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On the traversability of near common neighborhood graph of a graph

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Abstract

The near common neighborhood graph of a graph G, denoted by $ncn(G)$ is defined as the graph on the same vertices of G, two vertices are adjacent in $ncn(G)$, if there is at least one vertex in G not adjacent to both the vertices. In this research paper, the conditions for $ncn(G)$ to be disconnected are discussed and characterization for graph $ncn(G)$ to be hamiltonian and eulerian are obtained.
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1 Introduction

Let G be a graph. The near common neighborhood graph of G denoted by $ncn(G)$ is a graph with the same vertices as G in which two vertices u and v are adjacent if there exists at least one vertex $w \in V(G)$ not adjacent to both of u and v [Al-Kenani et al., 2016].

A cycle in a graph G that contains every vertex of G is called spanning cycle of G. Thus a hamiltonian cycle of G is a spanning cycle of G. A hamiltonian graph is a graph that contains a hamiltonian cycle.

An euler trail in a graph G is a trail that contains every edge of that graph. An euler tour is a closed euler trail. An eulerian graph is a graph that has an euler tour.

The graphs cosiderered in this paper are simple, undirected and connected with vertex set $v_i \in V(G)$, $i = 1, 2, 3, ..., n$. Let $deg(v_i)$ be the degree of vertices of G. Basic terminologies are referred from [Harary, 1969].

The common neighborhood graph (congraph) of G [Zadeh et al., 2014] which is exactly the opposite of near common neighborhood is denoted by $con(G)$ is a graph with vertex set, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. Here the common neighborhood of some composite graphs are computed and also the relation between hamiltonicity of graph G and $con(G)$ is investigated. [Hamzeh et al., 2018] computed the congraphs of some composite graphs and also results have been calculated for graph valued functions. [Sedghi et al., 2020] obtained the characteristics of congraphs under graph operations and relations between Cayley graphs and its congraphs.

[Al-Kenani et al., 2016] studied near common neighborhood of a graph and obtained results for paths, cycles and complete graphs. Motivated by the results on [Zadeh et al., 2014], [Hamzeh et al., 2018] and [Sedghi et al., 2020], in this paper, the conditions for $ncn(G)$ to be disconnected are discussed and also theorems stating necessary and sufficient conditions for a graph $ncn(G)$ to possess hamiltonian and eulerian cycle are studied.

2 Prelimnaries

Below mentioned some important results are used through out the paper for proving the theorems.

Proposition 2.1. *[Al-Kenani et al., 2016] For any path P_n,*

$$ncn(P_n) = \begin{cases} \overline{K_n}, & \text{if } n = 2, 3 \\ 2K_2, & \text{if } n = 4 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.2. *[Al-Kenani et al., 2016] For any path C_n,*

$$ncn(C_n) = \begin{cases} \overline{K_n}, & \text{if } n = 3, 4 \\ C_5, & \text{if } n = 5 \\ K_n, & \text{if } n \geq 7. \end{cases}$$

Proposition 2.3. *[Al-Kenani et al., 2016]. For any complete graph K_n and totally disconnected graph $\overline{K_n}$, we have*

$$\begin{aligned} 1.ncn(K_n) &= \overline{K_n} \\ 2.ncn(\overline{K_n}) &= K_n, n \geq 3 \end{aligned}$$

Theorem 2.1. *[Singh, 2010]. Let G be a simple graph with $p \geq 3$ and $\delta \geq p/2$. Then G is Hamiltonian.*

Theorem 2.2. *[Singh, 2010] A nonempty connected graph is Eulerian if and only if it has no vertices of odd degree.*

Remark 2.1. *[Singh, 2010] The complete graph K_p, for $p \geq 3$, is always Hamiltonian.*

3 Results

Theorem 3.1. *If G is a graph with n vertices, then $ncn(G)$ is disconnected if any one of the following conditions holds*
1. G is P_n, C_n, $n \leq 4$
2. G is K_n, $n \geq 3$
3. G has $\Delta(G) = n - 1$
4. G is a graph with two complete graphs connected by bridge
5. G is $K_n \bullet P_t$, $n \geq 3$, $t \leq 3$

