Product Signed Domination in Graphs

T. M. Velammal[*]
A. Nagarajan[†]
K. Palani[‡]

Abstract

Let $G = (V, E)$ be a simple graph. The closed neighborhood of v, denoted by $N[v]$, is the set $\{u: uv \in E\} \cup \{v\}$. A function $f: V \to \{-1,1\}$ is a product signed dominating function, if for every vertex $v \in V,\ f[v] = 1$where $f[v] = \prod_{u\in N[v]} f(u)$. The weight of f, denoted by $f(G)$, is the sum of the function values of all the vertices in G. $(i.e.)\ f(G) = \sum_{v\in V} f(v)$. The product signed domination number of $G,\ \gamma^*_{sign}(G)$ is the minimum positive weight of a product signed dominating function. In this paper, we establish bounds on the product signed domination number and estimate product signed domination number for some standard graphs.

Keywords: graphs, product signed dominating function, product signed domination number.

2010 AMS subject classification: 05C69[§].

[*]Research Scholar (Reg. No. 21212232092010), PG & Research Department of Mathematics, V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.avk.0912@gmail.com
[†]Associate Professor (Retd.), V.O. Chidambaram College, Thoothukudi-628008, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.nagarajan.voc@gmail.com
[‡]Associate Professor, A.P.C. Mahalaxmi College For Women, Thoothukudi-628002, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India. palani@apcmcollege.ac.in
[§] Received on June 8th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.923. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

The fundamental thought of graphs was first presented in eighteenth era by Swiss Mathematician Leonhard Euler. It has numerous applications in Natural Sciences, Technology, Information System Research and so on. The quickest developing region in theory of graph is domination. Ore introduced the terms "Dominating Set" and "Domination Number". Dunbar et al. introduced signed domination number [1],[2],[4],[5]. Hosseini gave a lower and upper bound for the signed domination number of any graph [3]. In this paper, we introduce the concept of product signed domination number and find bounds on product signed domination number.

2. Preliminaries

Definition 2.1: A comb graph $P_n \circ K_1$ is a graph obtained by joining a pendant edge to each vertex of a path.

Definition 2.2: A star graph $K_{1,n-1}$ is a tree on n vertices with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1.

Definition 2.3: A tree containing exactly two non-pendant vertices is called a double star. It is denoted by $D_{m,n}$

3. Main Results

Definition 3.1: Let $G=(V,E)$ be a simple graph. The closed neighborhood of v, denoted by $N[v]$, is the set $\{u: uv \in E\} \cup \{v\}$. A function $f: V \to \{-1,1\}$ is a product signed dominating function, if for every vertex $v \in V,\ f[v]=1$
where $f[v]=\prod_{u\in N[v]} f(u)$. The weight of f, denoted by $f(G)$, is the sum of the function values of all the vertices in G.

$$(i.e.)\, f(G)=\sum_{v\in V} f(v)$$

The product signed domination number of $G,\ \gamma^*_{sign}(G)$ is the minimum positive weight of a product signed dominating function.

Observation 3.2. (i) In a graph G, a pendant vertex u and its corresponding support vertex v get the same functional values (i.e.) either $+1$ or -1 since otherwise $f[u]=\prod_{x\in N[u]} f(x)=-1$.

(ii) In a product signed dominating function, all the vertices of a graph should not be assigned -1 since product signed domination number is positive.

(iii) In a product signed dominating function, for every vertex $v \in V, N[v]$ contains either zero or even number of vertices with functional value -1, since otherwise $f[v] = \prod_{x \in N[v]} f(x) = -1$.
(iv) If $N_f[1]$ and $N_f[-1]$ denote the number of vertices with functional values 1 and -1 respectively, then $N_f[1] - N_f[-1] \geq 1$.

