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On Real Roots of Complement Degree Polynomial of Graphs

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Abstract

Many studies have been carried out on the roots of graph polynomials such as the matching polynomials, the characteristic polynomial, the chromatic polynomial, and many others. In this paper, we study the real roots of the complement degree polynomials of some graphs. Moreover, we investigate the location of the roots of the complement degree polynomials of some graphs.
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1 Introduction

In mathematics, a graph polynomial is a graph invariant whose values are polynomials. In algebraic graph theory, invariants of this kind are explored [Shi et al., 2016]. These are some crucial graph polynomials: characteristic polynomial, chromatic polynomial, dichromatic polynomial, flow polynomial, Ihara zeta function, Martin polynomial, matching polynomial, reliability polynomial, Tutte polynomial. There is a lot of research on the roots of graph polynomials, including the characteristic polynomial, the chromatic polynomial, the matching polynomial, and many others. The location and nature of the roots have been important research areas for several graph polynomials. Recently, the present authors[Safeera and Kumar, a] introduced the complement degree polynomial of a graph.

Definition 1.1. *Let* $G = (V, E)$ *be a finite simple graph of order* n *and let* $CD(G, i)$ *be the set of vertices of degree* i *in complement graph* $\overline{G}$ *and let* $Cd_i(G) = |CD(G, i)|$*. Then complement degree polynomial of* G *is the polynomial:*

$$CD[G, x] = \sum_{i=\delta(\overline{G})}^{\Delta(\overline{G})} Cd_i(G)x^i, \tag{1}$$

where $\delta(\overline{G})$ *and* $\Delta(\overline{G})$ *respectively denote the minimum degree and maximum degree of the complement graph* $\overline{G}$ *[Safeera and Kumar, a].*

The authors also derived the complement degree polynomial of some well-known graphs and some graph operations [Safeera and Kumar, a,b]. In this paper, we study the real roots of the complement degree polynomial of some graphs obtained in [Safeera and Kumar, a]. In particular, we investigate the location of the roots of the polynomials so obtained.

2 Main Results

The roots of several graph polynomials have drawn a lot of interest, both for what they represent and what their nature and location indicated. In this section, we investigate the roots of the complementary degree polynomial investigated in [Safeera and Kumar, a].

Definition 2.1. *The roots of polynomial defined in equation* (1) *are called* cd-roots *of* G*. The number of real* cd-roots *of a graph* G *where the multiplicities counted, is denoted by* $cd(G)$*.*

Theorem 2.1. *Zero is a* cd-root *of a complement degree polynomial of a graph G with n vertices if and only if $\Delta(G) \leq n-2$.*

Proof. Let G be a graph of order n and zero is a *cd-root* of the polynomial $CD[G,x]$. If G has a vertex, say v which is adjacent to all other vertices, then v is an isolated vertex in $\overline{G}$. This implies that $CD[G,x]$ has a constant term. This is a contradiction because zero is a *cd-root* of $CD[G,x]$. Therefore, G has no vertices adjacent to all other vertices.

Conversely, assume that $\Delta(G) \leq n-2$. Then $\delta(\overline{G}) \geq 1$. Equivalently, $Cd_0(G) = 0$. This tells us that the constant term of $CD[G,x]$ is zero, and hence the result follows.□

Corolary 2.1. *If $\overline{G}$ has no isolated vertices, then zero is a root of $CD[G,x]$ with multiplicity $\delta(\overline{G})$.*

Theorem 2.2. *If G is the non complete graph of order n, then zero is the only cd-root of $CD[G,x]$ if and only if G is a regular graph.*

Proof. First, assume that zero is the only cd-root of a graph G with n vertices. Then it follows that the complement degree polynomial of G is $CD[G,x] = nx^r$. This implies that the degree of every vertex in $\overline{G}$ is the same. Equivalently, G is regular.

Conversely, assume that G is a r-regular graph. Then we have $CD[G,x] = nx^{n-r-1}$. It follows that zero is only cd-root of $CD[G,x]$. □

Corolary 2.2. *If G is the r-regular graph with n vertices, then $cd(G) = n-r-1$.*

Theorem 2.3. *Let G be a graph with n vertices. Then*

(1) $CD[G,x]$ is a strictly increasing function in $[0,\infty)$.

(2) Let G be a graph and H be any spanning subgraph of G. Then the degree of $CD[G,x]$) is less than or equal to the degree of $CD[H,x]$.

(3) Let G be a graph and H be any induced subgraph of G. Then the degree of $CD[G,x]$) greater than equal to the degree of $CD[H,x]$.

(4) Let G be a graph of order n with t isolated vertices in G and r isolated vertices in $\overline{G}$. Then $Cd_0(G) = r$ and $Cd_{n-1}(G) = t$.

Proof Proof of the above result follows from the definition of complement degree polynomial of a graph.

