The Upper and Forcing Fault Tolerant Geodetic Number of a Graph

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Abstract

A fault tolerant geodetic $S \subseteq V$ is said to be minimal fault tolerant geodetic set of G if no proper subset of S is a fault tolerant geodetic set of G is called the upper fault tolerant geodetic number of G is denoted by $\gamma_{gft}^{+}(G)$. Some general properties satisfied by this concept are studied. For connected graphs of order $n \geq 3$ with $\gamma_{gft}^{+}(G)$to be $n-1$ is given. It is shown that for every pair of $a, b$with $5 \leq a < b$, there exists a connected graph Gsuch that $\gamma_{gft}(G) = a$ and $\gamma_{gft}^{+}(G) = b$, where $\gamma_{gft}(G)$ is the fault tolerant geodetic number of G and $\gamma_{gft}^{+}(G)$ is the upper fault tolerant geodetic number of a graph. Let S be a gft-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique gft-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing fault tolerant geodetic number of S, denoted by$f_{gft}(S)$, is the cardinality of a minimum forcing subset of S. The forcing fault tolerant geodetic number of G, denoted by $f_{gft}(G)$, is $f_{gft}(G) = min\{f_{gft}(S)\}$, where the minimum is taken over all f_{gft}-sets in G. The forcing fault tolerant geodetic number of some standard graphs are determined. Some of its general properties are studied. It is shown that for every pair of positive integers a and b with $0 \leq a \leq b, b \geq 2$ and $b \geq 2a$, there exists a connected graph G such that $f_{gft}(G) = a$ and$g_{ft}(G) = b$.

Keywords: tolerant geodetic, connected graphs, minimum cardinality.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [9]. Two vertices u and v of said to be adjacent in G if $uv \in E(G)$. The neighborhood $N(v)$ of the vertex $v$in G is the set of vertices adjacent to v. The degree of the vertex v is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph $G$with $deg(u) = 1$ and $deg(v) > 1$, then we call e an end edge, u a leaf and v a support vertex. For any connected graph G, a vertex $v \in V(G)$ is called a cut vertex of G if $V(G) - v$ is disconnected. The subgraph induced by set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is called an extreme vertex of G if $\langle N(v) \rangle$ is *complete*.

A vertex x is an internal vertex of an $u - v$path P if x is a vertex of P and $x \neq u, v$. An edge e of G is an internal edge of an $u - v$ path P if e is an edge of P with both of its ends or in P. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An$u - v$ path of length $d(u, v)$ is called an *$u - v$geodesic*. A vertex x is said to lie on an $u - v$ geodesic P if x isa vertex of P including the vertices $u$and v. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $rad(G)$ and the maximum eccentricity is its diameter, $diam(G)$. We denote $rad(G)$ by r and $diam(G)$ by d. The closed interval $I[u, v]$ consists of u, v and all vertices lying on some $u - v$ geodesic of G. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \bigcup_{u,v \in S} I[u, v]$ is the closure of S. A set $S \subseteq V(G)$ is called a geodetic set if $I[S] = V(G)$. Thus every vertex of G is contained in a geodesic joining some pair of vertices in S. The minimum cardinality of a geodetic set of G is called the *geodetic number* of G and is denoted by $g(G)$. A geodetic set of minimum cardinalities is called g-set of G. For references on geodetic parameters in graphs see [4, 5, 6, 7, 10]. Let S be a geodetic set of G. Wbe the set of extreme vertices of G. Then S is said to be a fault tolerant geodetic set of G, if $S - \{v\}$ is also a geodetic set of G for every $v \in S \setminus W$. The minimum cardinality of a fault tolerant geodetic set is called fault tolerant geodetic number and is denoted by $g_{ft}(G)$. The minimum fault tolerant geodetic dominating set of G is denoted by g_{ft}-set of G.The following theorem is used in the sequel.

Theorem 1.1. [6] Each extreme vertex of a connected graph G belongs to every geodetic set of G.

2. The Upper Fault Tolerant Geodetic Number of a Graph

Definition 2.1. A fault tolerant geodetic $S \subseteq V$ is said to be minimal fault tolerant geodetic set of G if no proper subset of S is a fault tolerant geodetic set of G is called the upper fault tolerant geodetic number of G is denoted by $\gamma_{gft}^{+}(G)$.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_5, v_6\}$ is a γ_{gft}-setof G so that $\gamma_{gft}(G) = 5$. Let $S_1 = \{v_1, v_5, v_6, v_7, v_8, v_9\}$. Then S_1 is a minimal faulttolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq 6$.It is easily verified that there is no faulttolerant geodetic set of G with cardinality more than six. Therefore $\gamma_{gft}^{+}(G) = 6$.

