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Strong perfect cobondage number of standard graphs

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Abstract

Let G be a simple graph. A subset $S \subseteq V(G)$ is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$ ($|N_w(u) \cap S| = 1$) for every $u \in V(G) - S$ where $N_s(u) = \{v \in V(G) / uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G) / uv \in E(G), \deg v \leq \deg u\}$. The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)(\gamma_{wp}(G))$. The strong perfect cobondage number $bc_{sp}(G)$ of a nonempty graph G is defined to be the minimum cardinality among all subsets of edges $X \subseteq E(G)$ for which $\gamma_{sp}(G + X) < \gamma_{sp}(G)$. If $bc_{sp}(G)$ does not exist, then $bc_{sp}(G)$ is defined as zero. In this paper study of strong perfect cobondage number of standard graphs and some special graphs are determined.

Keywords: Strong perfect dominating set, strong perfect domination number and strong perfect cobondage number.

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1. Introduction

Let G be a simple graph with vertex set V(G) and edge set E(G). A dominating set D of G is a subset of V(G) such that every vertex in V – D is adjacent to at least one vertex in D [6]. A set D $\subseteq$ V(G) is a strong(weak) dominating set of G [5] if every vertex in V – D is strongly(weakly) dominated by at least one vertex in D. The strong (weak) domination number $\gamma_s(G)$ ($\gamma_w(G)$) is the minimum cardinality of a strong (weak) dominating set of G. A dominating set S is a perfect dominating set of G [1] if $|N(v) \cap S| = 1$ for each $v \in V – S$. Motivated by these definitions, the authors defined strong perfect domination in graphs [2]. In [4], V.R. Kulli and B. Janakiram introduced the concept of cobondage number of a graph. The cobondage number of graph G is the minimum cardinality among the set of edges X such that $\gamma(G+X) < \gamma(G)$. In this paper strong perfect cobondage number is defined and strong perfect cobondage number of paths, cycle, bistar, complete bipartite graph and some special graphs are determined. For all graph theoretic terminologies and notations Harary [3] is followed.

2. Preliminaries

Definition 2.1. [2] Let G be a simple graph. A subset S $\subseteq$ V(G) is called a strong (weak) perfect dominating set of G if $|N_s(u) \cap S| = 1$($|N_w(u) \cap S| = 1$) for every $u \in V(G) - S$ where $N_s(u) = \{v \in V(G)/uv \in E(G), \deg v \geq \deg u\}$ ($N_w(u) = \{v \in V(G)/uv \in E(G), \deg v \leq \deg u\}$.

Remark 2.2. [2] The minimum cardinality of a strong (weak) perfect dominating set of G is called the strong (weak) perfect domination number of G and is denoted by $\gamma_{sp}(G)(\gamma_{wp}(G))$.

Definition 2.3. Bi star is the graph obtained by joining the apex vertices of two copies of star $K_{1, n}$.

Definition 2.4. The wheel W_n is defined to be the graph $C_{n-1} + K_1$, $n \geq 4$.

Definition 2.5. The helm H_n is the graph obtained from the wheel W_n with n spokes by adding n pendant edges at each vertex on the wheel's rim.

Definition 2.6. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 (where G_i has p_i points and q_i lines) is defined as the graph G obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2.

Definition 2.7. The lily graph L_n, $n \geq 2$ can be constructed by two-star graphs $2K_{1, n}$, $n \geq 2$ joining two path graphs $2P_n$, $n \geq 2$ with sharing a common vertex.

Theorem 2.8. [2] For any path P_m, Then $\gamma_{sp}(P_m) = \begin{cases} n \text{ if } m = 3n, n \epsilon N \\ n+1 \text{ if } m = 3n+1, n \epsilon N \\ n+2 \text{ if } m = 3n+2, n\ \epsilon N \end{cases}$

Theorem 2.9 [2] For any cycle C_m**,**

Then $\gamma_{sp}(C_m) = \begin{cases} n \text{ if } m = 3n, n \epsilon N \\ n+1 \text{ if } m = 3n+1, n \epsilon N \\ n+2 \text{ if } m = 3n+2, n\ \epsilon N \end{cases}$

Theorem 2.10. [2] Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{sp}(G \odot K_1) = n$.

