$$\varsigma + \sigma_\varsigma + \sigma_\varsigma + \sigma_\varsigma \cap \mathfrak{B} = \emptyset$$

or that

$$\varsigma + \sigma_\varsigma + \sigma_\varsigma \cap \mathfrak{B} + \sigma_\varsigma = \emptyset. \qquad (*)$$

Continuing in a similar vein, we obtain a family of $\alpha-$open subsets,

$$\pi = \{\varsigma + \sigma_\varsigma : \sigma_\varsigma \in \Gamma,\ \varsigma \in \mathfrak{A}\}.$$

Since $\mathfrak{A}$ is $\alpha-$compact, we have a finite subset $\mathfrak{J} \subseteq \mathfrak{A}$ such that

$$\mathfrak{A} \subseteq \bigcup\{\varsigma + \sigma_\varsigma : \sigma_\varsigma \in \Gamma,\ \varsigma \in \mathfrak{J}\}.$$

Consider the set

$$\sigma = \bigcap\{\sigma_\varsigma : \varsigma \in \mathfrak{J}\}.$$

Then σ is $\alpha-$open subset of $\mathcal{R}$.

By virtue of $(*)$,

$$(\mathfrak{A} + \sigma) \cap (\mathfrak{B} + \sigma) = \emptyset.$$

□

Theorem 2.2. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\mathfrak{A} \subseteq \mathcal{R}$ an $\alpha-$compact, and $\mathfrak{B} \subseteq \mathcal{R}$ an $\alpha-$closed. Then $\mathfrak{A} + \mathfrak{B}$ is $\alpha-$closed subset of $\mathcal{R}$.*

Proof. We prove this theorem by contrapositive. Let $\varsigma \notin \mathfrak{A} + \mathfrak{B}$.

Then, by Theorem 2.1, for each $\nu \in \mathfrak{A}$, there exists a set $\mathfrak{U}_\nu \in \Gamma$ such that

$$(\varsigma+\mathfrak{U}_\nu)\cap(\nu+\mathfrak{B}+\mathfrak{U}_\nu)=\emptyset. \tag{*}$$

Consider the collection of sets obtained form this fixed ς,

$$\pi=\{\nu+\mathfrak{U}_\nu:\nu\in\mathfrak{A}\}$$

By Theorem 3.1 in [4], π is a cover of $\mathfrak{A}$ by $\alpha-$open subsets of $\mathcal{R}$. Therefore, it has a finite subcover

$$\pi'=\{\nu_i+\mathfrak{U}_{\nu_i}:\nu_i\in\mathfrak{A},\ i=1,\ 2,.....,\ n\}. \tag{**}$$

To obtain the sets,

$$\mathfrak{P}=\bigcap\{\mathfrak{U}_{\nu_i}:\ i=1,\ 2,\ ....,\ n\},\ \mathfrak{Q}=\bigcup\{\mathfrak{U}_{\nu_i}:\ i=1,\ 2,\ ......,\ n\}. \tag{@}$$

By construction, $\mathfrak{P}$ and $\mathfrak{Q}$ are $\alpha-$open subsets of $\mathcal{R}$.

Because of (*) and (**), we have

$$(\varsigma+\mathfrak{U}_{\nu_i})\cap(\nu_i+\mathfrak{B}+\mathfrak{U}_{\nu_i})=\emptyset,\forall\, i=1,\ 2,\ ....\ n. \tag{@@}$$

Then (@) and (@@) together complete the proof. □

Theorem 2.3. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\mathfrak{A}\subseteq\mathcal{R}$ an $\alpha-$compact, and $\mathfrak{B}\subseteq\mathcal{R}$ an $\alpha-$closed. Then $\mathfrak{A}+\mathfrak{B}$ is closed subset of $\mathcal{R}$.*

To prove Theorem 2.3, we need the following fact:

Theorem 2.4. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring, $\sigma\subseteq\mathcal{R}$ an $\alpha-$compact, and $\nabla\subseteq\mathcal{R}$ an $\alpha-$closed satisfying $\sigma\cap\nabla=\emptyset$. Then there exist disjoint open subsets $\mathfrak{P}$ and $\mathfrak{Q}$ of $\mathcal{R}$ such that $\sigma\subseteq\mathfrak{P}$ and $\nabla\subseteq\mathfrak{Q}$.*

Proof. In light of Theorem 2.1, there exist $\alpha-$open subsets $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathcal{R}$ such that

$$\sigma\subseteq\mathfrak{U},\ \nabla\subseteq\mathfrak{V}\text{ and }\mathfrak{U}\cap\mathfrak{V}=\emptyset.$$

Take $\mathfrak{P}=Int(Cl(Int(\mathfrak{U})))$ and $\mathfrak{Q}=Int(Cl(Int(\mathfrak{V})))$.