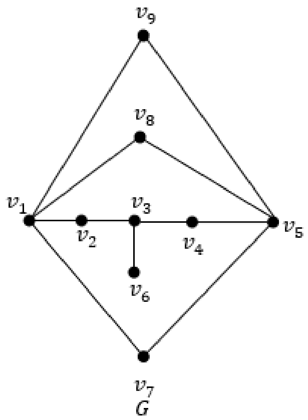

Figure 2.1

Observation 2.3. (i) For a connected graph of order $n \geq 2, 2 \leq \gamma_{gft}(G) \leq \gamma_{gft}^{+}(G) \leq n$.
(ii) No cut vertex of G belongs to any minimal fault tolerant geodetic set of G.
(iii) Each extreme vertex of G belong to any minimal fault tolerant geodetic set of G.

Theorem 2.4. For the complete graph $G = K_n n \geq 2, \gamma_{gft}^{+}(G) = n$.
Proof: This follows from Observation 2.3(iii). ■

Theorem 2.5. For any non-trivial tree, $\gamma_{gft}^{+}(G) =$ number of end vertices
Proof: This follows from Observation 2.3(ii) and (iii). ■

Theorem 2.6. For the cycle $G = C_n$ $(n \geq 4), \gamma_{gft}^{+}(G) = \begin{cases} 4 & ifnisodd \\ 5 & ifniseven \end{cases}$
Proof: Let n be even. Let v be the antipodal vertex of u and y be the antipodal vertex of x, where $u \neq x$. Then $S = \{u, v, x, y\}$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq 4$. We prove that $\gamma_{gft}^{+}(G)$= 4. On the contrary, suppose that$\gamma_{gft}^{+}(G) \geq 5$.

Then there exists a geodetic set S' such that $|S'| \geq 5$. Then there exist at least two pair of antipodal vertices of G. Hence it follows that $S \subset S'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G)$= 4.
Let n be odd. Let u and x be two adjacent vertices of G. Let v and w be antipodal vertices of $u, w$and y be two antipodal vertices of x. Then $M = \{u, v, w, x, y\}$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G)$≥ 5. We prove that $\gamma_{gft}^{+}(G)$= 5. On the contrary, suppose that $\gamma_{gft}^{+}(G) \geq 6$. Then there exists a fault tolerant geodetic set M' of G such that $|M'| \geq 6$.ThenM' contains two pair of antipodal vertices. Which implies $M \subset M'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G)$= 5. ■

Theorem 2.7. Let G be the complete bipartite graph $K_{r,s}, (2 \leq r \leq s), \gamma_{gft}^{+}(G)$ $= s + 2$.
Proof: Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the two bipartite sets of$G$. Let $S = Y \cup \{x_i, x_j\}$, where $2 \leq i \neq j \leq r$is a fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq s + 2$. We prove that $\gamma_{gft}^{+}(G)$= $s + 2$. On the contrary, suppose that $\gamma_{gft}^{+}(G)$≥ $s + 3$. Then there exists a fault tolerant geodetic set of G such that $|S'| \geq s + 3$. Which implies $S \subseteq S'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G) = s + 2$.■

Theorem 2.8. For every pair of $a, b$with $5 \leq a < b$, there exists a connected graph$G$ such that $\gamma_{gft}(G) = a$ and $\gamma_{gft}^{+}(G) = b$.
Proof: Let $P: v_1, v_2, v_3, v_4, v_5$ be a path on five vertices and $V(K_{b-a-2}) = \{z_1, z_2, \dots, z_{b-a-2}\}$. Let H be a graph obtained from P and $V(K_{b-a-2})$ by joining each z_i $(1 \leq i \leq b - a + 2)$ with $v_1$and v_5. Let G be the graph obtained from H by introducing the vertices $h_1, h_2, \dots, h_{a-4}$ and join each h_i $(1 \leq i \leq a - 4)$with v_3. The graph G is shown in Figure 2.2.
First, we prove that $\gamma_{gft}(G) = a$. Let $X = \{h_1, h_2, \dots, h_{a-4}\}$ be the set of end vertices of G. By Observation 2.3(iii), X is a subset of every fault tolerant geodetic set of G. It is easily verified that there is no fault tolerant geodetic set of cardinality less than a and so $\gamma_{gft}(G) \geq a$. Let $S = X \cup \{v_1, v_2, v_4, v_5\}$ is a fault tolerant geodetic set of G so that $\gamma_{gft}(G) = a$.
Next, we prove that $\gamma_{gft}^{+}(G) = b$. Let $M = X \cup \{z_1, z_2, \dots, z_{b-a-2}, v_1, v_4\}$. Then$M$ is a minimal fault tolerant geodetic set of G and so $\gamma_{gft}^{+}(G) \geq b$. We prove that$\gamma_{gft}^{+}(G) = b$. On the contrary, suppose that $\gamma_{gft}^{+}(G) \geq b + 1$. Then there existsa fault tolerant geodetic set $M'$of G such that $|M'| \geq b + 1$. Since $v_3 \notin M'$ and $|V(G)| = b + 3$, it follows that either $S \subset M'$ or $M \subset M'$, which is a contradiction. Therefore $\gamma_{gft}^{+}(G) = b$.■

