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Common Fixed Point Theorems for $(\phi,\mathfrak{F})$-Integral Type Conractive Mapping on C^*-Algebra Valued $\mathfrak{b}$-Metrix Space

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Abstract

The object of this paper, we establish the concept of integral type of common fixed point theorem for new type of generalized C^*-valued contractive mapping. The main theorem is an existence and uniqueness of common fixed-point theorem for self-mappings with $(\phi, \mathfrak{F})$-contractive conditions on complete C^*-algebra valued $\mathfrak{b}$-metric space. Moreover, some illustrated examples are also provided.

Keywords: C^*-algebra valued, common fixed point, b-metric spaces.
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1 Introduction

In 2002, Branciari [2002] introduced the concept of integral type contraction on fixed point solution. Many writers researched at the presence of fixed points for a variety of integral type contractive mappings, see Liu et al. [2018]. Especially, Liu et al. [2014] several more fixed point theorems for integral type contractive mappings in complete metric spaces. After that Ma et al. [2014] and Ma and Jiang [2015] presented the notion of C^*-algebra-valued metric space, C^*-algebra-valued $\mathfrak{b}$-metric space and investigated certain fixed point results for self-mapping under certain contractive conditions. Alsulami et al. [2016] investigated that fixed point theorem in the classical Banach fixed point theorem can be used to produce C^*-algebra-valued $\mathfrak{b}$- metric space in fixed point results Kamran et al. [2016]. We symbolize $\mathcal{A}$ as an unital C^*-algebra, and $\mathcal{A}_h = \{\mathfrak{a} \in \mathcal{A} : \mathfrak{a} = \mathfrak{a}^*\}$. Especially, an element $\mathfrak{a} \in \mathcal{A}$ is a positive factor, if $\mathfrak{a} = \mathfrak{a}^*$. A natural partial order on $\mathcal{A}_h$ given by $\mathfrak{a} \leq \mathfrak{b}\ iff\ \theta \leq (\mathfrak{b} - \mathfrak{a})$, where q signifies the zero element in $\mathcal{A}$. Then, let $\mathcal{A}^+$ and $\mathcal{A}'$ symbolize the set $\{\mathfrak{a} \in \mathcal{A} : \theta \leq \mathfrak{a}\}$ and the set $\{\mathfrak{a} \in \mathcal{A} : \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a},\ \forall\ \mathfrak{b} \in \mathcal{A}\}$, respectively and $|\mathfrak{a}| = (\mathfrak{a}^*\mathfrak{a})^{\frac{1}{2}}$.

2 Preliminaries

Definition 2.1 (Ma and Jiang [2015]). *Let χ be a non-empty set and $\omega \in \mathcal{A}$ such that $\omega \geq I$. Suppose that the mapping $D_b : X \times X \to \mathcal{A}$ is held, the following constraints exist.*

(i) $\theta \leq D_b(\zeta,\ \eta)$ *and* $D_b(\zeta,\ \eta) = \theta$ *iff* $\zeta = \eta$;

(ii) $D_b(\zeta,\ \eta) = D_b(\eta,\ \zeta)$;

(iii) $D_b(\zeta,\ \eta) \leq \omega(D_b(\zeta,\ \vartheta) + D_b(\vartheta,\ \eta))$ *for all* $\zeta,\ \eta,\ \vartheta \in \chi$.

Then, D_b is called C^*-algebra-valued $\mathfrak{b}$-metric on X and $(\chi,\ \mathcal{A}, D_b)$ is called C^*-algebra-valued $\mathfrak{b}$-metric space.

Definition 2.2 (Ma and Jiang [2015]). *Let $(\chi,\ \mathcal{A}, D_b)$ be C^*-algebra valued b-metric space. Assume that $\{\zeta_n\}$ is a sequence in χ and $\zeta \in \chi$. If for each $\epsilon > \theta$, there exists $\mathcal{N}$ such that $\forall\, n > \mathcal{N}$, $||d\,(\zeta_n,\ \zeta)|| \leq \epsilon$ then $\{\zeta_n\}$ is alleged to be convergent with regard to $\mathcal{A}$, and $\{\zeta_n\}$ converges to ζ, i.e., we take $\lim_{n\to\infty} \zeta_n = \zeta$. If for each $\epsilon > \theta$, there exists $\mathcal{N}$ such that $\forall\, l\, n, m > \mathcal{N}$, $||d\,(\zeta_n,\ \zeta_m)|| \leq \epsilon$, then $\{\zeta_n\}$ is referred to as a Cauchy sequence in χ. $(\chi,\ \mathcal{A}, D_b)$ is referred to as a complete C^*-algebra-valued $\mathfrak{b}$-metric space if every Cauchy sequence is convergent in χ.*

Definition 2.3 (Mustafa et al. [2021]). *Let the non-decreasing function $\mathfrak{F} : \mathcal{A}^+ \to \mathcal{A}^+$ be positive linear mapping satisfying following constraints:*

(i) $\mathfrak{F}$ *is continuous;*

(ii) $\mathfrak{F}(\mathfrak{a}) = \theta$ *iff* $\mathfrak{a} = \theta$*;*

(iii) $\lim_{n\to\infty} \mathfrak{F}^n(\mathfrak{a}) = \theta$.

