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Research Paper

Coefficient Bound of a New Generalized Class of Tilted Analytic Univalent Functions

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Abstract

This paper is concerned with the new generalized class of tilted analytic univalent functions, $S^*(\theta,\alpha,s,t)$ which denoted as.

$$Re\{e^{i\theta} f'(z)\frac{z}{m(z)}\} > \alpha,$$

for $\cos\theta > \alpha$, $|\theta| < \pi$, $0 \le \alpha < 1$, $m(z)=\frac{z}{(1-sz)(1-tz)}$, $s \ne 1$, $-1 \le t < 1$, $s \ne t$ and $s, t \in \mathbb{C}$ which is analytic in the unit disk Δ={w:|w|<1}. The coefficient bound as well as representation theorem of extremal properties is obtained $S^*(\theta,\alpha,s,t)$.

Keywords

Analytic Functions, Univalent Functions, Representation Theorem, Coefficient Bound

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1. INTRODUCTION

Let **U** represent the class of functions presented by

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots + a_nz^n + \ldots \quad (1)$$

which are regular in Δ={w:|w|<1} and is the subclass of **U** consisting of normalized univalent functions which are satisfying $f(0)=0$ and $f'(0)-1=0$ (Goodman, 1983; Duren, 2001). We also denote the subclass of **U** containing functions that are starlike, convex, and close-to-convex by *St*, *K* and *C* respectively. Silverman and Silvia (1996) obtained representation theorem and coefficient bound for the class $G(\delta, 0)$ satisfies $Re(e^{i\delta}\frac{f'z}{h'z}) > 0$ $|\delta| < \frac{\pi}{2}$, $h(z) \in K$ and $z \in \Delta$. Subsequently, Mohamad (1998) introduced the class $G(\lambda,\gamma)$ satisfies $Re(e^{i\lambda}\frac{f'z}{h'z}) > \gamma$,where $|\lambda| \le \pi$, $\cos\lambda - \gamma > 0$ and $z \in \Delta$ which is the same $h,(z)$ done by Silverman and Silvia (1996) and discovered representation theorem and coefficient bound of $G(\alpha,\delta)$. Afterward, Soh and Mohamad (2006) determined extremal properties for the class of functions which $h(z)= -z-2(\log(1-z))$ implies $n'(z)=(1+(z))(1-z)^{-1}$ where it is the extreme function produced by MacGregor (1962). Next, Cik Soh (2009) introduced the class $G(\lambda,\gamma,\gamma)$ with the form of $k_\gamma(z)= -z(1-2\gamma)-(\log(1-z))(2-2\gamma)$ where $0 \le \gamma \le 1$ $g'(z)= \frac{1+(1-2\gamma)z}{1-z}$ and the author presented some extremal properties of $G(\lambda,\gamma,\gamma)$. Kaharudin (2011) has expanded the results from Cik Soh (2009) by presenting a special case of $g_\beta(z)$ when $\beta=\frac{1}{2}$ given by Brickman et al. (1973) and found important properties of the coefficient bound and the rotation theorem for this class of functions. In addition, many researchers such as MacGregor (1962), Goel (1967), MacGregor (1964), Silverman (1972), Goel and Mehrok (1983), Silverman and Telage (1979), Fukui et al. (1987), Yahya et al. (2014), Kaplan (1952), Mohamad (2000), Akbarally et al. (2011), Wang (2010), Shashkin (1994), Çağlar et al. (2013), Magesh and Yamini (2013), Elhaddad and Darus (2020), Adegani et al. (2021), and Shaba and Wanas (2022) had studied geometrical properties of the different class of functions.

In the present paper, we introduce a new generalized class of tilted analytic univalent functions $S^*(\theta,\alpha,s,t)$ as the class of normalized functions $f \in$ **U** fulfilling the condition

$$Re\left(e^{i\theta}\frac{zf'(z)}{m(z)}\right) > \alpha, (z \in \Delta)$$

for $\cos\theta > \alpha$, $|\theta| < \pi$, $0 \le \alpha < 1$, $m(z)= \frac{z}{(1-sz)(1-tz)}$, $s \ne 1$, $-1 \le t < 1$, $s \ne t$ and $s, t \in \mathbb{C}$. Thus, our purpose in the present paper is to obtain representation theorem and coefficient bound of $S^*(\theta,\alpha,s,t)$f.

