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Determine the value *d(M(G))* for non-abelian *p*-groups of order *q = pnk* of Nilpotency *c*

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Abstract

In this paper we prove that if n, k and t be positive integer numbers such that $t < k < n$ and G is a non abelian p-group of order pnk with derived subgroup of order pkt and nilpotency class c, then the minimal number of generators of G is at most $p1\ 2\ ((nt+kt-2)(2c-1)(nt-kt-1)+n$. In particular, $|M(G)|$ _ $p1\ 2\ (n(k+1)-2)(n(k-1)-1)+n$, and the equality holds in this last bound if and only if $n = 1$ and $G = H \times Z$, where H is extra special p-group of order $p3n$ and exponent p, and Z is an elementary abelian p-group.

Keywords: Schur multiplier, elementary abelian, p-group, extra special

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1. Introduction

Let G be a finite group and $G = FR$ a presentation for G as a factor group of the free group F. Then Schur in [11], show that $M(G) = (F_0 \backslash R)\ [F,R]$.
(1.1) Recall that, for two finite groups A and B, AB _= (AA_0)(BB_0).
Michael R. Jones in years 1973 and 1974 for the finite group G, get some inequalities for $d(M(G))$ and $e(M(G))$, which $d(M(G))$ and $e(M(G))$ the minimal number of generators and exponent of finite group G, respectively. now in current paper we generalized and compute the value $d(M(G))$ and $e(M(G))$ for non-abelian pgroups of order $q = p_{nk}$ and nilpotency c.
Notation: The notation used in this paper is as follows:
(i) If G is a finite group then $E(G)$ denotes exponent of G and $D(G)$ denotes the minimal number of generators of G.
(ii) The the lower central series of a group G is denoted by $G = g_1(G)$ _ $g_2(G) = G_0$ _ $g_3(G)$ _ ..., where for j _ 1, $g_{i+1}(G) = [g_i(G),G]$.
And the upper central series of a group G is denoted by $1 = Z_0(G)$ _ $Z_1(G) = G_0$ _ $Z_2(G)$ _ ..., where for i _ 0, $Z_{i+1}Z_i$ _ $Z(\ GZ_i(G)\ )$.
The main theorem of this paper as follows.

Main Theorem: Let n, k and t be positive integer numbers such that $t < k < n$ and G is a non abelian p-group of order p_{nk} with derived subgroup of order p_{kt} and nilpotency class c, then the minimal number of generators of G, $(D|M(G)|)$ is $p_{12}\ ((2c-1)n_2-k(k-1)-3n+4$.

2. Some definition, lemma and theorems

The results of this section are several lemma and theorems, where the proofs of their in references [6], [7] and [8], and so we will be omitted.
2.1. Lemma: Let G be a finite group and B a normal subgroup. Set $A = GB$. Let $G = F\ R$ be a presentation for G as a factor group of the free group F and suppose $B = SR$ so that $A = FS$. Then $[F,S]\ [F,R][F,S,F]S_0$ is isomorphic with a factor group of AB.
Proof. See to ([6], Lemma 2.1).
2.2. Corollary. Further to the notation and assumptions of Lemma 2.1, let B 2 be a central subgroup of G. Then $[F,R]\ [F,R]S_0$ is an epimorphic image of AB.
Proof. See to ([6]).
2.3. Definition. Let G be a finite group. We say that G has (special) rank $r(G)$ if every subgroup of G may be generated by $r(G)$ elements and there is at least one subgroup that cannot be generated by fewer than $r(G)$ elements.
Let $G = F\ R$ be a presentation for the finite p-group G as a factor group of a free group F. Let $\Gamma_{i+1} = g_{i+1}(F)$ for all i. Since $G_0 = F_0R\ R$ we have by (1.1), that

$M(\frac{G}{G'}) \cong \frac{(F'\cap F'R)}{[F,F'R]} = \frac{F'}{[F,F'R]}$.
With this notation we have:

2.4. Theorem: Let G be a finite p-group of nilpotency class c and $Q_i = \frac{G}{\gamma_i(G)}$ for $2 \le i \le c$. Then (i) $|G'||M(G)| \le |M(\frac{G}{G'})\prod_{i=1}^{c-1} |Q_{i+1}\gamma_{i+1}(G)|$,
(ii)D(M(G)) $\le D(M(\frac{G}{G'}))+\sum_{i=1}^{c-1} D(Q_{i+1}\gamma_{i+1}(G))$,
(iii) $E(M(G)) \le E(M(\frac{G}{G'}))\prod_{i=1}^{c-1} E(Q_{i+1}\gamma_{i+1}(G))$.
(i) In the above notation, $|G'||M(G)| = |\frac{F'}{[F,R]}| = |M(\frac{G}{G'})| \frac{|[F,F'R]}{[F,R]|} = |M(\frac{G}{G'})| \frac{|[F,F_{i+2}R]}{[F,R]|} \prod_{k=1}^{I} |\frac{[F,\Gamma_{k+1}R|}{[F,\Gamma_{k+2}R}$, for all $i \ge 1$. Now, $1 = \gamma_{c+1}(G) = \frac{\Gamma_{c+1}R}{R}$ so that $\Gamma_{c+1} \le R$ and $[F,F_{c+1}R] = [F,R]$.
Next, $\gamma_i(G) = \frac{\Gamma_i R}{R}$ for all $i \ge 2$. Thus $[F,R](\Gamma_i R)'[F,\Gamma_i R,F] = [F,R]\Gamma_{i+2} = [F,\Gamma_{i+1}R]$ and (i) follows by Lemma 2.1. (ii) We have,
$r(\frac{F'}{[F,R]}) \le r(M(\frac{G}{G'}))+r(\frac{[F,\Gamma_2 R]}{[F,R]}$ so that $D(M(G)) \le D(M(\frac{G}{G'}))+\sum_{i=1}^{c-1} r(\frac{[F,\Gamma_{i+1}R]}{[F,\Gamma_{i+2}R]})$, and (ii) again follows by Lemma 2.1.
(iii) This follows as for (i) and (ii).

3. The proof of main Theorem

In this section we show that, Let n, k and t be positive integer numbers such that $t < k < n$ and G is a non abelian p-group of order p^{nk} with derived subgroup of order p^{kt} and nilpotency class c, then the minimal number of generators of G, $(D|M(G)|)$ is $p^{\frac{1}{2}((2c-1)n^2-k(k-1)-3n+4}$. For proof of this work we action as follows:
Proof. Let n, k and t be positive integer numbers such that $t < k < n$ and G is a non abelian p-group of order p^{nk} with derived subgroup of order p^{kt} and nilpotency class c. Then by using of Theorem 2.4(ii),we have
$D(M(G)) \le D(M(\frac{G}{G'}))+\sum_{i=1}^{c-1} D(Q_{i+1}\gamma_{i+1}(G))$.
If $D(M(G)) = n$ then the above relation will coming as follows:
$D(M(G)) \le \frac{1}{2}((n+k-2)(n-k-1)+1)+n(\sum_{i=1}^{c-1} \gamma_{i+1}(G))$.
$= \frac{1}{2}((n+k-2)(n-k-1)+1)+n^2(c-1)$. Which the result now follows.
In 1904, Schur [11,12] prove that for every finite groups H and K, then $M(H \times K) = M(H)\times M(K)\times \frac{H}{H'} \frac{K}{K'}$.
In 1957, Green [5] show that if G be a p-group of order p^n, then $|M(G)| \le p^{\frac{1}{2}n(n-1)}$.
In 1967, Gaschatz el al [4] prove that if G be a d-generator p-group of order p^n, G' has order p^c and $\frac{G}{Z(G)}$ is a d- generator group, then $|M(G)| \le p^{\frac{1}{2}d(2n-2c-d-1)+2(d-1)c}$.
In 1973, Jones [4-6] show that if G be a p-group of order p^n and $|G'| = p^k$, then $|M(G)| \le p^{\frac{1}{2}n(n-1)-k}$.
In 1982, Byel and Tappe [2] shown that if G be a Extra especial p-group of order p^{2m+1}, then