Definition 2.4 (Mustafa et al. [2021]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebra. $\mathcal{A}$ mapping $\mathfrak{F} :\mathcal{A}\to \mathcal{B}$ is said to be C^*-homomorphism if :*

(i) $\mathfrak{F}(\mathfrak{a}\zeta + \mathfrak{b}\eta) = \mathfrak{a}\mathfrak{F}(\zeta) + \mathfrak{b}\mathfrak{F}(\eta)$ *for all* $\mathfrak{a}, \mathfrak{b} \in C$ *and* $\zeta, \eta \in \mathcal{A}$*;*

(ii) $\mathfrak{F}(\zeta\eta) = \mathfrak{F}(\zeta)\mathfrak{F}(\eta)$ *for all* $\zeta, \eta \in \mathcal{A}$*;*

(iii) $\mathfrak{F}(\zeta^*) = \mathfrak{F}(\zeta)^*$ *for all* $\zeta \in \mathcal{A}$*;*

(iv) $\mathfrak{F}$ *maps the unit in $\mathcal{A}$ to the unit in $\mathcal{B}$.*

Lemma 2.1. *Let $(\chi, \mathcal{A}, D_b)$ be a C^*-algebra valued $\mathfrak{b}$-metric space such that $D_b(\zeta, \eta) \in \mathcal{A}$, for all $\zeta, \eta \in \chi$ where $\zeta \neq \eta$. Let $\phi : \mathcal{A}^+ \to \mathcal{A}^+$ be a function with the following properties:*

(i) $\phi(\mathfrak{a}) = \theta$ *if and only if* $\mathfrak{a} = \theta$*;*

(ii) $\phi(\mathfrak{a}) < \mathfrak{a}$*, for all* $\mathfrak{a} \in \mathcal{A}$*;*

(iii) either $\phi(\mathfrak{a}) \leq D_b(\zeta, \eta)$ *or* $D_b(\zeta, \eta) \leq \phi(\mathfrak{a})$*, where* $\mathfrak{a} \in \mathcal{A}$ *and* $\zeta, \eta \in \chi$.

Corolary 2.1. *Every C^*-homomorphism is bounded.*

Lemma 2.2. *Every $*$-homomorphism is positive.*

Definition 2.5 (Branciari [2002]). *The function $\xi : \chi \to \chi$ is called sub-additive integrable function iff* $\forall\, \mathfrak{a},\mathfrak{b} \in \chi$, $\int_0^{\mathfrak{a}+\mathfrak{b}} \xi d_t \leq \int_0^{\mathfrak{a}} \xi d_t + \int_0^{\mathfrak{b}} \xi d_t$.

3 Main Results

Definition 3.1. *Let $(\chi, \mathcal{A}, D_b)$ is a complete C^*-algebra valued $\mathfrak{b}$-metric space. Let $\mathfrak{L}, \mathfrak{M}, :\chi \to \chi$ be a integral C^*-valued contractive mapping and*

$$\mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta,\mathfrak{M}\eta)} \xi d_t\right) \leq \mathfrak{F}\left(\int_0^{\mathfrak{I}(\zeta,\eta)} \xi d_t\right) - \phi\left(\int_0^{D_b(\zeta,\eta)} \xi d_t\right) \qquad (1.1)$$

$$\mathfrak{I}(\zeta,\eta) \leq \begin{pmatrix} \alpha \int_0^{D_b(\zeta,\eta)} \xi d_t + \gamma \int_0^{[D_b(\zeta,\mathfrak{L}\zeta)+D_b(\eta,\mathfrak{M}\eta)]} \xi d_t \\ +\delta \int_0^{[D_b(\zeta,\mathfrak{M}\eta)+D_b(\eta,\mathfrak{L}\zeta)]} \xi d_t \end{pmatrix}$$

For all $\zeta, \eta \in \chi$, where $\omega \in \mathcal{A}_+^{'}$, $\alpha + \gamma + \delta \geq 0$ with $\omega\alpha + \gamma(\omega + 1) + \delta(\omega(\omega + 1)) < 1$. $\mathfrak{F} \in \Psi$ and $\phi \in \Phi$ and $\xi : \chi \to \chi$ is the Lebesgus-integral function.