2. REPRESENTATION THEOREM

Suppose that P is the class of all functions with a positive real part Δ={w:|w|<1}. Thus, we can write the function $p(z) \in P= p(z)= 1+ p_1z+ p_2z^2+...+ p_nz^n+...$ as that is analytic in $\Delta=\{w : |w| < 1\}$ such that $Re\{p(z)\}>0$. Next, we relate the functions $S^*(\theta,\alpha,s,t)$ to the functions in P. For any $f \in \mathbf{U}$, we have

$$e^{i\theta}\frac{\frac{zf'(z)}{m(z)} - \alpha - i\sin\theta}{\cos\theta - \alpha} = P, (z, \in, \Delta) \tag{2}$$

The approach of The Herglotz Representation Theorem will be carried on for the establishment of the representation theorem $S^*(\theta,\alpha,s,t)$

2.1 Theorem

Let $f \in S^*(\theta,\alpha,s,t)$, then some probability measure μ on the unit circle $X f$ can be shown as

$$f(z) = \int_x \Bigg[\left[\frac{\log(1-xsz)}{x} - \frac{\log(1-xtz)}{x}\right] + \left(2A_{\theta\alpha}e^{-1\theta}\right) \left[-\frac{\log(1-xz)}{(s-1)-(t-1)x} + \log(1-xsz)\frac{s}{(t-s)-(s-1)x} - \frac{t\log(1-xtz)}{(t-s)-(t-1)x}\right] \left(\frac{2\alpha e^{i\theta - e^{-2i\theta}}}{(t-s)}\right)\Bigg] d\mu(x) \tag{3}$$

and conversely, $if\ t$ is given by the (3), then $f \in S^*(\theta,\alpha,s,t)$.

Proof.

We have $p \in P$ which is $p(z)=\int_x (1+xz)(1-xz)^{-1}\, d\mu(x)$, $|x|=1$, for some probability measure μ on the unit circle X. Rearranging yields,

$$e^{i\theta}\frac{zf'(z)}{m(z)}z = p(z)(\cos\theta - \alpha) + \alpha + i\sin\theta,$$

and by replacing $\cos\theta - \alpha = A_{\theta\alpha}$ that is always positive, we have

$$f'(z) = \frac{m(z)}{ze^{i\theta}}\left[A_{\theta\alpha}p(z) + \alpha + i\sin\theta\right]$$

Therefore,

$$f'(z) = e^{-i\theta}\frac{m(z)}{z}\left[A_{\theta\alpha}p(z) + \alpha + i\sin\theta\right] \tag{4}$$

Substituting $m(z)=\frac{z}{(1-sz)(1-tz)}$ into (4), we have

$$f'(z) = e^{-i\theta}\left(\frac{z}{(1-sz)(1-tz)}\right)\left(\frac{1}{z}\right)\left[A_{\theta\alpha}p(z) + \alpha + i\sin\theta\right]$$

and

$$f'(z) = e^{-i\theta}\left(\frac{1}{(1-sz)(1-tz)}\right)\left[A_{\theta\alpha}p(z) + \alpha + i\sin\theta\right] \tag{5}$$

From, we have

$$f'(z) = e^{-i\theta}\left(\frac{1}{(1-sz)(1-tz)}\right)\left[A_{\theta\alpha}\int_x \frac{(1+xz)}{(1-xz)}d\mu(x) + \alpha + i\sin\theta\right]$$

Then, we obtain

$$f'(z) = \int_x e^{-i\theta}\left[\frac{(\alpha + i\sin\theta)(1-xz) + (1+xz)A_{\theta\alpha}}{(1-xsz)(1-xtz)(1-xz)}\right]d\mu(x)$$