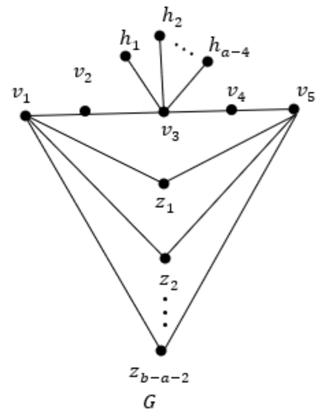

Figure 2.2

3. The Forcing Fault Tolerant Geodetic Number of a Graph

Definition 3.1. Let S be a g_{ft}-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique f_{gft}-set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing fault tolerant geodetic number of S, denoted by $f_{gft}(S)$, is the cardinality of a minimum forcing subset of S. The forcing fault tolerant geodetic number of G, denoted by $f_{gft}(G)$, is $f_{gft}(G) = min\{f_{gft}(S)\}$, where the minimum is taken over all f_{gft}- sets in G.

Example 3.2. For the graph G given in Figure 3.1, $S_1 = \{v_1, v_3, v_4, v_5, v_6\}, S_2 = \{v_1, v_2, v_4, v_5, v_7\}$are the only two g_{ft}-sets of G so that $f_{gft}(S_1) = f_{gft}(S_2) = 1$ so that $g_{ft}(G) = 1$ and $f_{gft}(G) = 1$.

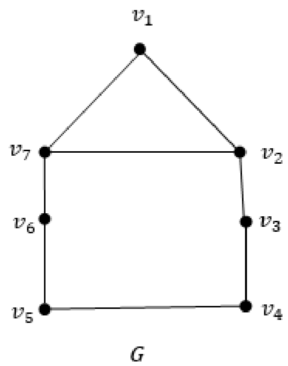

Figure 3.1

The next theorem follows immediately from the definition of the forcing fault tolerant geodetic number of the graph.

Theorem 3.3. For any connected graph $G, 0 \leq f_{gft}(G) \leq g_{ft}(G)$.
In the following we determine the forcing fault tolerant geodetic number of some standard graphs.

Theorem 3.4. For a non-trivial tree $T, f_{gft}(G) = 0$.
Proof: Since for a tree T, the set of end vertices of G is the unique g_{ft}-set of G. Hence it follows that $f_{gft}(G)$ = 0. ■

Theorem 3.5. For the complete graph $G = K_n$ $(n \geq 3), f_{gft}(G) = 0$.
Proof: Since $S = V(G)$ is the unique g_{ft}-set of G, $f_{gft}(G) = 0$.■

Theorem 3.6. For the cycle $G = C_n$ $(n \geq 4), f_{gft}(G)$ = $\begin{cases} 0 & if n = 4\ or\ 5 \\ 1 & if n is even and n \geq 6 \\ 3 & if n is odd and n \geq 7 \end{cases}$

Proof: Let $V(C_n) = \{v_1, v_2, \ldots, v_n, v_1\}$.
Case 1. Let n is even. For $n = 4$, $S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$ = 0. So, let $n \geq 6$. let$n = 2k$ $(k \geq 3)$. Let $S_1 = \{x, y, u, v\}$ be a g_{ft}-set of G, where y is the antipodal vertex of x and u is the antipodal vertex of u. Since $n \geq 6$, g_{ft}-set of G is not unique and so $f_{gft}(G)$= 1. Since S_1 is the unique g_{ft}-set of Gcontaining S_1, $f_{gft}(S_1)$ = 1 so that $f_{gft}(G)$= 1.
Case 2. Let n is odd. For $n = 5, S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0. So, let $n \geq 7$. let$n = 2k + 1$ $(k \geq 3)$. Then it is easily verified that no singleton or two element subsets of a g_{ft}-set S_1 is not a forcing subset of S_2. Let S_2 = $\{v_1, v_{k+1}, v_{k+2}\} \cup \{v_2, v_{k+3}\}$. Then S_2 is the g_{ft}-set of G containing $\{v_1, v_{k+1}, v_{k+2}\}$. Therefore $f_{gft}(G)$= 3. ■