Theorem 3.1. *Let $(\chi, \mathcal{A}, D_b)$ is a complete C^*-algebra valued $\mathfrak{b}$-metric space. 3.1 are $*-$homomorphisms and with the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$ and $\xi : \chi \to \chi$ is a Lebesgus-integral mapping which is summable, non-negative and such that for all $\varepsilon > 0$, $\int_0^{\varepsilon} \xi d_t > 0$. Then $\mathfrak{L}$ and $\mathfrak{M}$ have a unique common fixed point in χ.*

Proof. *Let $\zeta_0 \in \chi$ and define $\zeta_n = \mathfrak{L}\zeta_{n-1}$, $\zeta_{n+1} = \mathfrak{M}\zeta_n$ we have*

$$\mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_{n+1})} \xi d_t\right) = \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta_{n-1},\mathfrak{M}\zeta_n)} \xi d_t\right)$$
$$\leq \mathfrak{F}\left(\int_0^{I(\zeta_{n-1},\zeta_n)} \xi d_t\right) - \phi\left(\int_0^{D_b(\zeta_{n-1},\zeta_n)} \xi d_t\right)$$

$$= \mathfrak{F}\begin{pmatrix} \alpha \int_0^{D_b(\zeta,\eta)} \xi d_t + \gamma \int_0^{[D_b(\zeta,\mathfrak{L}\zeta)+D_b(\eta,\mathfrak{M}\eta)]} \xi d_t \\ +\delta \int_0^{[D_b(\zeta,\mathfrak{M}\eta)+D_b(\eta,\mathfrak{L}\zeta)]} \xi d_t \end{pmatrix} - \phi\left(\int_0^{D_b(\zeta_{n-1},\zeta_n)} \xi d_t\right)$$

$$= \begin{pmatrix} \mathfrak{F}(\alpha)\mathfrak{F}\left(\int_0^{D_b(\zeta_{n-1},\ \zeta_n)} \xi d_t\right) + \mathfrak{F}(\gamma)\mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\mathfrak{L}\zeta_{n-1})+D_b(\zeta_n,\mathfrak{M}\zeta_n)]} \xi d_t\right) \\ +\mathfrak{F}(\delta)\mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\mathfrak{M}\zeta_n)+D_b(\zeta_n,\mathfrak{L}\zeta_{n-1})]} \xi d_t\right) - \phi\left(\int_0^{D_b(\zeta_{n-1},\zeta_n)} \xi d_t\right). \end{pmatrix}$$

Therefore,

$$\| \mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_{n+1})} \xi d_t\right) \| = \| \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta_{n-1},\mathfrak{M}\zeta_n)} \xi d_t\right) \|$$

$$\leq \begin{pmatrix} \| \mathfrak{F}(\alpha)\| \| \mathfrak{F}\left(\int_0^{D_b(\zeta_{n-1},\zeta_n)} \xi d_t\right) \| \\ +\| \mathfrak{F}(\gamma)\| \| \mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\mathfrak{L}\zeta_{n-1})+D_b(\zeta_n,\mathfrak{M}\zeta_n)]} \xi d_t\right) \| \\ +\| \mathfrak{F}(\delta)\| \| \mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\mathfrak{M}\zeta_n)+D_b(\zeta_n,\mathfrak{L}\zeta_{n-1})]} \xi d_t\right) \| \\ - \| \phi\left(\int_0^{D_b(\zeta_{n-1},\zeta_n)} \xi d_t\right) \| \to 0 \; as \; n \to +\infty. \end{pmatrix}$$

Give that ϕ (2.1) and $\mathfrak{F}$ (2.4) are strongly monotone functions. We have

$$\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t = \int_0^{D_b(\mathfrak{L}\zeta_{n-1},\mathfrak{M}\zeta_n)}\xi d_t$$

$$\leq \begin{pmatrix} \alpha\int_0^{D_b(\zeta_{n-1},\zeta_n)}\xi d_t + \gamma\int_0^{[D_b(\zeta_{n-1},\mathfrak{L}\zeta_{n-1})+D_b(\zeta_n,\mathfrak{M}\zeta_n)]}\xi d_t \\ +\delta\int_0^{[D_b(\zeta_{n-1},\mathfrak{M}\zeta_n)+D_b(\zeta_n,\mathfrak{L}\zeta_{n-1})]}\xi d_t \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\int_0^{D_b(\zeta_{n-1},\zeta_n)}\xi d_t + \gamma\int_0^{[D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1})]}\xi d_t \\ +\delta\int_0^{[D_b(\zeta_{n-1},\zeta_{n+1})+D_b(\zeta_n,\zeta_n)]}\xi d_t \end{pmatrix}$$

$$\leq (\alpha+\gamma)\int_0^{D_b(\zeta_{n-1},\,\zeta_n)}\xi d_t + \gamma\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t + \delta\int_0^{D_b(\zeta_{n-1},\zeta_{n+1})}\xi d_t$$

$$\leq (\alpha+\gamma)\int_0^{D_b(\zeta_{n-1},\,\zeta_n)}\xi d_t + \gamma\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t + \omega\delta\int_0^{(D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1}))}\xi d_t$$

$$\leq (\alpha+\gamma+\omega\delta\,)\int_0^{D_b(\zeta_{n-1},\,\zeta_n)}\xi d_t + (\gamma+\omega\delta)\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t.$$