Then $\mathfrak{P},\ \mathfrak{Q}\in\Im$ with $\mathfrak{P}\cap\mathfrak{Q}=\emptyset$. It finishes the proof. □

Proof of Theorem 2.3. Let $\varsigma\notin\mathfrak{A}+\mathfrak{B}$. By Theorem 2.2, $\mathfrak{A}+\mathfrak{B}$ is $\alpha-$closed subset of $\mathcal{R}$.

In view of Theorem 2.4, there exists an open subset $\mathfrak{P}\subseteq\mathcal{R}$ such that

$$\varsigma\in\mathfrak{P}\text{ and }\mathfrak{P}\cap(\mathfrak{A}+\mathfrak{B})=\emptyset.$$

This proves that $\mathfrak{A}+\mathfrak{B}$ is closed subset of $\mathcal{R}$. Proof is over.

We now give two theorems, Theorem 2.5 and Theorem 2.6, having conceptual niceties or that indicate the relationship between the topological rings and the $\alpha-$irresolute topological rings.

Theorem 2.5. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring such that every nowhere dense subset of $\mathcal{R}$ is $\alpha-$closed. Then $(\mathcal{R},\ \Im)$ is a topological ring.*

Proof. Let $\mathfrak{B}$ be any $\alpha-$open subset of $\mathcal{R}$. Then $\mathfrak{B} = \mathfrak{O} - \nabla$ where $\mathfrak{O}$ is an open subset of $\mathcal{R}$ and ∇ is nowhere dense subset of $\mathcal{R}$. Now,

Let $\varsigma \in \mathfrak{B}$ be any element. Then $\varsigma \notin \nabla$. By hypothesis, ∇ is $\alpha-$closed subset of $\mathcal{R}$.

By dint of Theorem 2.4, there exist open subsets $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathcal{R}$ such that

$$\varsigma \in \mathfrak{U},\ \nabla \subseteq \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset.$$

Whence we easily conclude that $\mathfrak{B}$ is open subset of $\mathcal{R}$. Hence $(\mathcal{R},\ \Im)$ is a topological ring. □

Theorem 2.6. *Let $(\mathcal{R},\ \Im)$ be a topological ring such that every nowhere dense subset of $\mathcal{R}$ is closed. Then $(\mathcal{R},\ \Im)$ is $\alpha-$irresolute topological ring.*

Proof. Let $\mathfrak{B}$ be any $\alpha-$open subset of $\mathcal{R}$. Then $\mathfrak{B} = \mathfrak{P} - \nabla$ for some open subset $\mathfrak{P} \subseteq \mathcal{R}$ and nowhere subset $\nabla \subseteq \mathcal{R}$. By given condition, ∇ is a closed subset of $\mathcal{R}$.

Thus, for any $\varsigma \in \mathfrak{B}$, there exist $\mathfrak{U},\ \mathfrak{V} \in \Im$ with the property that

$$\varsigma \in \mathfrak{U},\ \ \nabla \subseteq \mathfrak{V} \text{ and } \mathfrak{U} \cap \mathfrak{V} = \emptyset.$$

Let $\mathfrak{O} = \mathfrak{P} \cap \mathfrak{U}$. Then evidently, $\mathfrak{O}$ is open subset of $\mathcal{R}$ containing ς such that $\mathfrak{O} \subseteq \mathfrak{B}$. Thereby it follows that $(\mathcal{R},\ \Im)$ is an $\alpha-$irresolute topological ring. □

Theorem 2.7. *Let $(\mathcal{R},\ \Im)$ be an $\alpha-$irresolute topological ring. Then, for every $\mathfrak{O} \in \Gamma$, there exists $\sigma \in \Gamma$ such that $\alpha Cl(\sigma) \subseteq \mathfrak{O}$.*