Remark 2.11. [2]

(i) $\gamma_{sp}(D_{r,s}) = 2$, r, s ϵ N

(ii) $\gamma_{sp}(K_{m,n}) = \begin{cases} 2 \text{ if } m = n, m, n \geq 2 \\ m+n \text{ if } m \neq n, m, n \geq 2 \end{cases}$

(iii) $\gamma_{sp}(H_n) = n$, $n \geq 5$ *and* $\gamma_{sp}(H_4) = 3$.

3. Main Results

Definition 3.1. The strong perfect cobondage number $bc_{sp}(G)$ of a graph is defined to be the minimum cardinality among all subsets of edges $X \subseteq E(G)$ for which $\gamma_{sp}(G + X) < \gamma_{sp}(G)$. If $bc_{sp}(G)$ does not exist, then $bc_{sp}(G)$ is defined as zero.

Example 3.2. Consider the graphs G and G + e in the following figures 1 and 2 respectively.

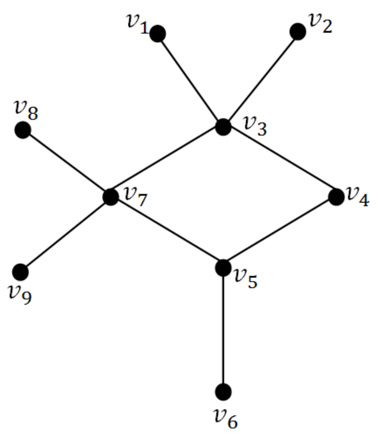

Fig. 1.

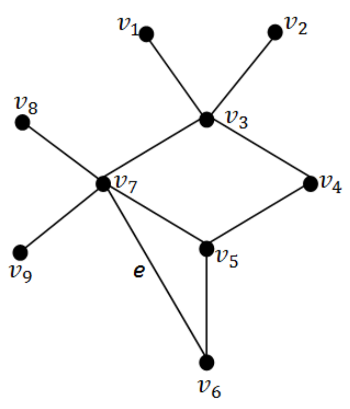

Fig. 2.

$\{v_3, v_7, v_6\}$, $\{v_3, v_7, v_5, v_4\}$ and $\{v_3, v_8, v_9, v_4, v_5\}$ are some strong perfect dominating sets of G. Hence $\gamma_{sp}(G) \leq 3$. Since there is no full degree vertex in G, $\gamma_{sp}(G) \geq 2$. There are five pendent vertices. Any strong perfect dominating sets contains either pendent vertices or their support vertices. v_3, v_7, v_5 are the support vertices adjacent with pendent vertices. Therefore $\gamma_{sp}(G) \geq 3$. Hence $\gamma_{sp}(G) = 3$. Let $e = v_6v_7$. $\{v_3, v_7\}$, $\{v_7, v_4, v_1, v_2\}$ are some strong perfect dominating sets of G + e. Therefore $\gamma_{sp}(G + e) \leq 2$. Since there is no full degree vertex in G + e, $\gamma_{sp}(G) \geq 2$. Hence $\gamma_{sp}(G + e) = 2 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Theorem 3.3. $bc_{sp}(P_m) = \begin{cases} 1 \text{ if } m = 3n+1, n \geq 1 \\ 1 \text{ if } m = 3n+2, n \geq 1 \\ 3 \text{ if } m = 3n, n \geq 2 \end{cases}$

Proof. Let $G = P_m$, $m \geq 4$. Let $V(G) = \{v_i / 1 \leq i \leq m\}$.