It follows that

$$f(z) = \int_0^z\int_x e^{-i\theta}\left[\frac{A_{\theta\alpha}(1+x\varphi) + (\alpha + i\sin\theta)(1-x\varphi)}{(1-xs\varphi)(1-xt\varphi)(1-x\varphi)}\right] d\mu(x)d\varphi$$

and

$$f(z) = \int_0^z\int_x e^{-i\theta}\left[\frac{(\cos\theta + x\varphi\cos\theta - \alpha - x\varphi\alpha + i\sin\theta - x\varphi i\sin\theta + \alpha - x\varphi\alpha)}{(1-xs\varphi)(1-xt\varphi)(1-x\varphi)}\right] d\mu(x)d\varphi$$

Then,

$$f(z) = \int_0^z\int_x e^{-i\theta}\left[\frac{(\cos\theta + i\sin\theta) + x\varphi(\cos\theta - i\sin\theta) - 2x\varphi + \alpha - x\varphi\alpha)}{(1-xs\varphi)(1-xt\varphi)(1-x\varphi)}\right]d\mu(x)d\varphi$$

Since $\cos a + i\sin a = e^{ia}$ and $\cos a - i\sin a = e^{-ia}$

$$f(z) = \int_0^z\int_x \left[\frac{1 + x\varphi\left[(e^{-2i\theta}\cos\theta) - 2\alpha(e^{-i\theta})\right]}{(1-xs\varphi)(1-xt\varphi)(1-x\varphi)}\right]d\mu(x)d\varphi \tag{6}$$

Upon simplification (6) , we have

$$f(z) = \int_0^z\int_x \left[\frac{(e^{-i\theta} - 2\alpha)e^{-i\theta} + 1}{(1-xs\varphi)(1-xt\varphi)(1-x\varphi)}\frac{-(e^{-i\theta} - 2\alpha)e^{-i\theta}}{(1-xs\varphi)(1-xt\varphi)}\right]d\mu(x)d\varphi$$

By changing the integration order, we obtain that

$$f(z) = \int_x \Bigg[\int_0^z \Bigg[\frac{(-e^{-2i\theta} + 2\alpha e^{-i\theta})}{(t-s)}\left(-\frac{s}{(1-xs\varphi)} + \frac{t}{(1-xt\varphi)}\right) + (2A_{\theta\alpha}e^{-i\theta})\left(\frac{1}{(s-1)(t-1)(1-x\varphi)} - \frac{s^2}{(t-s)(s-1)(1-xs\varphi)} + \frac{t^2}{(t-s)(t-1)(1-xt\varphi)}\right)\Bigg] d\varphi\Bigg] d\mu(x) \tag{7}$$

Integrating (7) ϕ concerning,

$$f(z) = \int_x \left[\left[\frac{\log(1-xsz)}{(x)} - \frac{\log(1-xtz)}{(x)}\right] + (2A_{\theta\alpha}e^{-i\theta})\left[-\frac{\log(1-xz)}{(s-1)(t-1)x} + \frac{s\log(1-xsz)}{(-s+t)(s-1)x} - \frac{t\log(1-xtz)}{(-s+t)(t-1)x}\right]\left(\frac{2\alpha e^{-i\theta}-e^{-2i\theta}}{(-s+t)}\right)\right] d\mu(x)$$

which $|x|=1$, and this is desired representation.

2.2 Corollary

The extreme points $S^*(\theta, \alpha, s, t)$ are the unit points masses

$$f_x(z) = \int_x \left[\left[\frac{\log(1-xsz)}{(x)} - \frac{\log(1-xtz)}{(x)}\right] + (2A_{\theta\alpha}e^{-i\theta}) \left[-\frac{\log(1-xz)}{(-1+s)(-1+t)x} + \frac{s\log(1-xsz)}{(-s+t)(-1+s)x} - \frac{t\log(1-xtz)}{(-s+t)(-1+t)x}\right]\left(\frac{2\alpha e^{-i\theta}-e^{-2i\theta}}{(-s+t)}\right)\right],$$

with $|x|=1$. Meanwhile,

$$f_x'(z) = \frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})}{(t-s)}\left(-\frac{s}{(1-xsz)} + \frac{t}{(1-xtz)}\right) + (2A_{\theta\alpha}e^{-i\theta})\left(\frac{1}{(-1+s)(-1+t)(1-xz)} - \frac{s^2}{(-s+t)(-1+s)(1-xsz)} + \frac{t^2}{(-s+t)(-1+t)(1-xtz)}\right),$$

is the derivatives of the extreme points for $S^*(\theta, \alpha, s, t)$ with $|(x)|=1$.