This implies that

$$\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t \leq \frac{\alpha+\gamma+\omega\delta}{\gamma+\omega\delta}\int_0^{D_b(\zeta_n,\zeta_{n-1})}\xi d_t$$

$$\int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t \leq h\int_0^{D_b(\zeta_n,\zeta_{n-1})}\xi d_t$$

where, $h = \frac{\alpha+\gamma+\omega\delta}{\gamma+\omega\delta} < 1$. *Thus, we have*

$$\parallel \int_0^{D_b(\zeta_{n-1},\zeta_n)}\xi d_t \parallel \parallel \int_0^{D_b(\zeta_n,\zeta_{n+1})}\xi d_t \parallel \leq \parallel h \parallel \parallel \int_0^{D_b(\zeta_n,\zeta_{n-1})}\xi d_t \parallel \to 0,$$

as$n, m \to +\infty$. *If* $n > m$

$$\int_0^{D_b(\zeta_n,\zeta_m)}\xi d_t \leq \begin{pmatrix} \omega\int_0^{D_b(\zeta_n,\zeta_{n-1})}\xi d_t + \omega^2\int_0^{D_b(\zeta_{n-1},\zeta_{n-2})}\xi d_t \\ +\ \ldots\ +\omega^{n-m}\int_0^{D_b(\zeta_{m-1},\zeta_m)}\xi d_t \end{pmatrix}.$$

Applying the constraint of theorem then,

$$\mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_m)}\xi d_t\right) \leq \begin{pmatrix} \mathfrak{F}\left(\omega\int_0^{D_b(\zeta_n,\zeta_{n-1})}\xi d_t\right) + \mathfrak{F}\left(\omega^2\int_0^{D_b(\zeta_{n-1},\zeta_{n-2})}\xi d_t\right) \\ +\ldots+\mathfrak{F}\left(\omega^{n-m}\int_0^{D_b(\zeta_{m-1},\zeta_m)}\xi d_t\right) \end{pmatrix}$$

$$\leq \begin{pmatrix} \mathfrak{F}(\omega)\mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t\right) + \mathfrak{F}\left(\omega^2\right)\mathfrak{F}\left(\int_0^{D_b(\zeta_{n-1},\zeta_{n-2})} \xi d_t\right) \\ + \ldots + \mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}\left(\int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t\right) \end{pmatrix}$$

$$\leq \begin{pmatrix} \mathfrak{F}\left(\omega \int_0^{\mathfrak{I}(\zeta_n,\zeta_{n-1})} \xi d_t\right) - \phi\left(\omega \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t\right) \\ + \mathfrak{F}\left(\omega^2 \int_0^{\mathfrak{I}(\zeta_{n-1},\zeta_{n-2})} \xi d_t\right) - \phi\left(\omega^2 \int_0^{D_b(\zeta_{n-1},\zeta_{n-2})} \xi d_t\right) \\ + \ldots + \mathfrak{F}\left(\omega^{n-m} \int_0^{\mathfrak{I}(\zeta_{m-1},\zeta_m)} \xi d_t\right) - \phi\left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t\right) \end{pmatrix}$$

$$= \begin{pmatrix} \mathfrak{F}\begin{pmatrix} \alpha\omega \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t \\ + \gamma\omega \int_0^{[D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1})]} \xi d_t \\ + \delta\omega \int_0^{[D_b(\zeta_{n-1},\zeta_{n+1})+D_b(\zeta_n,\zeta_n)]} \xi d_t \end{pmatrix} \\ -\phi\left(\omega \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t\right) + \ldots \\ + \mathfrak{F}\begin{pmatrix} \alpha\omega^{n-m} \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t \\ + \gamma\omega^{n-m} \int_0^{[D_b(\zeta_m,\zeta_{m-1})+D_b(\zeta_{m-1},\zeta_{m-2})]} \xi d_t \\ + \delta\omega^{n-m} \int_0^{[D_b(\zeta_m,\zeta_{m-2})+D_b(\zeta_{m-1},\zeta_{m-1})]} \xi d_t \end{pmatrix} \\ -\phi\left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t\right) \end{pmatrix}$$