Theorem 3.3:
$For\ p \leq 4,\ \ \gamma^*_{sign}(K_p) = p,$ the total number of vertices.

$$For\ p > 4,\ \gamma^*_{sign}(K_p) = \begin{cases} 1\ if\ p\ is\ odd\ and\ \dfrac{p-1}{2}\ is\ even \\ 2\ if\ p\ is\ even\ and\ \dfrac{p}{2}\ is\ odd \\ 3\ if\ p\ and\ \dfrac{p-1}{2}\ are\ odd \\ 4\ if\ p\ and\ \dfrac{p}{2}\ are\ even \end{cases}$$

Proof:
Let K_p be a complete graph on p vertices.
Let $V(K_p) = \{v_1, v_2, \dots, v_p\}$and $E(K_p) = \{v_i v_j | i \neq j, 1 \leq i, j \leq p\}$.
Since each pair of vertices is connected by an edge, in a product signed dominating function the number of vertices with functional value -1 must be even.
Case 1: $p \leq 4$
Define a function $f: V(G) \to \{-1, +1\}$ as follows.
When $p \leq 4$, every vertex should be assigned $+1$ under f, since otherwise f would not be a product signed dominating function with a positive weight.

$$Therefore, \qquad \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = p$$

$=$the total number of vertices.
Case 2: $p > 4$ and p is odd
Subcase 2.1:$\frac{p-1}{2}$ is even
Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+1}{2}$, $|V_2| = \frac{p-1}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1\ \forall\ v \in V_1 \\ -1\ \forall\ v \in V_2 \end{cases}$
Obviously, for every $v \in V$, $f[v] = 1$ and hence f is a product signed dominating function.

$$Therefore, \qquad \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v)$$

$$= \left(\frac{p+1}{2}\right).1 + \left(\frac{p-1}{2}\right).(-1) = 1$$

Subcase 2.2. $\frac{p-1}{2}$ is odd

Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+3}{2}$, $|V_2| = \frac{p-3}{2}$ and $V_1 \cap V_2 = \emptyset$. Here $|V_1|$ is odd and $|V_2|$ is even. Define $f: V(G) \to \{-1, +1\}$ as

$$f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$$

Clearly, for every $v \in V$, $f[v] = 1$.Hence f is a product signed dominating function. Also, $N_f[1] - N_f[-1] = 3$. Since $\frac{p-1}{2}$ is odd. This function f gives the minimum value for product signed domination number. $Therefore,\ \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = \left(\frac{p+3}{2}\right).1 + \left(\frac{p-3}{2}\right).(-1) = 3$

Case 3: $p > 4$and p is even

Subcase 3.1:$\frac{p}{2}$ is even

If we partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p}{2}$,$|V_2| = \frac{p}{2}$ and $V_1 \cap V_2 = \emptyset$ and assign $+1$ to all the vertices in V_1 and -1 to all the vertices in V_2, then the function would be a product signed dominating function but the weight would be zero.

Since $\frac{p}{2}$ is even, $\frac{p-2}{2}$ is odd. Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+4}{2}$, $|V_2| = \frac{p-4}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$

Therefore f is a product signed dominating function. Also $N_f[1] - N_f[-1] = 4$. Since $\frac{p}{2}$ is even, this function f gives the minimum value for product signed domination number as before. $Therefore,\ \gamma^*_{sign}(K_p) = \sum_{v \in V(K_p)} f(v) = \left(\frac{p+4}{2}\right).1 + \left(\frac{p-4}{2}\right).(-1)$
$= 4$

Subcase 3.2: $\frac{p}{2}$ is odd

We have $\frac{p-2}{2}$ is even.

Partition the vertex set V into two sets $V_1$and V_2 such that $|V_1| = \frac{p+2}{2}$, $|V_2| = \frac{p-2}{2}$ and $V_1 \cap V_2 = \emptyset$. Define $f: V(G) \to \{-1, +1\}$ as $f(v) = \begin{cases} 1 \; \forall \; v \in V_1 \\ -1 \; \forall \; v \in V_2 \end{cases}$

Correspondingly, for every $v \in V$, $f[v] = 1$. Hence f is a product signed dominating function. Also $N_f[1] - N_f[-1] = 2$. Proceeding as above, this function f gives the minimum value for product signed domination number.

$$Therefore, \quad \gamma^*_{sign}(K_p) = \sum_{v\in V(K_p)} f(v) = \left(\frac{p+2}{2}\right).1 + \left(\frac{p-2}{2}\right).(-1) = 2$$