Proof. The proof of the theorem is constructed by considering the following cases.
Case 1. Suppose G=P_n or C_n, $n \leq 4$.
We consider the following two subcases.
Subcase 1.1. Suppose G=P_n , $n \leq 4$.
If $n = 2, 3, 4$, then by the proposition 2.1, $ncn(G)$ is disconnected.
Subcase 1.2. Suppose G=C_n , $n \leq 4$.
If $n = 3, 4$, then from the proposition 2.2, $ncn(G)$ is disconnected.
Case 2. Suppose G=K_n, $n \geq 3$.
Then from the proposition 2.3, $ncn(G)$ is disconnected.
Case 3. Let G be a graph with vertex set $V(G) = \{v_i | i \in N\}$ and $\Delta(G) = n - 1$. Near common neighbourhood graph $ncn(G)$ is a graph with same vertices v_i as

G. The vertices v_i and v_j, $j = 1, 2, 3, ..., n, i \neq j$ of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both v_i and v_j.
Since $\Delta(G) = n - 1$ in G (that is v_i is adjacent to all other vertices of G), there does not exists any nonadjacent vertex for v_i and thus v_i cannot be connected to any vertex of $ncn(G)$. This results $ncn(G)$ into disconnected.
Case 4. Let G be a graph with two complete graphs K_m and K_n connected by bridge. Let $v_i \in V(G)$, $i = 1, 2, 3, ..., m$ be the vertex set of K_m and $v_j \in V(G)$, $j = m + 1, m + 2, m + 3, ..., n$ be the vertex set of K_n, where m and n are the vertices of bridge.
As G consists of two complete graphs, vertices v_i of K_m and v_j of K_n are respectively mutually adjacent. Nonadjacent vertices for all the vertices $v_i \in K_m$ exists in K_n and for $v_j \in K_n$ exists in K_m. Thus the vertices v_i of K_m and v_j of K_n are mutually connected in $ncn(G)$. This produces the disconnected graph with two components. Further, the end points of bridge are also mutually adjacent to all the vertices of K_m and K_n respectively. Hence nonadjacent vertex does not exists for end points of bridge. This produces the graph $ncn(G)$ into disconnected.
Case 5. Let G be a $K_n \bullet P_t$, $n \geq 3$ and $t \leq 3$.
$ncn(G)$ has the same vertices as G. Two vertices of $ncn(G)$ are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both the vertices.
Let v_i, $i = 1, 2, 3, ..., n, n + 1, n + 2$ be the vertex set of $K_n \bullet P_t$ .The vertices of K_n are $v_1, v_2, v_3, ..., v_n$ and vertices of P_t are v_n, v_{n+1}, v_{n+2}. The vertex v_n is the common vertex which connects K_n and P_t.

We consider the following subcases.

Subcase 5.1 Suppose $t = 2$ that is P_t is path with two vertices, then $G = K_n \bullet P_2$. Since $\Delta(G) = n - 1$, there exists at least one vertex which is adjacent to all the other vertices of G. From theorem 3.1 (Case 3), $ncn(G)$ is disconnected.
Subcase 5.2 Suppose $t = 3$ that is $t = n, n + 1, n + 2$.
In G, all the pairs of vertices of K_n have the nonadjacent vertex as v_{n+2} and can be mutually connected in $ncn(G)$. Also the vertices of P_t, v_{n+1} and v_{n+2} have the nonadjacent vertices in K_n and can be connected in $ncn(G)$. As there does not exist any nonadjacent vertices for the pair of common vertex n and the vertices of P_t in G, they cannot be connected in $ncn(G)$. This produces the the graph $ncn(G)$ with two components. Thus $ncn(G)$ is disconnected.

$\square$.

