Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 8, No. 2, April 2023

Research Paper

Extension of Exponential Pareto Distribution with the Order Statistics: Some Properties and Application to Lifetime Datasets

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Abstract
The Exponential Pareto (EP) model has been extended by applied and theoretical statisticians for wider applications and new knowledge using different techniques but the Weibull-X technique has not been considered. This article proposed a new extension of the EP model called the Weibull-Exponential Pareto (WEP) distribution to provide better modeling that fits real-life datasets and to explore the statistical theory of order statistics from the proposed distribution. Statistical properties investigated include the Shannon and Renyi entropies; the moments and moment generating function. Distribution of order statistics and the moment of order statistics were derived including the mean and variance of order statistics. WEP distribution has unimodal, decreasing, and increasing failure rates; and it can be negatively or positively skewed and approximately symmetric with the potential for fitting platykurtic, mesokurtic, and leptokurtic lifetime data. The parameters of the distribution were estimated using the method of maximum likelihood estimation (MLE), which was examined for consistency through a simulation study. The performance of the proposed distribution was investigated by application to flood peaks exceedances and some lifetime datasets from engineering. The results from data analysis using the R-software revealed that the WEP distribution has the potential to provide a superior model that fits the three data sets better than some notable existing distributions and previous extensions of the EP model in the literature. The statistical property of order statistics extended in the study established some important results that characterized some notable lifetime distributions in the literature.

Keywords
Exponential Pareto Model, Weibull-X Technique, Weibull-Exponential Pareto Distribution, Shannon and Renyi Entropies, Distribution of Order Statistics, Moment of Order Statistics

Received: 9 November 2022, Accepted: 6 February 2023
https://doi.org/10.26554/sti.2023.8.2.265-279

1. INTRODUCTION

Statisticians and researchers in general are motivated by burning desires to discover new distributions that are adequate and more flexible in terms of application to real-life problems. Extension of classical distributions and some other existing models have been suggested and successfully implemented in the statistical literature using various techniques. Previous efforts towards the actualization of the objectives are contained in Lee et al. (2007) using the beta-generator technique to extend the Weibull distribution, the T-X family of distribution was introduced by Alzaatreh et al. (2013b) and by taking T to be a Weibull distribution random variable, the Weibull-X was defined by Alzaatreh et al. (2013b) as a sub-family of the T-X family. Several important distributions have been proposed using the Weibull-X technique including the Weibull-Pareto model by (Alzaatreh et al., 2013a). The Weibull-Rayleigh distribution was developed by Akarawak et al. (2013) and later by Ahmad et al. (2017) with different motives and diverse applications when X follows the Rayleigh random variable.

Al Kadim and Boshi (2013) defined and studied the Exponential Pareto (EP) distribution with the cumulative distribution function (CDF) and probability density function (PDF) defined respectively as

$$F(x) = 1 - e^{-\beta\left(\frac{x}{k}\right)^{\theta}} \quad (1)$$

$$f(x) = \frac{\beta\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} e^{-\beta\left(\frac{x}{k}\right)^{\theta}}; \beta, k, \theta > 0 > 0; x > 0 \quad (2)$$

β,k,θ are the parameters of the distribution The EP distribution has further gained extensive studies with applications from Luguterah and Nasiru (2015), using the quadratic rank transmutation map (QRTM) to develop the Transmuted Exponential Pareto (TEP) distribution, the beta-G framework was used by Aryal (2019) and later by Rashwan and Kamel (2020) for the construction of Beta Exponential Pareto (BEP) model. Kumaraswamy Exponential Pareto (KEP) distribution was introduced by Elbatal and Aryal (2017) using the Kum-G technique and most recently, the Gompertz-G technique was explored for developing the Gompertz Exponential Pareto distribution by (Adeyemi et al., 2021). Dikko and Faisal (2017) proposed the generalized exponential Weibull (GEW) distribution and the Topp Leone Weibull distribution (TLWD) was introduced by (Tuoyo et al., 2021).

In another dimension, the new Weibull Pareto distribution (NWP) was developed by Nasiru and Luguterah (2015); and thereafter, with the aid of the QRTM, Tahir and Akhter (2018) extended the NWP to produce the Transmuted new Weibull Pareto (TNWP) and most recently, Aljuhani et al. (2022) introduced the Alpha Power Exponentiated New Wuibull Pareto distribution from the NWP model and Hassan et al. (2022) developed the Kumaraswamy extended Exponential (KwEE). Nevertheless, instances abound where these existing distributions have not been able to explain some of the real-life problems adequately through the analysis of their corresponding datasets, making this study gap a challenge that is constantly been addressed by researchers.

This current study is aimed at exploring the versatility of the Weibull distribution as a generator by using the Weibull-X technique to extend the EP model. The proposed model called the Weibull Exponential Pareto (WEP) distribution is expected to provide more flexibility for addressing various forms of kurtosis and skewness associated with real-life datasets describing some of the random events in our environments. The remaining part of the study is organized as follows; Section 2 is devoted to describing and developing the proposed distribution with the sub-models and some of the statistical properties. Section 3 discussed the procedure for the estimation of parameters and simulation study. Applications to three real-life datasets were carried out to assess the importance of the distribution in Section 3. Section 4 concludes the work and Section 5 for acknowledgment.