$Therefore,\ p \leq 4,\ \gamma^*_{sign}(K_p) = p,$ the total number of vertices.

$$For\ p > 4, \gamma^*_{sign}(K_p) = \begin{cases} 1\ if\ p\ is\ odd\ and\ \frac{p-1}{2}\ is\ even \\ 2\ if\ p\ is\ even\ and\ \frac{p}{2}\ is\ odd \\ 3\ if\ p\ and\ \frac{p-1}{2}\ are\ odd \\ 4\ if\ p\ and\ \frac{p}{2}\ are\ even \end{cases}$$

Theorem 3.4: For the comb graph, $P_n \circ K_1$, the product signed domination number $\gamma^*_{sign}(P_n \circ K_1) = 2n$, the total number of vertices.

Proof:
Let G be a comb graph $P_n \circ K_1$.
Let $V = \{v_i, u_i | 1 \leq i \leq n\}$ be the vertex set with u_i's representing the pendant vertices and $E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n\}$ be the edge set.
Since u_i is the pendant vertex to v_i, both $f(u_i)$ and $f(v_i)$ must be either $+1$ or -1 for $1 \leq i \leq n$ (by 3.2(i)).
Case 1: n is odd
Define $f: V(G) \to \{-1, +1\}$ as follows.
If u_1 and v_1 are both assigned -1, then u_2 and v_2 should be assigned $+1$ since otherwise $f[v_1]$ would be -1. Further if $f(u_3) = f(v_3) = 1$, again $f[v_2] = -1$. Hence $f(u_1) = f(v_1) = -1$, $f(u_2) = f(v_2) = 1$, $f(u_3) = f(v_3) = -1$, $f(u_4) = f(v_4) = 1$ and so on. Then $f(u_n) = f(v_n) = -1$ and f is a product signed dominating function. Correspondingly, the weight of the graph is $-2 =$ a negative integer which is a contradiction to the weight is positive.
Hence, let us start with $f(u_1) = f(v_1) = 1$.
Then $f(u_i) = f(v_i) = 1\ \forall\ i = 2\ to\ n$, since otherwise f is not a product signed dominating function.
Hence $f(u_i) = f(v_i) = 1\ \forall\ i = 1\ to\ n$ is the only product signed dominating function having a positive weight. Hence it is the unique product signed dominating function.

$Therefore,\ \gamma^*_{sign}(G) = \sum_{v\in V(G)} f(v) = 2n =$the total number of vertices of G

Case 2: n is even
Define $f: V(G) \to \{-1, +1\}$ as follows.
If u_1 and v_1 are both assigned -1, then u_2 and v_2 should be assigned $+1$ since otherwise $f[v_1]$ would be -1. Further if $f(u_3) = f(v_3) = 1$, again $f[v_2] = -1$. Hence

$f(u_1) = f(v_1) = -1,\ f(u_2) = f(v_2) = 1,\ f(u_3) = f(v_3) = -1,\ f(u_4) = f(v_4) = 1$ and so on. Then $f(u_n) = f(v_n) = 1$ and hence $f[v_n] = -1$.
Therefore, this f is not a product signed dominating function.
Hence, let us start with $f(u_1) = f(v_1) = 1$. Then $f(u_i) = f(v_i) = 1\ \forall\ i = 2\ to\ n$, since otherwise f is not a product signed dominating function.
Hence $f(u_i) = f(v_i) = 1\ \forall\ i = 1\ to\ n$ is the unique product signed dominating function.
$Therefore,\ \gamma_{sign}^*(G) = \sum_{v \in V(G)} f(v) = 2n =$ the total number of vertices of G
By cases 1 and 2, $\gamma_{sign}^*(G) = 2n$.

Observation 3.5: For any graph G, $1 \leq \gamma_{sign}^*(G) \leq p\ ;p =$ total number of vertices of G. Here the bounds are sharp since $\gamma_{sign}^*(K_5) = 1$ and $\gamma_{sign}^*(P_n \circ K_1) = 2n =$ total number of vertices.