Therefore

$$\begin{aligned} \mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_m)} \xi d_t\right) &= \mathfrak{F}(\alpha)\mathfrak{F}(\omega)\mathfrak{F}\left(\int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t\right) \\ &+ \mathfrak{F}(\gamma)\mathfrak{F}(\omega)\mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1})]} \xi d_t\right) \\ &+ \mathfrak{F}(\delta)\mathfrak{F}(\omega)\mathfrak{F}\left(\int_0^{[D_b(\zeta_{n-1},\zeta_{n+1})+D_b(\zeta_n,\zeta_n)]} \xi d_t\right) \\ &- \phi\left(\omega \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t\right) + \ldots \\ &+ \mathfrak{F}(\alpha)\mathfrak{F}\left(\omega^{n-m}\right)\mathfrak{F}\left(\int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t\right) \\ &+ \mathfrak{F}(\gamma)\mathfrak{F}\left(\omega^{n-m}\right)\mathfrak{F}\left(\int_0^{[D_b(\zeta_m,\zeta_{m-1})+D_b(\zeta_{m-1},\zeta_{m-2})]} \xi d_t\right) \\ &+ \mathfrak{F}(\delta)\mathfrak{F}\left(\omega^{n-m}\right)\mathfrak{F}\left(\int_0^{[D_b(\zeta_m,\zeta_{m-2})+D_b(\zeta_{m-1},\zeta_{m-1})]} \xi d_t\right) \\ &- \phi\left(\omega^{n-m} \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t\right). \end{aligned}$$

Since the property of ϕ (2.1) and $\mathfrak{F}$ (2.4) is strongly monotone, we have

$$\begin{aligned}\int_0^{D_b(\zeta_n,\zeta_m)} \xi d_t \quad & \leq \alpha\omega \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t + \gamma\omega \int_0^{[D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1})]} \xi d_t \\ & +\delta\omega \int_0^{[D_b(\zeta_{n-1},\zeta_{n+1})+D_b(\zeta_n,\zeta_n)]} \xi d_t + \ \ldots \\ & +\alpha\omega^{n-m} \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t \\ & +\gamma\omega^{n-m} \int_0^{[D_b(\zeta_m,\zeta_{m-1})+D_b(\zeta_{m-1},\zeta_{m-2})]} \xi d_t \\ & +\delta\omega^{n-m} \int^{[D_b(\zeta_m,\zeta_{m-2})+D_b(\zeta_{m-1},\zeta_{m-1})]} \xi d_t\end{aligned}$$

so we get

$$\int_0^{D_b(\zeta_n,\zeta_m)} \xi d_t \leq \begin{pmatrix} \| \alpha \|\| \omega \|\| \int_0^{D_b(\zeta_n,\zeta_{n-1})} \xi d_t \| \\ + \| \gamma \|\| \omega \|\| \int_0^{[D_b(\zeta_{n-1},\zeta_n)+D_b(\zeta_n,\zeta_{n+1})]} \xi d_t \ \| \\ + \| \delta \|\| \omega \|\| \int_0^{[D_b(\zeta_{n-1},\zeta_{n+1})+D_b(\zeta_n,\zeta_n)]} \xi d_t \| + \ \ldots \\ + \| \alpha \|\| \omega^{n-m} \|\| \int_0^{D_b(\zeta_{m-1},\zeta_m)} \xi d_t \| \\ + \| \gamma \|\| \omega^{n-m} \|\| \int_0^{[D_b(\zeta_m,\zeta_{m-1})+D_b(\zeta_{m-1},\zeta_{m-2})]} \xi d_t \| \\ + \| \delta \|\| \omega^{n-m} \|\| \int_0^{[D_b(\zeta_m,\zeta_{m-2})+D_b(\zeta_{m-1},\zeta_{m-1})]} \xi d_t \| \end{pmatrix} \to 0,$$

as $n.m \to +\infty$.

Then $\{\zeta_n\}$ is Cauchy sequence. Since $(\chi, \mathcal{A}, D_b)$ is a complete C^*-algebra valued b-metric space there exists $u \in \chi$ such that $\zeta_n \to u$ as $n \to \infty$. Now since

$$\begin{aligned}\int_0^{D_b(u,\mathfrak{M}u)} \xi d_t & \leq \omega \left[\int_0^{D_b(u,\zeta_{n+1})} \xi d_t + \int_0^{D_b(\zeta_{n+1},\mathfrak{M}u)} \xi d_t \right] \\ & = \omega \left[\int_0^{D_b(\zeta_{n+1},\mathfrak{M}u)} \xi d_t + \int_0^{D_b(u,\zeta_{n+1})} \xi d_t \right] \\ & = \omega \left[\int_0^{D_b(\mathfrak{L}\zeta_n,\mathfrak{M}u)} \xi d_t + \int_0^{D_b(u,\zeta_{n+1})} \xi d_t \right]\end{aligned}$$