Proof. Let $\mathfrak{O} \in \Gamma$. By Theorem 2.1, there exists a set $\sigma \in \Gamma$ such that

$$\sigma \cap (\mathfrak{O}^c + \sigma) = \emptyset.$$

This implies that

$$\sigma \subseteq (\mathfrak{O}^c + \sigma)^c.$$

Consequently, $\alpha Cl(\sigma) \subseteq \mathfrak{O}$. □

Definition 2.2. *A topological space $(X,\ \Im)$ is said to be*

(1) $\alpha - T_0$ if for distinct points x and y in X, there exists an $\alpha-$open set U in X such that either $x \in U,\ y \notin U$ or $y \in U,\ x \notin U$.

(2) $\alpha - T_1$ if for distinct points x and y in X, there exist $\alpha-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

(3) $\alpha - T_2$ if for distinct points x and y in X, there exist disjoint $\alpha-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

Theorem 2.8. *Let* $(\mathcal{R}, \Im)$ *be an* $\alpha-$*irresolute topological ring. Then the following are equivalent:*

(1) $(\mathcal{R}, \Im)$ *is* $\alpha - T_0$ *space.*
(2) $(\mathcal{R}, \Im)$ *is* $\alpha - T_1$ *space.*
(3) $(\mathcal{R}, \Im)$ *is* $\alpha - T_2$ *space.*

Proof. We prove only *(1) implies (3)*. Other implications are obvious.

(1) implies (3): Let $\varsigma \neq 0$. Without loss of generality, we may assume that there is $\sigma \in \Gamma$ such that $\varsigma \notin \sigma$. Then, by Theorem 2.7, there exists $\mathfrak{O} \in \Gamma$ such that $\varsigma \notin \alpha Cl(\mathfrak{O})$.

By Theorem 2.1, there exists $\mathfrak{U} \in \Gamma$ such that

$$(\varsigma + \mathfrak{U}) \cap (\alpha Cl(\mathfrak{O}) + \mathfrak{U}) = \emptyset.$$

Thence we infer that $\varsigma + \mathfrak{U}$ and $\mathfrak{U}$ are disjoint $\alpha-$open subsets of $\mathcal{R}$ containing ς and 0 respectively. Hence $(\mathcal{R}, \Im)$ is $\alpha-$Hausdorff space. □

We end this note with some open questions. Imbibing theorems, Theorem 2.3, Theorem 2.5, and Theorem 2.6, we point out several interesting and pertinent questions which are worthy to the healthy discussion about the interconnection between the $\alpha-$irresolute topological rings and the topological rings.

Question 1 . Does there exist an $\alpha-$irresolute topological ring $(\mathcal{R}, \Im)$ which is also a topological ring but $\Im \neq \Im^{\alpha}$?

Question 2 . Can we have a finite $\alpha-$irresolute topological ring which is not a topological ring?

Question 3 . Is there any $\alpha-$irresolute topological ring which is not a topological ring showing the Essentiality of each condition in Theorem 2.3?

Question 4 . Does there exist a topological ring which is not an $\alpha-$irresolute topological ring satisfying Theorem 2.3?

Question 5 . Can we replace '$\alpha-$closedness' in the statement of Theorem 2.5 by a weaker form of it like semi-closedness, pre-closedness, etc.?

Question 6 . Can we replace 'closedness' in the hypothesis of Theorem 2.6 by a weaker form of it like $\alpha-$closedness, semi-closedness, etc.?

Question 7 . Does there exist an $\alpha-$regular topological ring which is not an $\alpha-$irresolute topological ring?

3 Conclusions

In the literature it is a well known result that the algebraic sum of a compact set and a closed set in a topological ring is closed set. In this note, we showed that the algebraic sum $\mathfrak{A} + \mathfrak{B}$ of an $\alpha-$compact subset $\mathfrak{A} \subseteq \mathcal{R}$, and an $\alpha-$closed subset $\mathfrak{B} \subseteq \mathcal{R}$ of an $\alpha-$irresolute topological ring $\mathcal{R}$ is a closed subset of $\mathcal{R}$. In addition, we have shown the equivalence of $\alpha - T_i$ spaces in $\alpha-$irresolute topological rings for $i = 0,\ 1,\ 2$.

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