Theorem 3.2. *For any graph G, $ncn(G)$ is hamiltonian if and only if*
1. G contains all the vertices of $deg(v_i) < n - 1$ except for $C_4 \bullet P_2$ and G is a graph with two complete graphs connected by bridge.
2. G = P_n or C_n, $n \geq 5$.
3. G is $K_n \bullet P_t$, $n \geq 3$, $t \geq 4$.

Proof. Let G be graph with vertex set $V(G)$. Suppose $ncn(G)$ is hamiltonian.

In light of the above theorem 3.1 that $ncn(G)$ is disconnected if G is P_n, C_n, $n \leq 4$, K_n, $n \geq 3$, G has $\Delta(G) = n-1$, G is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t$, $n \geq 3$, $t \leq 3$.
Now we consider the graphs for which $ncn(G)$ is connected.
Case 1. Suppose G contains all the vertices of $deg(v_i) < n-1$.
Let G be a graph $v_i \in V(G)$ vertices with $degree(v_i) < n-1$ (v_i is not adjacent to all the vertices) and $ncn(G)$ be the graph with same set of vertices as G.
As $deg(v_i) < n-1$ in G, there exists at least one nonadjacent vertex for any pair of vertices of G. Hence by definition of $ncn(G)$, those vertices in $ncn(G)$ can be connected which produces connected $ncn(G)$ graph.
Further, since for each pair of vertices of G there exists a nonadjacent vertex, $ncn(G)$ contains a cycle and $\delta \geq n/2$. From the theorem 2.1 $ncn(G)$ is hamiltonian.
Next suppose $G = C_4 \bullet P_2$. Let v_i, $i = 1, 2, 3, 4, 5$ be the vertex set of $C_4 \bullet P_2$ with one common vertex between C_4 and P_2. Among the four vertices of C_4 of G, three vertices (except the common vertex) can be connected mutually adjacent as they have nonadjacent vertex (not common vertex) in P_2.
Similarly, a vertex of P_2 which is not common can be connected with only three vertices of C_4 in $ncn(G)$ as there exists a nonadjacent vertex for these each pair of vertices. Whereas the common vertex can be connected only with a vertex of P_2 in $ncn(G)$ which is not common, since there exists a nonadjacent vertex in C_4 for this pair and there does not exist nonadjacent vertex for the pair of vertices with C_4.
This results $ncn(G)$ into connected graph with one pendent vertex and consequently does not contain hamiltonian cycle.
Thus, $ncn(C_4 \bullet P_2)$ is connected but not hamiltonian.
Case 2. Suppose G is P_n or C_n, $n \geq 5$.
Let G be a P_n or C_n, $n \geq 5$. From the propositions 2.1 and 2.2, $ncn(P_n)$ and $ncn(C_n)$, $n = 5, 6$ are connected which contains a cycle and $\delta \geq n/2$. For $n \geq 7$, $ncn(G)$ is K_n. From the theorem 2.1 and remark 2.1, $ncn(G)$ is hamiltonian.
Case 3. Suppose G is $K_n \bullet P_t$, $n \geq 3$, $t \geq 4$.
Let G be a $K_n \bullet P_t, n \geq 3$, $t \geq 4$, where n is the number of vertices of K_n and t is the number of vertices of P_t.
Let $v_i \in V(G)$, $i = 1, 2, 3, ..., n$ be the vertex set of K_n and $v_j \in V(G)$, $j = n, n+1, n+2, ..., t$ be the vertex set of P_t, where n is the common vertex of K_n and P_t.
As there is increase in the number of vertices (path length) in P_t of G, there exists a nonadjacent vertex for each pair of vertices of K_n and vertices of P_t, which produces connected graph $ncn(G)$ with cycles and also $\delta \geq n/2$. From the theorem 2.1, $ncn(G)$ is hamiltonian.
Converse is obvious.

□

Theorem 3.3. *For any graph G, $ncn(G)$ is eulerian if only if G is*
1. $P_n, n \geq 7$.
2. $K_n \bullet P_t$, $n \geq 3$, $t \geq 5$.
3. $G = C_n$, $n = 5, 6$.