Theorem 2.4. *For a cosplitting graph $CS(G)$ of r-regular graph G with n vertices, $cd(CS(G)) = n$.*

Proof. Observe that $CD[CS(G), x] = nx^{n-1}(1 + x^r)$ [Safeera and Kumar, a]. It is clear that $x = 0$ is a cd-root of G with multiplicity $n - 1$. Note that the polynomial $1 + x^r$ has no real roots if r is even and one real root if r is odd. Thus we have,

$$cd(CS(G)) = \begin{cases} n-1 & \text{if } r \text{ is even,} \\ n & \text{if } r \text{ is odd} \end{cases}$$

. This completes the proof.□

Theorem 2.5. *For a path graph P_n,*

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n-2, & n \geq 3. \end{cases}$$

Proof. For a path graph P_n, we have [Safeera and Kumar, a]:

$$CD[P_n, x] = \begin{cases} 2x^{n-2}, & n = 2 \\ (n-2)x^{n-3} + 2x^{n-2}, & n \geq 3 \end{cases}.$$

Here we consider two cases:
If $n = 2$, then $CD[P_2, x] = 2$, which has no zeros. If $n > 2$, then we have $CD[P_n, x] = x^{n-3}(2x + n - 2)$. Obviously, $x = 0$ is the cd-root of $CD[P_n, x]$ with multiplicity $n - 3$ and $x = -(n - 2)/2$ is the another cd-root of $CD[P_n, x]$. Thus

$$cd(P_n) = \begin{cases} 0, & n = 2 \\ n-2, & n \geq 3. \end{cases}$$

This completes the proof.□

Theorem 2.6. *Let G be a graph with order n and* **G**=$G \cup G \cup \ldots \cup G$ *(m times). Then cd(**G**) $= n(m-1) + cd(G)$.*

Proof. Let G be a graph of order n and **G**=$G \cup G \cup \ldots \cup G$ (m times). Then, $CD[\mathbf{G}, x] = mx^{(m-1)n}CD[G, x]$. Observe that $x = 0$ is a zero of $CD[\mathbf{G}, x]$ of multiplicity $n(m-1)$. Consequently, $cd(\mathbf{G}) = n(m-1) + cd(G)$. This completes the proof.□

Theorem 2.7. *For a ladder graph L_n, $cd(L_n) = 2n - 3$ for $n \geq 2$.*

Proof. Obviously, *cd-roots* of $CD[L_n, x]$ are $x = 0$ with multiplicity $2n - 4$ and $x = -n - 2/2$ with multiplicity one. Hence the result follows.□

Theorem 2.8. *For a cocktail party graph CP_n, $cd(CP_n) = 3$ for $n \geq 2$.*

Proof. In [Safeera and Kumar, a], the authors proved that $CD[CP_n, x] = 2x^2(2+(n-2)x)$. It follows that $CD[CP_n, x]$ has *cd-roots* $x = 0$ with multiplicity 2 and $x = -2/(n-2)$ with multiplicity one. Thus $cd(CP_n) = 3$. Hence the proof follows.□

Theorem 2.9. *If $S(G)$ is a splitting graph of a graph G with n vertices, then $cd(S(G)) \geq 1$.*

Proof. Observe that $CD[S(G), x]$ do not have a constant term(see [Safeera and Kumar, a]). Hence the result□.

Theorem 2.10. *For a bull graph Bl, $cd(Bl) = 1$.*

Proof. Note that $CD[Bl, x] = x(2x^2 + x + 2)$ [Safeera and Kumar, a]. The roots of this polynomial are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$. Obviously, $x = 0$ is the only real root of $CD[Bl, x]$. Hence the result follows.□

Theorem 2.11. *For a sunlet graph Sl_n, $cd(Sl_n) = 2n - 4$ for $n \geq 3$.*

Proof. Note that $CD[Sl_n, x] = nx^{2n-4}(1 + x^2)$ [Safeera and Kumar, a]. Then the *cd-roots* are $x = 0$ with multiplicity $2n - 4$ and $x = \pm i$. Thus $cd(Sl_n) = 2n - 4$.□

Theorem 2.12. *For a tadpole graph $T_{m,n}$, $cd(T_{m,n}) = m + n - 2$ for $m \geq 3$ and $n \geq 1$.*

Proof. Note that $CD[T_{m,n}, x] = x^{m+n-4}(x^2 + (m + n - 2)x + 1)$ [Safeera and Kumar, a]. Since the discriminant of the polynomial $x^2 + (m+n-2)x+1$ is always greater than or equal to zero, it follows that $cd(T_{m,n}) = m + n - 2$. This completes the proof.□

Theorem 2.13. *For a bistar graph $B_{n,n}$ ($n \geq 1$),*

$$cd(B_{n,n}) = \begin{cases} n & \textit{if n is even} \\ n+1 & \textit{if n is odd.} \end{cases}$$