Theorem 3.6:
The product signed domination number of a path on n vertices is equal to n.
Proof:
Let P_n be a path on n vertices.
If $f(v_i) = -1\ (2 \leq i \leq n-1)$
Then $f(v_{i-1}) = -1\ and\ f(v_{i+1}) = +1$
(or)
$f(v_{i-1}) = +1\ and\ f(v_{i+1}) = -1$
If $f(v_i) = +1\ (2 \leq i \leq n-1)$
Then $f(v_{i-1}) = f(v_{i+1}) = -1$
(or) $f(v_{i-1}) = f(v_{i+1}) = +1$
By the above observation, if $v_1$is assigned -1, then $v_2$must be assigned -1 so that $f[v_1] = +1$. Then v_3 must be assigned $+1$ so that $f[v_2] = +1$. So v_4 must be assigned -1 so that $f[v_3] = +1$.
Proceeding like this, we define a function $f: V(G) \to \{-1, +1\}$ as follows.

$$\text{For } 1 \leq i \leq n,\ f(v_i) = \begin{cases} +1\ if\ i \equiv 0 (mod 3) \\ -1\ otherwise \end{cases}$$

So when $n = 3k, f$ is not a product signed dominating function since $f[v_n] = -1$.
When $n = 3k+1, f$ is not a product signed dominating function since $f[v_n] = -1$
When $n = 3k+2,\ f$ is a product signed dominating function having a negative weight.
So let us try with $+1$ assigned to v_1.
If v_1 is assigned $+1$, v_2 must be assigned $+1$ so that $f[v_1] = +1$. Again v_3 must be assigned $+1$ so that $f[v_2] = +1$. Again v_4 must be assigned $+1$ so that $f[v_3] = +1$ and so on.
Therefore, $f(v_i) = +1\ \forall\ i = 1\ to\ n$. And f is a minimum positive weight product signed dominating function.

The weight of this function $= n$, the total number of vertices.
Therefore, $\gamma^*_{sign}(P_n) = n$, the total number of vertices.

Theorem 3.7:
The product signed domination number of a cycle on n vertices is equal to n.

Proof:
Let C_n be a path on n vertices.
If $f(v_i) = -1$ $(2 \leq i \leq n-1)$
Then $f(v_{i-1}) = -1 \; and \; f(v_{i+1}) = +1$
(or)
$f(v_{i-1}) = +1 \; and \; f(v_{i+1}) = -1$
If $f(v_i) = +1$ $(2 \leq i \leq n-1)$
Then $f(v_{i-1}) = f(v_{i+1}) = -1$
(or)
$f(v_{i-1}) = f(v_{i+1}) = +1$
By the above observation, if $v_1$is assigned -1, then $v_2$must be assigned -1 so that $f[v_1] = +1$. Then v_3 must be assigned $+1$ so that $f[v_2] = +1$. So v_4 must be assigned -1 so that $f[v_3] = +1$.
Proceeding like this, we define a function $f: V(G) \to \{-1, +1\}$ as follows.
For $1 \leq i \leq n, f(v_i) = \begin{cases} +1 \; if \; i \equiv 0 (mod 3) \\ -1 \; otherwise \end{cases}$
So, when $n = 3k$, f is a product signed dominating function having negative weight.
When $n = 3k + 1$, f is not a product signed dominating function since $f[v_1] = -1$.
When $n = 3k + 2$, f is not a product signed dominating function since $f[v_1] = f[v_n] = -1$.
So let us try with $+1$ assigned to v_1.
If v_1 is assigned $+1$, v_2 must be assigned $+1$ so that $f[v_1] = +1$. Again v_3 must be assigned $+1$ so that $f[v_2] = +1$. Again v_4 must be assigned $+1$ so that $f[v_3] = +1$ and so on.
Therefore, $f(v_i) = +1 \; \forall \; i = 1 \; to \; n$. And f is a minimum positive weight product signed dominating function.
The weight of this function $= n$, the total number of vertices.
Therefore, $\gamma^*_{sign}(C_n) = n$, the total number of vertices.