Thus f is normalized functions since

$$f(0) = \int_x \left[\left[\frac{\log(1-xs(0))}{(x)} - \frac{\log(1-xt(0))}{(x)}\right] + (2A_{\theta\alpha}e^{-i\theta}) \left[-\frac{\log(1-x(0))}{(-1+s)(-1+t)x} + \frac{s\log(1-xs(0))}{(-s+t)(-1+s)x} - \frac{t\log(1-xt(0))}{(-s+t)(-1+t)x}\right]\left(\frac{2\alpha e^{-i\theta}-e^{-2i\theta}}{(-s+t)}\right)\right] = 0$$

and

$$f'(0) = \frac{(-e^{-2i\theta}+(2\alpha)e^{-i\theta})}{(-s+t)}\left(\frac{t}{(1-xt(0))} - \frac{s}{(1-xs(0))}\right) + (2A_{\theta\alpha}e^{-i\theta})\left(\frac{1}{(s-1)(t-1)(1-x(0))} - \frac{s^2}{(t-s)(s-1)(1-xs(0))} + \frac{t^2}{(t-s)(t-1)(1-xt(0))}\right) = 1$$

3. MAIN RESULT

Now, we proceed on finding coefficient bound of $S^*(\theta, \alpha, s, t)$f.

3.1 Theorem

if $f \epsilon S^*(\theta, \alpha, s, t)$, then

$$|a_n| \leqq \frac{s.s^n + A_{\theta\alpha}(2s^n - 4s.s^n)}{n(-s+t)(-s+1)} + \frac{A_\theta\alpha(2t^n - 4t.t^n) - t^n + t.t^n}{n(-s+1)(-1+t)} + \frac{2A_\theta\alpha}{n(-1+s)(-1+t)}$$

for n=2,3,4,...and equality is attained for each n when f is an extreme point of $S^*(\theta, \alpha, s, t)$.

Proof.

From (7), we have

$$f(z) = \int_0^z \left[\int_x \left[-\frac{(-e^{-2i\theta}+e^{-i\theta}(2\alpha))s}{(t-s)(1-xs\varphi)} - \frac{(e^{-i\theta}(2A_{\theta\alpha}))s^2}{(t-s)(s-1)(1-xs\varphi)} + \frac{(-e^{-i\theta}+(2\alpha))te^{-i\theta}}{(-s+t)(1-xt\varphi)} - \frac{e^{-i\theta}(2A_{\theta\alpha})t^2}{(-s+t)(-1+t)(1-xt\varphi)} + \frac{2e^{-i\theta}(A_{\theta\alpha})}{(-1+s)(-1+t)(1-x\varphi)}\right] d\mu(x)\right] d\varphi$$

and

$$f(z) = \int_0^z \left[\int_x \left[\left(-\frac{(-e^{-2i\theta}+e^{-i\theta}(2\alpha))s}{(-s+t)} - \frac{(e^{-i\theta}(2A_{\theta\alpha}))s^2}{(-s+t)(-1+s)}\right) \frac{1}{(1-xs\varphi)} + \left(\frac{(e^{-i\theta}(-e^{1\theta}+2\alpha)t}{(-s+t)(1-xt\varphi)} - \frac{e^{-i\theta}(2A_{\theta\alpha})t^2}{(-s+t)(-1+t)}\right) \frac{1}{(1-xt\varphi)} + \left(\frac{e^{-i\theta}(2A_{\theta\alpha})}{(-1+s)(-1+t)}\right)\frac{1}{(1-x\varphi)}\right] d\mu(x)\right] d\varphi \quad (8)$$