Proof. Note that $CD[B_{n,n}, x] = 2x^n(nx^n + 1)$. If n is even, then $nx^n + 1$ has only complex roots. If n is odd, then $nx^n + 1$ has only one real root and $n - 1$ complex roots. Hence ,

$$cd(B_{n,n}) = \begin{cases} n & \text{if n is even} \\ n+1 & \text{if n is odd.} \end{cases}$$

This completes the proof.□

Theorem 2.14. *For a web graph Wb_n, $cd(Wb_n) = 3n-4$ for $n \geq 3$.*

Proof. Note that $CD[Wb_n, x] = nx^{3n-5}(x^3+x+1)$. Obviously, the *cd-roots* of $CD[Wb_n, x]$ are 0 with multiplicity $3n-5$, $-0.68233, 0.34116 \pm 1.16154i$. Hence $cd(Wb_n) = 3n-4$.□

Theorem 2.15. *For a armed crown graph $C_n \bigodot P_m$,*

$$cd(C_n \bigodot P_m) = \begin{cases} n(m+1)-4, & \textit{if m=1,2} \\ n(m+1)-2, & \textit{if m} \geq \textit{3.} \end{cases}$$

Proof. Note that $CD[C_n \bigodot P_m, x] = x^2 + (m-1)x + 1$. For $m = 1, 2$, the zeros of $x^2 + (m-1)x + 1$ are complex numbers. If $m > 2$, the zeros of $x^2 + (m-1)x + 1$ are real numbers. Thus

$$cd(C_n \bigodot P_m) = \begin{cases} n(m+1)-4 & \text{if m=1,2} \\ n(m+1)-2 & \text{if m} \geq 3. \end{cases}$$

This completes the proof.□

Theorem 2.16. *For a sungraph S_n, $n \geq 3$,*

$$cd(S_n) = \begin{cases} n-2, & \textit{if n is odd} \\ n-1, & \textit{if n is even}. \end{cases}$$

Proof. Note that $CD[S_n, x] = nx^{n-2}(x^{n-1}+1)$. Since $x^{n-1}+1$ has real roots if and only if n is even. This tells us that the real *cd-roots* of $CD[S_n, x]$ are 0 and -1 if n is even. If n is odd $x = 0$ is the only real root of $CD[S_n, x]$. Therefore, we have

$$cd(S_n) = \begin{cases} n-2, & \text{if n is odd} \\ n-1, & \text{if n is even}. \end{cases}$$

This completes the proof.□

Theorem 2.17. *For a bipartite cocktaill party graph $b_n(n \geq 2)$, we have $cd(B_n) = n$.*

Proof. The result follows from the fact that $CD[B_n, x] = 2nx^n$.□

3 Location of the *cd*-roots of the some graphs

In this section, we investigate the location of the roots of some complement degree polynomials. Here need the following result [Prasolov, 2009].

Theorem 3.1. *Let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, where $a_i \in \mathbb{C}$. Then, inside the circle $|z| = 1 + max|a_i|$, there are exactly n roots of f, multiplicities counted.*

Theorem 3.2. *All the* cd-roots *of the gear graph G_n lie inside the circle with center $(0, 0)$ and radius $n + 1$.*

Proof. Observe that $CD[G_n, x] = x^n + nx^{2n-3} + nx^{2n-2}$. In this case $max|a_i| = n$, where $a_i's$ are the coefficients of $CD[G_n, x]$ for $i = 1, 2, \ldots, 2n-2$. Then by theorem 3.1, the result follows.□

Theorem 3.3. *All the* cd-roots *of the wheel graph W_n lie inside the circle with center $(0, 0)$ and radius n.*

Proof. It follows from the fact that $CD[W_n, x] = (n - 1)x^{n-4} + 1$.□

Theorem 3.4. *All the* cd-roots *of the bull graph Bl lie on the unit circle centered at the origin.*

Proof. Note that the *cd-roots* of Bl are $x = 0, \frac{-1 \pm i\sqrt{15}}{4}$. These three roots lie on the unit circle centered at the origin.□

Theorem 3.5. *All the* cd-roots *of the sunlet graph Sl_n lies in the disk $|z| \leq 1$.*

Proof. The *cd-roots* of the sunlet graph $CD[Sl_n, x]$ are $x = 0$ and $x = \pm i$. Hence the result follows. □

Theorem 3.6. *All the* cd-roots *of the sun graph S_n lies in the disk $|z| \leq 1$.*

Proof. Note that $CD[S_n, x] = nx^{2n-3} + nx^{n-2} = nx^{n-2}(x^{n-1} + 1)$. Obviously, roots of $x^{n-1} + 1$ lie on the unit circle. Hence the result.□

4 Conclusions

In this paper, we introduced cd-roots of the complement degree polynomial of some graphs. Moreover, we investigated the location of the cd- roots of some complementary degree polynomials.

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