$$\mathfrak{F}\left(\int_0^{D_b(u,\mathfrak{M}u)} \xi d_t \right) = \omega \left[\mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta_n,\mathfrak{M}u)} \xi d_t \right) + \mathfrak{F}\left(\int_0^{D_b(u,\zeta_{n+1})} \xi d_t \right) \right]$$

$$\leq \omega \left[\mathfrak{F}\left(\int_0^{\Im(\zeta_n,u)} \xi d_t \right) - \phi\left(\int_0^{D_b(\zeta_n,u)} \xi d_t \right) \right] + \omega \left[\mathfrak{F}\left(\int_0^{D_b(u,\zeta_{n+1})} \xi d_t \right) \right]$$

$$\| \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta_n,\mathfrak{M}u)} \xi d_t\right) \| \leq \begin{pmatrix} \| \omega \| \| \mathfrak{F}\left(\int_0^{D_b(u,\zeta_{n+1})} \xi d_t\right) \| \\ + \| \omega \| \| \mathfrak{F}\alpha \| \| \int_0^{D_b(\zeta_n,u)} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F}\gamma \| \| \int_0^{[D_b(\zeta_n,\zeta_{n+1})+D_b(u,\mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F}\delta \| \| \int_0^{[D_b(\zeta_n,\mathfrak{M}u)+D_b(u,\mathfrak{L}\zeta_n)]} \xi d_t \| \\ - \| \omega \| \| \phi\left(\int_0^{D_b(\zeta_n,u)} \xi d_t\right) \| . \end{pmatrix}$$

Using the property of ϕ (2.1), we get

$$\| \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}\zeta_n,\mathfrak{M}u)} \xi d_t\right) \| \leq \begin{pmatrix} \| \omega \| \| \mathfrak{F}\left(\int_0^{D_b(u,\zeta_{n+1})} \xi d_t\right) \| \\ + \| \omega \| \| \mathfrak{F}\alpha \| \| \int_0^{D_b(\zeta_n,u)} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F}\gamma \| \| \int_0^{[D_b(\zeta_n,\zeta_{n+1})+D_b(u,\mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \mathfrak{F}\delta \| \| \int_0^{[D_b(\zeta_n,\mathfrak{M}u)+D_b(u,\mathfrak{L}\zeta_n)]} \xi d_t \| \end{pmatrix}.$$

Where $\mathfrak{F}$ (2.4) is strongly monotone, then

$$\| \int_0^{(D_b(\mathfrak{L}\zeta_n,\mathfrak{M}u))} \xi d_t \| \leq \begin{pmatrix} \| \omega \| \| \int_0^{(D_b(u,\zeta_{n+1}))} \xi d_t \| \\ + \| \omega \| \| \alpha \| \| \int_0^{D_b(\zeta_n,u)} \xi d_t \| \\ + \| \omega \| \| \gamma \| \| \int_0^{[D_b(\zeta_n,\zeta_{n+1})+D_b(u,\mathfrak{M}u)]} \xi d_t \| \\ + \| \omega \| \| \delta \| \| \int_0^{[D_b(\zeta_n,\mathfrak{M}u)+D_b(u,\mathfrak{L}\zeta_n)]} \xi d_t \| \end{pmatrix}$$

$$= \| \omega \| \| \int_0^{(D_b(u,\zeta_{n+1}))} \xi d_t \| + \| \omega \| \begin{bmatrix} \| \alpha \| \| \int_0^{D_b(\zeta_n,u)} \xi d_t \| \\ + \| \gamma \| \| \int_0^{[D_b(\zeta_n,\zeta_{n+1})+D_b(u,\mathfrak{M}u)]} \xi d_t \| \\ + \| \delta \| \| \int_0^{[D_b(\zeta_n,\mathfrak{M}u)+D_b(u,\zeta_{n+1})]} \xi d_t \| \end{bmatrix}$$

as $\zeta_n \to u$ and $\zeta_{n+1} \to u$ as $n \to \infty$, we get

$$\| 1-\omega\gamma-\omega\delta \| \| \int_0^{D_b(u,\mathfrak{M}u)} \xi d_t \| \leq \begin{bmatrix} \| \omega \| \| \alpha \| \| \int_0^{D_b(\zeta_n,u)} \xi d_t \| \\ + \| \omega \| \| 1+\delta \| \| \int_0^{D_b(u,\zeta_{n+1})} \xi d_t \| \end{bmatrix} \to 0$$

$as\ n \to \infty.$

Hence $\| \int_0^{D_b(\mathfrak{M}u,u)} \xi d_t \| = 0$ *since* $\| 1-\omega\gamma-\omega\delta \| > 0$. *As a result,* $\mathfrak{M}u = u$ *that is* u *is a fixed point of* $\mathfrak{M}$. *Similarly we are able to demonstrate that* $\mathfrak{L}u = u$. *Hence* $\mathfrak{L}u = \mathfrak{M}u = u$. *This demonstrates that* u *is common fixed point of* $\mathfrak{L}$ *and* $\mathfrak{M}$.