Theorem 3.8:
The product signed domination number of a star graph $K_{1,n-1}$on n vertices is equal to n.
Proof:
Let $K_{1,n-1}$ be a star graph on n vertices.
Let $V(K_{1,n-1}) = \{v, v_1, v_2, \dots, v_{n-1}\}$and $E(K_{1,n-1}) = \{vv_i | 1 \leq i \leq n-1\}$.
By 3.2(i), v and $v_i (1 \leq i \leq n-1)$ should be assigned same functional value.

If $f(v) = -1$, then the weight of f is negative.
Therefore $f(v)$ must be equal to $+1$ and hence define $f: V(G) \to \{-1, +1\}$ as $f(v_i) = 1\ for\ 1 \leq i \leq n-1$. And obviously f is the minimum positive weight product signed dominating function.
Therefore, $\gamma^*_{sign}(K_{1,n-1}) = n$, the total number of vertices.

Theorem 3.9:
The product signed domination number of a double star graph $D_{m,n}$is equal to $m+n$.
Proof:
Let $D_{m,n}$ be a double star graph on $m+n$ vertices.
Let $V(D_{m,n}) = \{v, v_1, v_2, \dots, v_{n-1}, u, u_1, u_2, \dots, u_{m-1}\}$ and
$E(D_{m,n}) = \{vv_i | 1 \leq i \leq n-1\} \cup \{uu_j | 1 \leq j \leq m-1\} \cup \{uv\}$.
Case 1: Number of pendant vertices to atleast one of u, v is odd.
Without loss of generality, assume that number of pendant vertices to u is odd.
If we assign $-1$to u, then all the pendant vertices to u must be assigned -1 (by 3.2(i)). Since number of pendant vertices to u is odd, v must be assigned $+1$. Hence again by 3.2(i), all the pendant vertices to $v$get $+1$. But here $f[v] = -1$. So this f is not a product signed dominating function.
Hence define $f: V(G) \to \{-1, +1\}$as $f(v) = +1\ \forall\ v \in V(D_{m,n})$
Clearly, f is the minimum positive weight product signed dominating function.

$$Hence, \sum_{v \in V(D_{m,n})} f(v) = \sum_{v \in V(D_{m,n})} 1 = m+n$$

Therefore, $\gamma^*_{sign}(D_{m,n}) = m+n$, the total number of vertices.
Case 2: Number of pendant vertices to both u and v is even.
If we assign $-1$to u, then all the pendant vertices to u must be assigned -1 (by 3.2(i)). Since number of pendant vertices to u is even, v must be assigned -1. Hence again by 3.2(i), the pendant vertices to $v$get -1. Here this f is a product signed dominating function having a negative weight.
So, the only possible positive weight product signed dominating function $f: V(G) \to \{-1, +1\}$is $f(v) = +1\ \forall\ v \in V(D_{m,n})$

$$Hence, \sum_{v \in V(D_{m,n})} f(v) = \sum_{v \in V(D_{m,n})} 1 = m+n$$

Therefore, $\gamma^*_{sign}(D_{m,n}) = m+n$, the total number of vertices.

References

[1] J. Dunbar, S.T. Hedetniemi. Henning, and P.J. Slater (1995), Signed Domination in Graph Theory, In: Graph Theory, Combinatorics and Applications, John Wiley & Sons, New York, 311-322.

[2] Ernest J. Cockayneand Christina M. Mynhardt (1996), On A Generalisation Of Signed Dominating Funtions Of Graphs, Ars Combinatoria, 43, 235-245.

[3] S.M. Hosseini Moghaddam (2015), New Bounds On The Signed Domination Numbers Of Graphs, Australasian Journal Of Combinatorics, 61(3), 273-280.

[4] Izak Broere, Johannes H. Hattingh, Michael A. Henning, and Alice A. McRae (1995), Majority Domiation In Graphs, Discrete Mathematics, 138, 125-135. https://doi.org/10.1016/0012-365X(94)00194-N

[5] Odile Favaron (1996), Signed Domination In Regular Graphs, Discrete Mathematics, 158, 287-293. https://doi.org/10.1016/001-365X(96)00026-X