Theorem 3.7. For the fan graph $G = K_1 + P_{n-1}$ $(n \geq 4), f_{gft}(G)$ = 0.
Proof: Let $V(K_1) = \{x\}$ and $S = V(G) - \{x\}$ is the unique g_{ft}-set of G so that $f_{gft}(G)$ = 0. ■

Theorem 3.8. For the wheel graph $G = K_1 + C_{n-1}$ $(n \geq 4)$, $f_{gft}(G)$= 0.
Proof: Let $V(K_1) = \{x\}$ and $S = V(G) - \{x\}$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0. ■

Theorem 3.9. For the complete bipartite graph $G = K_{m,n}$ $(2 \leq r \geq s)$,
$f_{gft}(G)$= $\begin{cases} 0 & 2 \leq r \leq s \leq 3 \\ 3 & r = 3, s \geq 4 \\ 4 & 4 \leq r \leq s \end{cases}$
Proof: Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the two partite sets of G. Let $S = 2$. Then $G = C_4$. By Theorem, $f_{gft}(G)$= 0.

For $r = 2$and $s \geq 3$. Then $S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0. So let $r = 3, s = 3$. Then $S = V(G)$ is the unique g_{ft}-set of G so that $f_{gft}(G)$= 0.For$r = 3, s \geq 4$, let S be a g_{ft}-set of G. Then any two element subsets of S is a forcing subset of G and so $f_{gft}(G)\geq 3$. Let $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Then $\{x_1, x_2, y_1, y_2\}$ is a forcing subset of S so that $f_{gft}(G)$= 3. Since this is true for al g_{ft}-set of G, $f_{gft}(G)$= 3. Let $m \geq 4$and $n \geq 4$. Let S be a g_{ft}-set of G. Then one or two or three element subsets of S is a forcing subset of$S$ and so $f_{gft}(G)\geq 4$. Let $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Then $\{x_1, x_2, y_1, y_2\}$ is a forcing subset of S so that $f_{gft}(G)$= 4. Since this is true for all g_{ft}-set $S$of G, $f_{gft}(G)$= 4. ■

Theorem 3.10. For every pair of positive integers $a$and b with $0 \leq a \leq b, b \geq 2$and $b \geq 2a$, there exists a connected graph G such that $f_{gft}(G)$= a and $g_{ft}(G) = b$.
Proof: Let $P : x, y, z$ be a path on three vertices. Let P_i: u_i, v_i, w_i $(1 \leq i \leq a)$ be a copy of path on three vertices. Let G be the graph obtained from P and P_i $(1 \leq i \leq a)$ by adding new vertices $z_1, z_2, \ldots, z_{b-2a}$ and introducing the edges xx_i $(1 \leq i \leq a), zv_i$ $(1 \leq i \leq a)$and zz_i $(1 \leq i \leq b - 2a)$. The graph is shown in Figure 3.2.
First, we prove that $g_{ft}(G) = b$. Let $Z = \{z_1, z_2, \ldots, z_{b-2a}\} \cup \{u_1, u_2, \ldots, u_a\}$. Then $Z$is a subset of every g_{ft}-set of G. Let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq a)$. Then every g_{ft}-set of Gcontains at least one vertex from each H_i $(1 \leq i \leq a)$ and so $g_{ft}(G)\geq b - 2a + a + a = b$. Let $S = Z \cup \{u_1, u_2, \ldots, u_a\}$. Then S is a g_{ft}-set of G so that $g_{ft}(G)$= b.
Next, we prove that $f_{gft}(G) = a$. By Theorem$f_{gft}(G) \leq g_{ft}(G) - |Z| = b - (b - a) = a$. Now since every Z is a subset of every g_{ft}-set of G and every g_{ft}-set contains at least one vertex from each H_i $(1 \leq i \leq a)$, every g_{ft}-set S is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be a forcing subset with $|T| < a$. Then there exists $H_i$for some $i$such that $T \cap H_i = \phi$. Therefore $f_{gft}(G) = a$.■

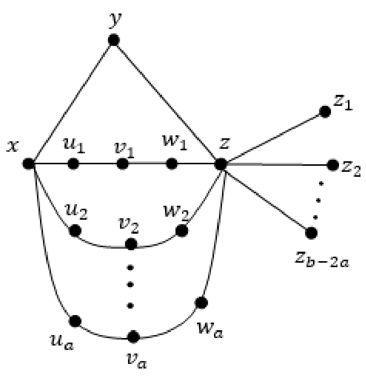

Figure 3.2

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