Let v *be a different fixed point common to* $\mathfrak{L}$ *and* $\mathfrak{M}$. *(i.e)* $\mathfrak{L}v = \mathfrak{M}v = v$ *such that* $u \neq v$ *we have* $\int_0^{D_b(u,v)} \xi d_t = \int_0^{D_b(\mathfrak{L}u,\mathfrak{M}v)} \xi d_t$ *then*

$$\begin{aligned}\mathfrak{F}\left(\int_0^{D_b(u,v)} \xi d_t\right) &= \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}u,\mathfrak{M}v)} \xi d_t\right)\\ &\leq \mathfrak{F}\left(\int_0^{\mathfrak{I}(u,v)} \xi d_t\right) - \phi\left(\int_0^{D_b(u,v)} \xi d_t\right)\end{aligned}$$

$$\| \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}u,\mathfrak{M}v)} \xi d_t\right) \| \leq \begin{pmatrix} \| \mathfrak{F}\alpha \| \| \int_0^{D_b(u,v)} \xi d_t \| \\ + \| \mathfrak{F}\gamma \| \| \int_0^{[D_b(u,\mathfrak{L}u)+D_b(v,\mathfrak{M}v)]} \xi d_t \| \\ + \| \mathfrak{F}\delta \| \| \int_0^{[D_b(u,\mathfrak{M}v)+D_b(v,\mathfrak{L}u)]} \xi d_t \| \\ - \| \phi\left(\int_0^{D_b(u,v)} \xi d_t\right) \| \end{pmatrix}$$

Using the property of ϕ *(2.1), we get*

$$\| \mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}u,\mathfrak{M}v)} \xi d_t\right) \| \leq \begin{pmatrix} \| \mathfrak{F}\alpha \| \| \int_0^{D_b(u,v)} \xi d_t \| \\ + \| \mathfrak{F}\gamma \| \| \int_0^{[D_b(u,\mathfrak{L}u)+D_b(v,\mathfrak{M}v)]} \xi d_t \| \\ + \| \mathfrak{F}\delta \| \| \int_0^{[D_b(u,\mathfrak{M}v)+D_b(v,\mathfrak{L}u)]} \xi d_t \| \end{pmatrix}$$

where $\mathfrak{F}$ *(2.4) is strongly monotone, then*

$$\begin{aligned}\| \int_0^{(D_b(\mathfrak{L}u,\mathfrak{M}v))} \xi d_t \| &\leq \begin{pmatrix} \| \alpha \| \| \int_0^{D_b(u,v)} \xi d_t \| \\ + \| \gamma \| \| \int_0^{[D_b(u,\mathfrak{L}u)+D_b(v,\mathfrak{M}v)]} \xi d_t \| \\ + \| \delta \| \| \int_0^{[D_b(u,\mathfrak{M}v)+D_b(v,\mathfrak{L}u)]} \xi d_t \| \end{pmatrix}\\ &\leq \| \alpha + 2\delta \| \| \int_0^{D_b(u,v)} \xi d_t \| \\ &\leq \| \omega\alpha + (\omega+1)\gamma + \omega(\omega+1)\delta \| \| \int_0^{D_b(u,v)} \xi d_t \| \\ &< \| \int_0^{D_b(u,v)} \xi d_t \| \end{aligned}.$$

Which is a contradiction. Hence $\| \int^{D_b(u,v)} \xi d_t \| = 0$ *and* $u = v$. *Thus* u *is a unique common fixed point of* $\mathfrak{L}$ *and* $\mathfrak{M}$.

Corolary 3.1. *Let* $(\chi, \mathcal{A}, D_b)$ *is a complete* C^**-algebra valued* $\mathfrak{b}$*-metric space. Let* $\mathfrak{L} : \chi \to \chi$ *be a contractive mapping and*

$$\mathfrak{F}\left(\int_0^{D_b(\mathfrak{L}^n\zeta,\mathfrak{L}^n\eta)} \xi d_t\right) \leq \mathfrak{F}\left(\int_0^{\mathfrak{I}(\zeta,\eta)} \xi d_t\right) - \phi\left(\int_0^{D_b(\zeta,\eta)} \xi d_t\right)$$

$$\mathfrak{I}(\zeta,\eta) \leq \begin{pmatrix} \alpha \int_0^{D_b(\zeta,\eta)} \xi d_t + \beta \int_0^{\frac{[1+D_b(\zeta,\mathfrak{L}^n\zeta)]D_b(\eta,\mathfrak{L}^n\eta)}{1+D_b(\zeta,\eta)}} \xi d_t \\ +\gamma \int_0^{[D_b(\zeta,\mathfrak{L}^n\zeta)+D_b(\eta,\mathfrak{L}^n\eta)]} \xi d_t \\ +\delta \int_0^{[D_b(\zeta,\mathfrak{L}^n\eta)+D_b(\eta,\mathfrak{L}^n\zeta)]} \xi d_t \end{pmatrix}$$