Case (1). Let $m = 3n+1$, $n \geq 1$. $\gamma_{sp}(G) = n+1$. Let $e = v_{3n-1}v_{3n+1}$, $\deg v_{3n-1} = \triangle(G+e) = 3$. $T = \{v_2, v_5, v_8, \ldots, v_{3n-1}\}$ is a strong perfect dominating set of G + e. $|T| = n$. Therefore $\gamma_{sp}(G + e) \leq n$. Let S be any strong perfect dominating set of G + e. Since v_{3n-1} is the unique maximum degree vertex in G + e, v_{3n-1} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3n-2}$ is P_{3n-3}. Therefore $\gamma_{sp}(P_{3n-3}) = n - 1$. $|S| = n$. Hence $\gamma_{sp}(G + e) \geq n$. Therefore $\gamma_{sp}(G + e) = n < n+1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let $m = 3n+2$, $n \geq 1$. $\gamma_{sp}(G) = n+2$. Let $e = v_{3n}v_{3n+2}$, $\deg v_{3n} = \triangle(G + e) = 3$. $T_1 = \{v_1, v_3, v_6, \ldots, v_{3n}\}$, $T_2 = \{v_2, v_5, \ldots, v_{3n-4}\, v_{3n-3}, v_{3n}\}$ are some strong perfect dominating sets of G + e. $|T_1| = |T_2| = n + 1$. Therefore $\gamma_{sp}(G + e) \leq n+1$. Let S be any strong perfect dominating set of G +e. v_{3n} is strongly dominates v_{3n+2}. Therefore $v_{3n} \epsilon$ S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3n-2}$ is P_{3n-2}. Therefore $\gamma_{sp}(P_{3n-2}) = \gamma_{sp}(P_{3(n-1)+1}) = n$. Therefore $|S| = n + 1$. $\gamma_{sp}(G + e) \geq n+1$. Therefore $\gamma_{sp}(G + e) = n+1 < n+2 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (3). Let $G = P_{3n}$, $n \geq 2$. $\gamma_{sp}(G) = n$. When one or two edges added with P_{3n} strong perfect domination number of the resulting graph either increases or remains same. Hence $bc_{sp}(G) \geq 3$. Let $e_1 = v_1v_3$, $e_2 = v_3v_5$, $e_3 = v_3v_6$. Let $X = \{e_1, e_2, e_3\}$. $T = \{v_3, v_8, v_{11}, \ldots, v_{3n-1}\}$is a strong perfect dominating set of G+X. $|T| = n - 1$. Therefore $\gamma_{sp}(G+X) \leq n - 1$. Let S be any strong perfect dominating set of G + X. $\deg v_3 = 5$. v_3 is the unique maximum degree vertex of G + X. v_3 belongs to S. The subgraph induced by the vertices $v_7, v_8, \ldots, v_{3n}$ is a P_{3n-6}. Therefore $\gamma_{sp}(P_{3n-6}) = n - 2$. Therefore $|S| = n - 2 + 1 = n - 1$. $\gamma_{sp}(G + X) \geq n - 1$. Therefore $\gamma_{sp}(G + X) = n - 1 < n = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 3$.

Remark. Let m = 3. $\gamma_{sp}(G) = 1$. Hence $bc_{sp}(G) = 0$.

Theorem 3.4. $bc_{sp}(C_m) = \begin{cases} 1 \text{ if } m = 3n+1, n \geq 1 \\ 1 \text{ if } m = 3n+2, n \geq 1 \\ 3 \text{ if } m = 3n, n \geq 2 \end{cases}$

Proof. Let $G = C_m$, $m \geq 4$. Let $V(G) = \{v_i / 1 \leq i \leq m\}$.

Case (1). Let $m = 3n+1$, $n \geq 1$. $\gamma_{sp}(G) = n+1$. Let $e = v_1v_3$. $T = \{v_1, v_5, v_8, \ldots, v_{3n-1}\}$ is a strong perfect dominating set of G + e. $|T| = n$. Therefore $\gamma_{sp}(G + e) \leq n$. Let S be any strong perfect dominating set of G + e. Since v_1, v_3 are maximum degree vertices in

G+ e, either v_1 belongs to S or v_3 belongs to S. Suppose v_1 belongs to S (Proof is similar if v_3 belongs to S). The subgraph induced by the vertices $v_4, v_5, \ldots, v_{3n}$ is the path P_{3n-3}. $\gamma_{sp}(P_{3n-3}) = n - 1$. Hence $\gamma_{sp}(G + e) \geq n$. Therefore $\gamma_{sp}(G + e) = n < n+1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let m = 3n+2, $n \geq 2$. $\gamma_{sp}(G) = n+2$. Let $e = v_1v_{3n+1}$. $T = \{v_1, v_4, v_7, \ldots, v_{3n-2}, v_{3n-1}\}$ is a strong perfect dominating set of G + e. $|T| = n+1$. Therefore $\gamma_{sp}(G + e) \leq n+1$. Let S be any strong perfect dominating set of G + e. Since v_1, v_{3n+1} are maximum degree vertices in G + e, either v_1 belongs to S or v_{3n+1} belongs to S. Suppose v_1 belongs to S (Proof is similar if v_{3n+1} belongs to S). The subgraph induced by the vertices $v_3, v_4, \ldots, v_{3n}$ is the path P_{3n-2}. Therefore $\gamma_{sp}(P_{3n-2}) = \gamma_{sp}(P_{3(n-1)+1}) = n$. Hence $\gamma_{sp}(G + e) \geq n+1$. Therefore $\gamma_{sp}(G + e) = n+1 < n+2 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Subcase (2a). If n = 1, $\gamma_{sp}(C_5) = 3$. Let $e = v_1v_4$. $\{v_1, v_2\}$ is the unique γ_{sp} - set of G + e. $\gamma_{sp}(G + e) = 2 < \gamma_{sp}(G)$. Hence $bc_{sp}(C_5) = 1$.