Next, (8) be written as

$$f(z) = \int_0^z \left[\left(-\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})s}{(-s+t)} - \frac{(2A_{\theta\alpha}e^{-i\theta})s^2}{(-s+t)(-1+s)}\right) \int_x \sum_0^\infty \left[(sx)^n\right] d\mu(x)(\varphi)^n + \left(\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})t}{(-s+t)} - \frac{(2A_{\theta\alpha}e^{-i\theta})t^2}{(-s+t)(-1+t)}\right)\int_x \sum_0^\infty \left[(tx)^n\right] d\mu(x)(\varphi)^n + \left(\frac{(2A_{\theta\alpha}e^{-i\theta})}{(-1+s)(-1+t)}\right)\int_x \sum_0^\infty \left[x^n d\mu\right](x)(\varphi)^n\right] d\varphi \quad (9)$$

From (9), substituting n=0, then

$$f(z) = -\frac{(-e^{-i\theta}+2\alpha)e^{-i\theta}(-1+s)(-1+t)s}{(-s+t)(-1+s)(-1+t)} - \frac{e^{-i\theta}(2A_{\theta\alpha})(-1+t)s^2}{(-s+t)(-1+s)(-1+t)} + \frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})(-1+s)(-1+t)t}{(-s+t)(-1+s)(-1+t)} - \frac{(2A_{\theta\alpha}e^{-i\theta})(-1+s)t^2}{(-s+t)(-1+s)(-1+s)} + \frac{(2A_{\theta\alpha}e^{-i\theta})(-s+t)}{(-s+t)(-1+s)(-1+t)}$$

and

$$\begin{aligned} f(z) =&\frac{1}{(t-s)(s-1)(t-1)}(e^{-2i\theta}s^2t - 2\alpha s^2 t - (e^{-2i\theta})s^2 \\ &- e^{-2i\theta}(st) + (2\alpha s^2)e^{-i\theta} + (2\alpha st)e^{-i\theta} - 2\alpha s e^{-i\theta} \\ &+ se^{-2i\theta}) - (2s^2t - 2s^2)A_{\theta\alpha}e^{-i\theta} + (st^2(-e^{-2i\theta}) \\ &+ 2\alpha e^{-i\theta}st^2 + e^{-2i\theta}st + e^{-2i\theta}t^2 - e^{-i\theta}(2\alpha st) \\ &- e^{-i\theta}(2\alpha t^2) - e^{-2i\theta}t + e^{-i\theta}(2\alpha t) + (2st^2 - 2t^2) \\ &A_{\theta\alpha}e^{-i\theta} + (2t-2s)A_{\theta\alpha}e^{-i\theta}) \end{aligned} \tag{10}$$

Upon simplification the equation (10), we have

$$\begin{aligned} f(z) =&\frac{1}{(t-s)(s-1)(t-1)}\left[(s^2t - s^2 + s - st^2 + t^2 - t)e^{-2i\theta}\right. \\ &+(-2s^2t - 2s + 2s^2 + 2st^2 - 2t^2 + 2t)\alpha e^{-i\theta} \\ &+(-2s^2t - 2s + 2s^2 + 2st^2 - 2t^2 + 2t)\alpha e^{-i\theta} \\ &\left.+(-2s^2t + 2s^2 + 2st^2 + 2t - 2t^2 - 2s)A_{\theta\alpha}e^{-i\theta}\right] \end{aligned}$$

Substituting $A_{\theta\alpha}$= cosθ-α and cosθ=$\frac{e^{i\theta}-e^{-i\theta}}{2}$, thus

$$\begin{aligned} f(z) =&\frac{1}{(t-s)(s-1)(t-1)}\left[(s^2t - s^2 + s - st^2 + t^2 - t)e^{-2i\theta}\right. \\ &+(-2s^2t + 2s^2 - 2s + 2st^2 - 2t^2 + 2t)\alpha e^{-i\theta} \\ &+(-2s^2t + 2s^2 + 2st^2 - 2t^2 + 2t - 2s) \\ &\left.\left(\frac{1 + 2e^{-i\theta} - 2\alpha e^{-i\theta}}{2}\right)\right] \end{aligned}$$