(i) If $m \leq n$, than $|M(G)| = p^{2m^2-m-1}$.
(ii) If $m = 1$, then the order of Schur multiplier of D_8, Q_8, E_1 and E_2 are equal 2, 1, p^2 and 1, respectively.
In 1991, Berkovich [1] show that if G be a p-group of order p^n, then $t(G) = 0$ if and only if $G \cong Z_p^{(n)}$, and also $t(G) = 1$ if and only if $G \cong Z^{(2)}$ or $G \cong E_1$.
In 1994, Zhou [14]prove that if G be a p-group of order p^n, then $t(G) = 2$ if and only if $G \cong Z \times Z_{p^2}$ or $G \cong D_8$, $G \cong E_1 \times Z_p$.
In 1999, Ellis [3]show that if G be a p-group of order p^n, then $t(G)=3$ if and only if $G \cong Z_{p^3}$, $G \cong Z_p^{(2)} \times Z_{p^2}$ or $G \cong Q_8$, $G \cong E_2$, $G \cong D_8 \times Z_2$ or $G \cong E_1 \times Z_p^{(2)}$.
In 2009, P.Niroomand [10] show that if G be a non-abelian finite p-group of order p^n and $|G'| = p^k$, then $|M(G)|$ is $p^{\frac{1}{2}((n+k-2)(n-k-1))+1}$. In particular, $|M(G)| \leq p^{\frac{1}{2}(n-2)(n-1)+1}$, and the equality holds in this last bound if and only if $G = E_1 \times Z$, where Z is an elementary abelian p-group.
The Schur multiplier of abelian groups may be calculated easily by a result [12] which was obtained by Schur. So in this paper, we focus on non-abelianp-groups.
This paper is devoted to the derivation of certain upper bound for the Schur multiplier of non-abelian p-groups of order p^{nk} with derived subgroup of order p^k. We prove that $|M(G)| \leq p^{\frac{1}{2}(nk+nt-2)(nk-nt-1)+n}$. In particular, if $|M(G)| = p^{\frac{1}{2}(n(k+1)-2)(n(k-1)-1)+n}$, we characterize the structure of the group G. If G is a p-group of order p^n, Jones [4] proved that $|M(G)||G'| \leq p^{\frac{1}{2}n(n-1)}$ which shows that $|M(G)| \leq p^{\frac{1}{2}n(n-1)+1}$ when G is a non-abelian p-group of order p^n. So, the general bound given above is better than Joness bound unless $|G| = p^3$, in which case the two bounds are the same.The principal result of this paper is presented in the following theorem.

Main Theorem. Let G be a non-abelian finite p-group of order p^{nk}. If $|G'| = p^{nt}$, then we have $M(G) \leq p^{\frac{1}{2}(nk+nt-2)(nk-nt-1)+n}$. In particular $M(G) \leq p^{\frac{1}{2}(n(k+1)-2)(n(k-1)-1)+n}$,
and the equality holds in this last bound if and only if $n-1$ and $G = H \times Z$, where H is an extra special p-group of order p^{3n} and exponent p, and Z is an elementary abelian p-group.

Preliminaries and Elementary Theorems.
In this section, we want to several Theorems and Lemmas whose proved in references
[1-14]. At first we list the following theorems, which are used in our proofs.
Our method for the proof is similar to P. Niroomand (2009) and Berkovich, Ya.G.
(1991), which we compute for groups of order p^{nk}.

Theorem 2.1.(See [7,theorem 3.1 and Theorem 4.1].) Let G be a finite p- group and let N be a central subgroup of G. Then $|M(G\,N| \_ |M(G)||G0 \backslash N| \_ |M(G\,N||M(N)||G\,N\,N|$.

Theorem 2.2.(See[9, Theorem 3.3.6].) Let G be an extra special p-group of order $p2m+1$. Then:
(i) If m _ 2, then $M(G) = p2m2-m-1$.
(ii) If m=1, then $M(G)$ _ $p2$, and the equality holds if and only if G is of exponent p.
Theorem 2.3.(See [9, Theorem 2.2.10].) For every finite groups H and K, we Have $M(H \times K$ _= $M(H) \times M(K) \times H\,H0\ K\,K0$.
Corollary 2.4. If G _= $Cm1 \times Cm2 \times ... \times Cmk$, where $mi+1$ divides mi for all i, 1 _ i _ k, then $M(G)$ _=$Cm2 \times C(2)\,m3 \times ... \times C(k-1)\,mk$.
Proof of the Main Theorem
In this section we want to prove our result. The following technical lemmas shorten the proof of our main Theorem.
Lemma 3.1. Let G be a finite p-group of order pn such that $G\,G0$ is elementary 6 of order $pn-1$, then G is a central product of an extra special p-group H and $Z(G)$ such that $H \backslash Z(G) = G0$.
Proof. Let $H\,G0$ be the complement of $Z(G)\,G0$ in $G\,G0$. Then $G = HZ(G)$, so $G0 = H0$ and $Z(H) = Z(G) \backslash H$. On the other hand, 1 6= $Z(G) \backslash H$ _ $G0$, and the result follows.
Lemma 3.2. Let G be an abelian p-group of order pn which is elementary abelian. Then $M(G)$ _ $p1\,2\,(n-1)(n-2)$.
Proof. the result is obtained obviously if G is cyclic. So, let G _=$Cpm1 \times Cpm2 \times ... \times Cpmk$ such that $\sum k\ i=1 mi = n$ and $m1$ _ $m2$ _ ... _ mk. We know that $m1$ _ 2, and then, by using *Corollary*2.4, $|M(G)| = pm2+2m3+...+(k-1)mk$
_ $p(m2+m3+...+mk)+(m3+...+mk)+...+mk$ _ $p1\,2\,(n-1)(n-2)$.