Case (3). Let m = 3n, $n \geq 2$. $\gamma_{sp}(G) = n$. When one or two edges added with C_{3n} strong perfect domination number of the resulting graph either increase or remain same. Hence $bc_{sp}(G) \geq 3$. Let $e_1 = v_1v_3$, $e_2 = v_1v_4$, $e_3 = v_1v_5$. $X = \{e_1, e_2, e_3\}$.$T = \{v_1, v_7, v_{10}, \ldots, v_{3n-2}\}$ is a strong perfect dominating set of G + X. $|T| = n - 1$.Therefore $\gamma_{sp}(G + X) \leq n - 1$. Let S be any strong perfect dominating set of G + X. Since v_1 is the unique maximum degree vertex in G + X, v_1 belongs S. The subgraph induced by the vertices $v_6, v_7, \ldots, v_{3n-1}$ is the path P_{3n-6}. Therefore $\gamma_{sp}(P_{3n-6}) = \gamma_{sp}(P_{3(n-2)}) = n - 2$. Hence $\gamma_{sp}(G + X) \geq n - 1$. Therefore $\gamma_{sp}(G + X) = n - 1 < n = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 3$.

Theorem 3.5. $bc_{sp}(D_{r,s}) = s$, $r \geq s$, $r, s \geq 1$.

Proof. Let $G = D_{r,s}$, $r, s \geq 1$. Let $V(G) = \{u, v, u_i, v_j / 1 \leq i \leq r, 1 \leq j \leq s\}$, $E(G) = \{uv, uu_i, vv_j / 1 \leq i \leq r, 1 \leq j \leq s\}$. $\gamma_{sp}(G) = 2$. Let $r \geq s$, $r, s \geq 1$, $\gamma_{sp}(G + X) = 1$ if and only if G + X must have a full degree vertex. It is possible only if u is adjacent with $v_1, v_2, \ldots, v_s$ or v is adjacent with $u_1, u_2, \ldots, u_r$. Since $r \geq s$, $X = \{uv_j / 1 \leq j \leq s\}$. u is the unique full degree vertex of G + X. Therefore $\gamma_{sp}(G + X) = 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = s$.

Theorem 3.6. $bc_{sp}(H_n) = 1$, $n \geq 4$.

Proof: Let $G = H_n$, $n \geq 4$. Let $V(G) = \{v, v_i, u_i / 1 \leq i \leq n - 1\}$, $E(G) = \{vv_i, v_iu_i / 1 \leq i \leq n - 1\} \cup \{v_nv_1\} \cup \{v_iv_{i+1} / 1 \leq i \leq n-2\}$. $\gamma_{sp}(G) = n$, $n \geq 5$. deg v = n - 1, deg v_i = 4, deg u_i = 1, $1 \leq i \leq n - 1$.

Case (1). Let $n \geq 5$. Let $e = u_1u_{n-1}$. $T = \{v, u_1, u_2, \ldots, u_{n-2}\}$ and $\{v, u_2, u_3, \ldots, u_{n-1}\}$ are some strong perfect dominating sets of G + e. $|T| = n - 1$. Therefore $\gamma_{sp}(G + e) \leq n - 1$. Let S be any strong perfect dominating set of G + e. There are n – 3 pendent vertices, either n – 3 pendent vertices belong to S or their support vertices belong to S. v is the unique maximum degree vertex of G + e. Hence v belongs to S. Since u_1 and u_{n-1} are adjacent either u_1 or u_{n-1} belong to S. Therefore $\gamma_{sp}(G + e) \geq n - 1$. Hence $\gamma_{sp}(G + e) = n - 1 < \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let n = 4. $\gamma_{sp}(G) = 3$. Let $e = v_1u_3$. $\{v_1, u_2\}$ is the unique γ_{sp} - set of G + e. Therefore $\gamma_{sp}(G + e) = 2 < 3 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Theorem 3.8. $bc_{sp}(K_{m,n}) = \begin{cases} n-1 \; if \; m=n, m,n \geq 2 \\ 1 \; if \; m=2, n \geq 3, m \neq n, m<n \\ n-m \; if \; m \neq n, m<n, m \geq 3, n \geq 4 \end{cases}$

Proof. Let G = $K_{m,n}$, m, n $\geq$ 2. V(G) = {v_i, u_j / 1≤ i≤ m, 1≤ j≤ n}.