and

$$\begin{aligned} f(z) =& \frac{1}{(t-s)(s-1)(t-1)} + \left[(-2s^2t + 2s^2 - 2s + 2st^2 - 2t^2\right. \\ &+2t)\alpha e^{-i\theta} + (-s^2t + s^2 - s + st^2 - t^2 + t)e^{-2i\theta} \\ &+(2s^2t - 2s^2 + 2s - 2st^2 + 2t^2 - 2t) + (-s^2t + s^2 \\ &\left.-s + st^2 - t^2 + t)\right] \\ =& \frac{1}{(-s+t)(s-1)(t-1)}\left[(-s+t)(s-1)(t-1)\right] \\ =& 1 \end{aligned}$$

Rewrite the equation (9) will yield to

$$\begin{aligned} f(z) =& \int_0^w \left[1 - \left(\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})s}{(t-s)} - \frac{(2A_{\theta\alpha}e^{-i\theta})s^2}{(t-s)(s-1)}\right)\right. \\ &\int_x \sum_0^\infty [(sx)^n]\, d\mu(x)(\varphi)^n + \left(\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})t}{(t-s)}\right. \\ &\left.-\frac{(2A_{\theta\alpha}e^{-i\theta})t^2}{(t-s)(t-1)}\right)\int_x \sum_0^\infty [(tx)^n]\, d\mu(x)(\varphi)^n \\ &\left.+\left(\frac{(2A_{\theta\alpha}e^{-i\theta})}{(s-1)(t-1)}\right)\int_x \sum_0^\infty [x^n d\mu](x)(\varphi)^n\right] d\varphi \end{aligned}$$

Integrating ϕ concerning gives us,

$$\begin{aligned} f(z) =&z - \left(\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})s}{(t-s)} - \frac{(2A_{\theta\alpha}e^{-i\theta})s^2}{(t-s)(s-1)}\right) \\ &\int_x \sum_2^\infty \left[\frac{(sx)^{n-1}}{n}\right] d\mu(x)z^n + \left(\frac{(-e^{-2i\theta}+2\alpha e^{-i\theta})t}{(t-s)}\right. \\ &\left.-\frac{(2A_{\theta\alpha}e^{-i\theta})t^2}{(t-s)(t-1)}\right)\int_x \sum_2^\infty \left[\frac{(tx)^{n-1}}{n}\right] d\mu(x)z^n \\ &+\left(\frac{(2A_{\theta\alpha}e^{-i\theta})}{(s-1)(t-1)}\right)\int_x \sum_2^\infty \left[\frac{(x)^{n-1}}{n}\right] d\mu(x)z^n \end{aligned}$$

As a comparison above equation (1) , we have

$$\begin{aligned} a_n =& -\left(\frac{e^{-i\theta}(-e^{-i\theta}+2\alpha)s}{(-s+t)n} - \frac{e^{-i\theta}(2A_{\theta\alpha})s^2}{(-s+t)(s-1)n}\right) \\ &\int_x \frac{(sx)^n}{nsx} d\mu(x) + \left(\frac{e^{-i\theta}(-e^{-i\theta}+2\alpha)t}{(-s+t)n} - \frac{e^{-i\theta}(2A_{\theta\alpha})t^2}{(-s+t)(t-1)n}\right) \\ &\int_x (tx)^{n-1} d\mu(x) + \left(\frac{e^{-i\theta}(2A_{\theta\alpha})}{n(s-1)(t-1)}\right)\int_x \frac{(x)^n}{nx} d\mu(x) \end{aligned}$$

and

$$\begin{aligned} |a_n| =& \left|\left[-\left(\frac{e^{-i\theta}(-e^{-i\theta}+2\alpha)s}{(-s+t)n} - \frac{e^{-i\theta}(2A_{\theta\alpha})s^2}{(-s+t)(s-1)n}\right)(s)^{n-1}\right.\right. \\ &+\left(\frac{e^{-i\theta}(-e^{-i\theta}+2\alpha)t}{(-s+t)n} - \frac{e^{-i\theta}(2A_{\theta\alpha})t^2}{(-s+t)(t-1)n}\right)(t)^{n-1} \\ &\left.\left.+\left(\frac{e^{-i\theta}(2A_{\theta\alpha})}{n(s-1)(t-1)}\right)\right]\left[\int_x (x)^{n-1} d\mu(x)\right]\right|. \end{aligned}$$