Lemma 3.3. Let G be a non- abelian p-group of order pnk with derived subgroup of order p such that $G\,G0$ is not elementary abelian, then $M(G) < p12\,(nk-1)(nk-2)+1$.
Proof. by using Theorem 2.1 and Lemma 3.2,
$M(G)$ _ $p-1|M(G\,G0)||G\,G0\,G0|$ _ $p-1p1\,2\,(nk-2)(nk-3)p(nk-1) < p1\,2\,(nk-1)(nk-2)+1$.
which completes the proof.
Lemma 3.4. let G be a non- abelian p-group of order pnk, such that $G\,G0$ is elementary abelian of order $pnk-1$, then $M(G)$ _ $p1\,2\,(nk-1)(nk-2)+1$ and the equality holds if and only if $G = H \times Z$, where H is extra special p- group of order $p3n$ and exponent p, and Z is elementary abelian p-group.
Proof. By Lemma 3.1, G is central product of H and $Z(G)$, and Theorem 2.2, 7 we may assume that $|Z(G)|$ _ $p2$. Let $|H| = p2m+1$, so $|Z(G)| = pn-2m$.

Suppose first that $m \geq 2$. If $Z(G)$ is elementary abelian, let T be a group such that $Z(G) \cong G' \times T$. By using Theorems 2.2 and 2.3, we have $|M(G)| = |M(H \times T)| = |M(H)||M(T)|| \frac{H}{H'} \otimes T| = p^{2m^2-m-1} p^{\frac{(n-2m-1)(n-2m-2)}{2}} p^{2m(n-2m-1)} = p^{\frac{1}{2}(n^2-3m)} < p^{\frac{1}{2}(n-1)(n-2)+1}$.

Now assume that $Z(G)$ is not elementary abelian. Theorems 2.1 and 2.3 imply That $|M(G)| \leq p|M(H \times Z(G)| = p|M(H)||M(Z(G))|| \frac{H}{H'} Z(G)|$.

Hence by using Theorem 2.2 and Lemma 3.2, we have

$|M(G)| \leq p p^{2m^2-m-1} p^{\frac{1}{2}(n-2m-1)(n-2m-2)} p^{2m(n-2m-1)} < p^{\frac{1}{2}(n-1)(n-2)+1}$.

If H is extra special of order p^{3n} and $Z(G)$ is not elementary abelian, then Theorem 2.1 implies that $|M(G)| \leq p^{-1}|M(\frac{G}{Z(G)})||M(Z(G))|| \frac{G}{Z(G)} Z(G)| \leq p^{\frac{1}{2}nk(nk-3)+1} < p^{\frac{1}{2}(nk-1)(nk-2)+1}$.

By Theorem 2.2, it is easy to see that if $Z(G)$ is elementary abelian, then $|M(G)| = p^{\frac{1}{2}(nk-1)(nk-2)+1}$ if H is extra special of order p^{3n} and exponent p; and in other cases $|M(G)| < p^{\frac{1}{2}(nk-1)(nk-2)+1}$.

Proof of the Main Theorem we prove the theorem by induction on t. if $t = 1$ the result is obtained by Lemma 3.2 and 3.4. Let G be a non-abelian p-group of order p^{nk} with derived subgroup of order $p^{nt} (t \geq 2)$. Choose K in $G' \setminus Z(G)$ of order $p-1$. By using induction hypothesis, we have $|M(\frac{G}{K})| \leq p^{\frac{1}{2}(nk+nt-4)(nk-nt-1)+n}$.

On the other hand, By using Theorem 2.1, implies that $|M(G)| \leq p^{-1}|M(\frac{G}{K})||M(K)||(\frac{G}{G'} K)| \leq p^{-1} p^{\frac{1}{2}(nk+nt-4)(nk-nt-1)} p^{n-1} p^{(nk-nt)} \leq p^{\frac{1}{2}(nk+nt-4)(nk-nt-1)} p^{n-1} p^{(nk-nt)} p^{\frac{1}{2}(nk+nt-2)(nk-nt-1)+n}$.

Now let G be a p-group of order p^{nk} such that $|M(G)| = p^{\frac{1}{2}(nk-1)(nk-2)+n}$. If $|G'| \geq p^{2k}$, then $|M(G)| \leq p^{\frac{1}{2}(n(k-1)-1)(n(k+1)-2)}$, which is a contradiction.

Since $|G'| = p^k$, Lemma 3.3 implies that G/G' is elementary abelian. Hence Lemma 3.4 shows that $G = H \times Z$, where H is an extra special p- group of order p^{3n} and exponent p, and Z is an elementary abelian p-group, so the result follows.

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