Case (1). Let m = n, m, n $\geq$ 2. $\gamma_{sp}(G) = 2$. $\gamma_{sp}(G + X) = 1$ if and only if G + X must have a full degree vertex. It is possible only if any u_i is adjacent with all other u_j or v_i is adjacent with all other v_j, i ≠ j,1 ≤ j≤ n. Therefore, u_i or v_i is a full degree vertex of G + X. Let X = {u_iu_j /1 ≤ j≤n and j ≠i} or {v_iv_j /1 ≤ j≤ n and i ≠j}. |X| = n – 1. Therefore $\gamma_{sp}(G + X)$ =1 < $\gamma_{sp}(G)$. Hence $bc_{sp}(G)$ = n - 1.

Case (2). Let m ≠ n, m <n, m = 2, n $\geq$ 2. G = $K_{2,n}$. $\gamma_{sp}(G)$ = n +2. Let e = v_1v_2. v_1 and v_2 are full degree vertices of G + e. Therefore $\gamma_{sp}(G + e)$ =1 < $\gamma_{sp}(G)$. Hence $bc_{sp}(G)$ = 1.

Case (3). Let m ≠ n, m < n, m ≥ 3, n ≥ 4. $\gamma_{sp}(G)$ = m + n. If any two vertices v_i and v_j are adjacent then G′ is the new graph $\gamma_{sp}(G')$ = m + n, otherwise the vertices u_j are strongly dominated by two vertices v_i and v_j, a contradiction. Suppose degree of the vertex u_j is increased so that deg v_i = deg u_j. u_j strongly dominates the vertices v_i, 1 ≤i≤m and also strongly dominates (n – m) u_i's. Without loss of generality, let u_1 be adjacent with u_2, u_3, …, u_{n-m+1}. Let X = {u_1u_2,u_1u_3, …, u_1u_{n-m+1}}. |X| = n – m. T = {u_1, u_{n-m+2}, u_{n-m+3},…,u_n} is a strong perfect dominating set of G + X. $\gamma_{sp}(G + X)$ ≤ m. Let S be any strong perfect dominating set of G. v_i's ,1≤i≤m are maximum degree vertices and they are mutually non adjacent if they belong to S then u_1 is strongly dominated by more than one vertex in S, a contradiction. If u_1 is included along with v_i's then u_j, j ≠ i, is dominated by more than one vertex, a contradiction. Hence S contains all the vertices of G + X. |S| = n + m. Any γ_{sp} – set does not contain v_1,v_2,…,v_m. Therefore, T is the minimum strong perfect dominating set of G + X. Therefore $\gamma_{sp}(G + X)$ = m < m + n = $\gamma_{sp}(G)$. Hence $bc_{sp}(G)$ = n - m.

Theorem 3.9. Let G ⨀ K_1 = G' be a connected graph. Then $bc_{sp}(G')$ = 1.

Proof. Let V(G') = {v_i, u_i / 1≤i≤n}. $\gamma_{sp}(G')$ = n. deg v_i ≥ 2, deg u_i = 1,1 ≤ i≤ n. Let T = {v_i/1 ≤ i ≤ n} be a strong perfect dominating set of G′. |T| = n. Join any vertex v_i with u_j, without loss of generality, let e = v_2u_1. v_2 strongly dominates v_1,v_3,u_1 and u_2. The subgraph induced by the remaining vertices v_i, u_i, 3 ≤ i ≤ n, is H⨀ K_1, where |V(G')| = n – 2. Therefore $\gamma_{sp}(H ⨀K_1)$ = n – 2. These n – 2 vertices along with v_1 form a strong perfect dominating set of G⨀ K_1. $\gamma_{sp}(G')$ ≥ n – 2 + 1 = n – 1. Let T = { v_i/ 2 ≤i ≤n } is a strong perfect dominating set of the resulting graph G' + e. Therefore $\gamma_{sp}(G' + e) < \gamma_{sp}(G')$. Hence $bc_{sp}(G')$ = 1.