Upon simplification $-e^{-2i\theta}$ + $2\alpha e^{-i\theta}$= 1-2 $A_{\theta\alpha}e^{-i\theta}$, we have

$$|a_n| = \frac{1}{n}\left|\left[\int_x (x)^{n-1} d\mu(x)\right]\left[\left(\frac{(2A_{\theta\alpha})s^2 e^{-i\theta}}{(-s+t)(s-1)} - \frac{(1-[e^{i\theta}]\,2A_{\theta\alpha})s}{(-s+t)}\right)(s)^{n-1} + \left(\frac{(1-2e^{-i\theta}A_{\theta\alpha})t}{(-s+t)} - \frac{e^{-i\theta}(2A_{\theta}\alpha)t^2}{(-s+t)(t-1)}\right)(t)^{n-1} + \left(\frac{e^{-i\theta}(2A_{\theta}\alpha)}{(s-1)(t-1)}\right)\right]\right|.$$

and

$$|a_n| = \frac{1}{n}\left|\left[\int_x (x)^{n-1} d\mu(x)\right]\left[-\left(\frac{s.s^n - 4A_{\theta\alpha}s.s^n e^{-i\theta} - s^n + 2s^n A_{\theta\alpha}e^{-i\theta}}{(-s+t)(-1+s)}\right) + \left(\frac{t.t^n - 4A_{\theta\alpha}t.t^n e^{-i\theta} - t^n + 2t^n A_{\theta\alpha}e^{-i\theta}}{(-s+t)(-1+t)}\right) + \left(\frac{2A_{\theta\alpha}e^{-i\theta}}{(-1+s)(-1+t)}\right)\right]\right|$$

Then, since $|e^{-i\theta}|=1$,

$$|a_n| = \frac{1}{n}\left|\left[\int_x (x)^{n-1} d\mu(x)\right]\left[-\left(\frac{s.s^n - s^n + (2s^n - 4s.s^n)A_{\theta\alpha}e^{-i\theta}}{(-s+t)(-1+s)}\right) + \left(\frac{t.t^n - t^n + (2t^n - 4t.t^n)A_{\theta\alpha}e^{-i\theta}}{(-s+t)(-1+s)}\right) + \left(\frac{2A_{\theta\alpha}e^{-i\theta}}{(-s+t)(-1+s)}\right)\right]\right| \le \frac{1}{n}\int_x |(x)^{n-1}| d\mu(x)$$

$$\left[\frac{s.s^n - s^n + (2s^n - 4s.s^n)A_{\theta\alpha}}{(-s+t)(-1+s)} + \frac{t.t^n - t^n + (2t^n - 4t.t^n)A_{\theta\alpha}}{(-s+t)(-1+s)} + \frac{2A_{\theta\alpha}}{(-s+t)(-1+s)}\right]$$

$$= \frac{s^{n+1} - s^n + (2s^n - 4s^{n+1})A_{\theta\alpha}}{(-s+t)(-1+s)n} + \frac{t^{n+1} - t^n + (2t^n - 4t^{n+1})A_{\theta\alpha}}{(-s+t)(-1+s)n} + \frac{2A_{\theta\alpha}}{(-s+t)(-1+s)n}$$

as required. Equality is attained for n=2,3,4,...when f is an extreme point of $S^*(\theta, \alpha, s, t)$

3.2 Remark

if $f \in S^*(\theta, \alpha, 0, 0)$, we reduce to the class $G(\theta,\alpha)$ studied by Mohamad (1998) and Mohamad (2000). The coefficient bound is given by

$$|a_n| \le \frac{2A_{\theta\alpha}}{n} \text{ for } n=2,3,4,...$$

4. CONCLUSIONS

This paper has applied Herglotz Representation Theorem in finding representation theorem and coefficient bound for $S^*(\theta, \alpha, s, t)$. The results will contribute new knowledge and technique in geometric function theory.

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