Proof. Let G is a graph with vertex set $v_i \in G$, $i = 1, 2, 3, ..., n$.
Suppose $ncn(G)$ is eulerian, then degree of each v_i of $ncn(G)$ is even. From the theorems 3.1 and 3.2, $ncn(G)$ is disconnected if G is P_n, C_n, $n \leq 4$, K_n, $n \geq 3$, G has $\Delta(G) = n-1$, G is a graph with two complete graphs connected by bridge, G is $K_n \bullet P_t$, $n \geq 3$, $t \leq 3$ and is connected only if G contains all the vertices of $deg(v_i) < n - 1$, $G = P_n$ or C_n, $n \geq 5$ and G is $K_n \bullet P_t$, $n \geq 3$, $t \geq 4$.
From the proposition 2.1, $ncn(G = P_n)$ is K_n with even degree; $n \geq 7$, where $n = 2s + 1$, $s = 2, 3, 4, ...$
From the proposition 2.2, $ncn(G = C_n)$, $n = 5, 6$ is K_n of even degree.
From the theorem 3.2, $ncn(G = K_n \bullet P_t)$ is K_n with even degree; $n \geq 3$, $t \geq 5$, where $n = 2s + 1$, $s = 1, 2, 3, ...$
Hence from the theorem 2.2, $ncn(G)$ is eulerian.
Conversely, $ncn(G)$ is a graph with same vertices as G.
From theorem 3.1, $ncn(G)$ is disconnected if G is P_n, C_n; $n \leq 4$, K_n; $n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t$; $n \geq 3$, $t \leq 3$ in all other cases it is connected.

We consider the following cases.

Case 1. Suppose $ncn(G)$ is K_n, $n = 1, 2, 3, ..., n$, then from the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $G = P_n, C_n$, $n \geq 5$ and $K_n \bullet P_t$; $n \geq 3$, $t \geq 5$.
Subcase 1.1 Suppose $G = P_n$ or C_n, then from the propositions 2.1 and 2.2, $ncn(G)$ is K_n, $n \geq 7$. The degree of each vertex of K_n is even and $n = 2s + 1$, $s = 1, 2, 3, ..., n$. From theorem 2.2, $ncn(G)$ is eulerian.
Subcase 1.2. Suppose $G = K_n \bullet P_t$, $n \geq 3$, $t \geq 5$, then from the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3$, $t \geq 5$. The degree of K_n is even if n is odd.
From the theorem 2.2, $ncn(G)$ is eulerian.
Case 2. Suppose $G = C_n$, $n = 5, 6$.
Subcase 2.1 Suppose $G = C_n$, $n = 5$, then from the proposition 2.2, $ncn(G)$ is C_5 or 2-regular graph. From the theorem 2.2 $ncn(G)$ is eulerian.
Subcase 2.2 Suppose $G = C_n$, $n = 6$, then from the proposition 2.2, $ncn(G)$ is 2-regular graph. Hence from the theorem 2.2, $ncn(G)$ is eulerian.
Case 3. Suppose $G = K_n \bullet P_t$, $n \geq 3$, $t \geq 5$.
From the theorem 3.2, if $ncn(G)$ is connected and it is K_n for $n \geq 3$, $t \geq 5$.
The degree of K_n is even if n is odd. From the theorem 2.2, $ncn(G)$ is eulerian.

4 Conclusions

In this paper the study on near common neighborhood graph of a graph is extended and various general conditions for which $ncn(G)$ to be disconnected are discussed. It is disconnected for the graphs P_n, C_n; $n \leq 4$, K_n; $n \geq 3$, G has $\Delta(G) = n - 1$, G is a graph with two complete graphs connected by bridge and G is $K_n \bullet P_t$; $n \geq 3$, $t \leq 3$ in all other cases it is connected. We have also obtained the necessary and sufficient condition for $ncn(G)$ to possess hamiltonian and eulerian cycles. It contains hamiltonian cycle whenever it is connected except for $C_4 \bullet P_2$. It contains eulerian cycle if $ncn(G)$ is K_n with odd vertices and G is $C_n, n = 5, 6$.

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