Theorem 3.10. $\gamma_{sp}(L_n) = \begin{cases} 2k+1 \; if \; n=3k, 3k+2, k \geq 1 \\ 2k+3 \; if \; n=3k+1, k \geq 1 \end{cases}$

Proof. Let G = L_n, n ≥ 3. V(G) = {u_i, v_j/1≤i≤2n, 1≤j≤2n - 1}. |V(G)| = 4n – 1. deg v_n = 2n + 2 = △(G), deg u_i =1,1≤ i ≤2n, deg v_1 = deg v_{2n-1} = 1, deg v_j = 2, 2 ≤ j ≤ 2n - 2.

Case (1). Let n = 3k, k ≥ 1. T = {v_2, v_3, v_6, …, v_{3k-3}, v_{3k}, v_{3k+3}, v_{3k+6}, …, v_{6k-6}, v_{6k-3}, v_{6k-2}} is a strong perfect dominating set of G. |T| = 1+ $\frac{3k-3-3}{3} + 2 + \frac{6k-6-3k-3}{3} +$

$3 = 2k + 1$. $\gamma_{sp}(G) \leq 2k + 1$. Let S be any strong perfect dominating set of G. Since v_{3k} is unique the maximum degree vertex in G. v_{3k} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3k-2} \cup v_{3k+2}, v_{3k+3}, \ldots, v_{6k-1}$ is 2 P_{3k-2}. $\gamma_{sp}(2P_{3k-2}) = 2\gamma_{sp}(P_{3(k-1)+1}) = 2k$. $|S| = 2k+1$. Therefore $\gamma_{sp}(G) \geq 2k + 1$. Hence $\gamma_{sp}(G) = 2k + 1$.

Case (2). Let n = 3k+1, $k \geq 1$. T = $\{v_2, v_3, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}, v_{3k+4}, v_{3k+7}, \ldots, v_{6k-2}, v_{6k-1}, v_{6k}\}$ is a strong perfect dominating set of G. $|T| = 1 + \frac{3k-2-4}{3} + 3 + \frac{6k-2-3k-4}{3} + 3 = 2k + 3$. $\gamma_{sp}(G) \leq 2k + 3$. Let S be any strong perfect dominating set of G. Since v_{3k+1} is unique the maximum degree vertex in G. v_{3k+1} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3k-1} \cup v_{3k+3}, v_{3k+4}, \ldots, v_{6k+1}$ is 2 P_{3k-1}. $\gamma_{sp}(2P_{3k-1}) = 2\gamma_{sp}(P_{3(k-1)+2}) = 2k + 2$. $|S| = 2k+3$. Therefore $\gamma_{sp}(G) \geq 2k + 3$. Hence $\gamma_{sp}(G) = 2k + 3$.

Case (3). Let n = 3k+2, $k \geq 1$. T = $\{v_2, v_5, \ldots, v_{3k-1}, v_{3k+2}, v_{3k+5}, v_{3k+8}, \ldots, v_{6k+2}\}$ is a strong perfect dominating set of G. $|T| = 1 + \frac{3k-1-2}{3} + 1 + \frac{6k+2-3k-5}{3} + 1 = 2k + 1$. $\gamma_{sp}(G) \leq 2k + 1$. Let S be any strong perfect dominating set of G. Since v_{3k+2} is unique the maximum degree vertex in G. v_{3k+2} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3k} \cup v_{3k+4}, v_{3k+5}, \ldots, v_{6k+3}$ is 2 P_{3k}. $\gamma_{sp}(2P_{3k}) = 2\gamma_{sp}(P_{3k}) = 2k$. $|S| = 2k+1$. Therefore $\gamma_{sp}(G) \geq 2k + 1$. Hence $\gamma_{sp}(G) = 2k + 1$.

Remark. $L_2 = K_{1,6}$. Therefore $\gamma_{sp}(L_2) = 1$.

Theorem 3.11. $bc_{sp}(L_n) = \begin{cases} 1 & if\ n = 3k, 3k+1, k \geq 1 \\ 3 & if\ n = 3k+2, k \geq 1 \end{cases}$

Proof. Let G = L_n, $n \geq 3$. V(G) = $\{u_i, v_j / 1 \leq i \leq 2n, 1 \leq j \leq 2n - 1\}$.