for all $\zeta,\eta \in \chi$, where $\omega \in \mathcal{A}'_+$, $\alpha+\beta+\gamma+\delta \geq 0$ with $\omega\alpha+\beta+\gamma(\omega+1)+\delta(\omega(\omega+1)) < 1$. $\mathfrak{F}$ and ϕ are $*-$homomorphisms and with the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$ and $\xi : \chi \to \chi$ is a Lebesgus-integral mapping which is summable, non-negative and such that for all $\varepsilon > 0$, $\int_0^{\varepsilon} \xi d_t > 0$. Then $\mathfrak{L}$ have a unique fixed point in χ.

Example 3.1. *Let $\chi = [0,1]$ and $\mathcal{A} = \mathcal{R}^2$ with a norm $\parallel \zeta \parallel = |\zeta|$ be a real C^*-algebra. We define $\mathfrak{p} = \{(\zeta,\eta) \in \mathcal{R}^2 \,:\, \zeta \geq 0,\ \eta \geq 0\}$. The partial order $\leq$ with respect to the C^*-algebra $\mathcal{R}^2$. $\zeta_1 \leq \zeta_2$ and $\eta_1 \leq \eta_2$ $for\ all$ $(\zeta_1,\eta_1),(\zeta_2,\eta_2) \in \mathcal{R}^2$. Let $D_b : \chi\times\chi \to \mathcal{R}^2$ suppose that $D_b(\zeta,\eta) = 2(|\zeta-\eta|,\ |\zeta-\eta|)$ $for\ \zeta,\eta \in \chi$.*

Then, $(\chi,\mathcal{A},D_b)$ is a C^*-algebra valued $\mathfrak{b}$-metric space where $\omega = 1$ in theorem 3.1.

Let $\mathfrak{F},\phi : \mathfrak{p} \to \mathfrak{p}$ be the mappings defined as follows: For $\mathcal{T} = (\zeta,\eta) \in \mathfrak{p}$

$$\mathfrak{F}(\mathcal{T}) = \begin{cases} (\zeta,\eta), & if\ \zeta \leq 1\ and\ \eta \leq 1, \\ (\zeta^2,\eta), & if\ \zeta > 1\ and\ \eta \leq 1, \\ (\zeta,\eta^2), & if\ \zeta \leq 1\ and\ \eta > 1, \\ (\zeta^2,\eta^2), & if\ \zeta > 1\ and\ \eta > 1. \end{cases}$$

and for $\mathcal{S} = (\mathcal{S}_1,\mathcal{S}_2) \in \mathfrak{p}$ with $\mathcal{V} = \min\{\mathcal{S}_1,\mathcal{S}_2\}$,

$$\phi(\mathcal{S}) = \begin{cases} \left(\frac{\mathcal{V}^2}{2},\frac{\mathcal{V}^2}{2}\right), & if\ \mathcal{V} \leq 1 \\ \left(\frac{1}{2},\frac{1}{2}\right), & if\ \mathcal{V} > 2 \end{cases}$$

Then, $\mathfrak{F}$ and ϕ have the properties mentioned in (2.4) and (2.1). Let $\mathfrak{L},\mathfrak{M} : \chi \to \chi$ be defined as follows:

$$\mathfrak{L}(\zeta) = \begin{cases} \frac{1}{32}, & if\ 0 \leq \zeta \leq \frac{1}{2} \\ 0, & if \frac{1}{2} < \zeta \leq 1 \end{cases} \;;\; M(\zeta) = \frac{1}{32},\ for\ \zeta \in \chi$$

Then, $\mathfrak{L}$ and $\mathfrak{M}$ have the required properties mentioned in theorem 3.1.
Let $\alpha = \frac{1}{16}$, $\beta = 0$, $\gamma = \frac{1}{64}$ and $\delta = \frac{1}{64}$. It can be verified that:

$$\mathfrak{F}(D_b(\mathfrak{L}\zeta,\mathfrak{M}\eta)) \leq \mathfrak{F}(N(\zeta,\eta)) - \phi(D_b(\zeta,\eta)),\ \forall\ \zeta,\eta \in \chi\ with\ \eta \leq \zeta$$

. Hence, Theorem 3.1 is satisfied. Then demonstrate that 0 is a unique common fixed point of $\mathfrak{L}$ and M.

4 Conclusions

In Theorem 3.1 we have formulated a contractive conditions to modify and extend the concept of common fixed point theorem for C^*-algebra valued b-metric space via $(\phi, \mathfrak{F})$-integral type contractive mapping. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the contractive condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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