Case (1). Let n = 3k, $k \geq 1$. Let e = $v_{3k-2}v_{3k}$. deg $v_{3k} = 6k + 3 = \triangle(G + e)$, deg $u_i = 1$, $1 \leq i \leq 6k$, deg v_1 = deg $v_{6k-1} = 1$, deg $v_j = 2$, $2 \leq j \leq 6k - 2$, $j \neq 3k - 2$, deg $v_{3k-2} = 3$. T = $\{v_2, v_5, \ldots, v_{3k-4}, v_{3k}, v_{3k+3}, v_{3k+6}, \ldots, v_{6k-3}, v_{6k-2}\}$ is a strong perfect dominating set of G + e. $|T| = \frac{3k-4-2}{3} + 2 + \frac{6k-3-3k-3}{3} + 2 = 2k$. $\gamma_{sp}(G + e) \leq 2k$. Let S be any strong perfect dominating set of G + e. Since v_{3k} is unique the maximum degree vertex in G+ e. v_{3k} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3k-3} \cup v_{3k+2}, v_{3k+3}, \ldots, v_{6k-1}$ is $P_{3k-3} \cup P_{3k-2}$. $\gamma_{sp}(P_{3(k-1)}) + \gamma_{sp}(P_{3(k-1)+1}) = 2k - 1$. $|S| = 2k$. Therefore $\gamma_{sp}(G + e) \geq 2k$. Hence $\gamma_{sp}(G + e) = 2k$. Therefore $\gamma_{sp}(G + e) < 2k + 1 = \gamma_{sp}(G)$. Hence $bc_{sp}(G) = 1$.

Case (2). Let n = 3k+1, $k \geq 1$. Let e = $v_{3k-1}v_{3k+1}$. deg $v_{3k+1} = 6k + 5 = \triangle(G + e)$, deg $u_i = 1, 1 \leq i \leq 6k + 2$, deg v_1 = deg $v_{6k+1} = 1$, deg $v_j = 2$, $2 \leq j \leq 6k$, $j \neq 3k - 1$, deg $v_{3k-1} = 3$. T = $\{v_2, v_3, v_6, \ldots, v_{3k-3}, v_{3k+1}, v_{3k+4}, \ldots, v_{6k-2}, v_{6k-1}, v_{6k}\}$ is a strong perfect dominating set of G + e. $|T| = 1 + \frac{3k-3-3}{3} + 2 + \frac{6k-2-3k-4}{3} + 3 = 2k + 2$. $\gamma_{sp}(G + e) \leq 2k + 2$. Let S be any strong perfect dominating set of G +e. Since v_{3k+1} is unique the maximum degree vertex in G +e. v_{3k+1} belongs to S. The subgraph induced by the vertices $v_1, v_2, \ldots, v_{3k-2} \cup v_{3k+3}, v_{3k+4}, \ldots, v_{6k+1}$ is $P_{3k-2} \cup P_{3k-1}$. $\gamma_{sp}(P_{3(k-1)+1}) + \gamma_{sp}(P_{3(k-1)+2})$

= 2k + 1. |S| = 2k+ 2. Therefore γ_{sp} (G + e) ≥ 2k + 2. Hence γ_{sp} (G + e) = 2k + 2< 2k + 3 = γ_{sp}(G). Hence bc_{sp}(G) = 1.

Case (3). Let n = 3k+2, k ≥ 1. γ_{sp}(G) = 2k +1. To reduce the strong perfect domination number, at least one edge must be added. Since v_{3k+2} is the unique maximum degree vertex, it belongs to any strong perfect dominating set. Make v_{3k+2} adjacent with v_{3k}, v_{3k-1}, v_{3k-2}. So that strong perfect domination number decreases by at least one. Let X = $\{v_{3k}v_{3k+2}, v_{3k-1}v_{3k+2}, v_{3k-2}v_{3k+2}\}$. T = $\{v_2, v_5, \ldots, v_{3k-4}, v_{3k+2}, v_{3k+5}, v_{3k+8}, \ldots, v_{6k+2}\}$ is a strong perfect dominating set of G + X. Hence γ_{sp} (G + X) = 2k< 2k +1 = γ_{sp}(G). Hence bc_{sp}(G) = 3.

4. Conclusion

In this paper, strong perfect cobondage number of some standard graphs and some special graphs are studied. Further study can be observed for path related graphs, cycle related graphs, subdivision graphs, middle graphs and product graphs. Bounds for strong perfect cobondage number of graphs and Nordhaus Gaddum type results can be determined.

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