2. EXPERIMENTAL SECTION

This section is used for the design of the new distributions from some existing resources in the statistical literatures; properties of the WEP distribution constructed are also investigated.

2.1 Materials and Method

A new method of generating continuous distribution proposed by Alzaatreh et al. (2013b) has the CDF for the T-X class of distribution defined as

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt = R\{-\log(1-F(x))\} \quad (3)$$

Where R(t) the CDF of a non-negative continuous random variable is T defined on [0,∞) and F(x) is the CDF of a random variable X. The PDF associated with Equation (3) is given by,

$$G(x) = \frac{f(x)}{1-F(x)} r\{-\log(1-F(x))\} \quad (4)$$

Where r(t) is the PDF of random variable T and the derivative of R(t). Let T be a random variable from the Weibull distribution with parameters α and γ having the CDF given by,

$$R(t) = 1 - \exp\left(-\left(\frac{t}{\gamma}\right)^{\alpha}\right); \alpha, \gamma > 0, t > 0 \quad (5)$$

The CDF of the Weibull-X family is derived by inserting t=-log(1-F(x)) into Equation (5) to obtain Alzaatreh and Ghosh (2015) given by

$$G(x) = 1 - \exp\left(-\left(\frac{-\log(1-F(x))}{\gamma}\right)^{\alpha}\right) \quad (6)$$

And the PDF can be derived by taking the first derivative of Equation (6) to get

$$g(x) = \frac{\alpha}{\gamma}\frac{f(x)}{1-F(x)}\left(\frac{-\log(1-F(x))}{\gamma}\right)^{\alpha-1} \exp \left(-\left(\frac{-\log(1-F(x))}{\gamma}\right)^{\alpha}\right) \quad (7)$$

2.2 The New Extension of Exponential Pareto Distribution

The CDF of the proposed lifetime distribution called Weibull Exponential Pareto (WEP) distribution when X follows the EP distribution in Equation (1) is developed as;

$$G_1(x) = 1 - \exp\left(-\left(\frac{\beta\left(\frac{x}{k}\right)^{\theta}}{\gamma}\right)^{\alpha}\right) \quad (8)$$

Then by replacing β/γ with λ, the CDF in Equation (8) can be written as

$$G(x) = 1 - \exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \quad (9)$$

The derivative of Equation (9) is the corresponding PDF of the Weibull Exponential Pareto (WEP) distribution given by

$$g(x) = \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right); \alpha, \lambda, \theta, k > 0; x > 0 \quad (10)$$

A random variable X that follows the Weibull Exponential Pareto distribution with parameters α,λ,θ, and k is characterized with the density function g(x) in Equation (10) and is denoted by WEP (α,λ,θ,k).

2.3 Sub-Models of WEP Distribution

The important sub-models derived from the proposed distribution are presented in Table 1.

Plots of the shapes for the CDF and PDF of the distribution for some parameter values are displayed in Figure 1. Figure 2 provides the visual view of the shapes of the hazard rate function of the proposed distribution for some values of the parameter (α,λ,θ,k).

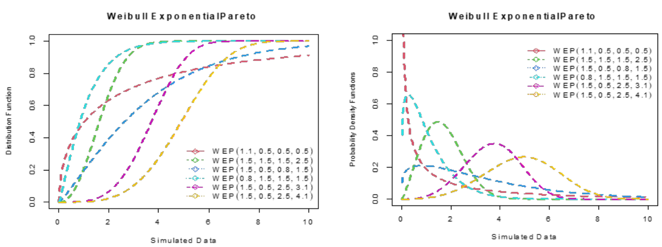

Figure 1. Plots of the CDFs and PDFs of WEP Distributions for Some Values of the Parameters

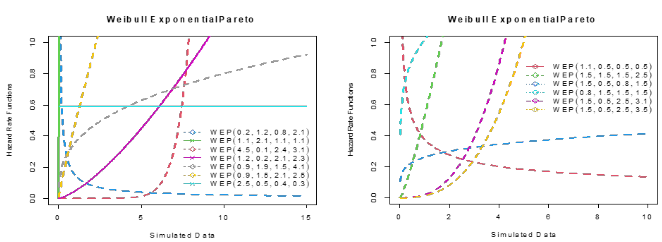

Figure 2. Plots of Hazard Rate Functions of WEP Distributions for Some Values of the Parameters

2.4 Properties of the Weibull Exponential Pareto Distribution

Some of the properties of the distribution are discussed in this section

2.4.1 The Reliability and the Hazard Rate Function

The reliability function is defined in similar works including Adeyemi et al. (2021) and is given by

$$S(x) = 1 - G(x) = \exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{11}$$

The hazard rate function is derived from Equation (9) and Equation (10) and is given by

$$h(x) = \frac{g(x)}{1 - G(x)} = \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \tag{12}$$

2.4.2 Asymptotic Behavior of WEP

The Asymptotic properties of the proposed model are investigated by taking limits of the density function, CDF, and the hazard rate function as x→∞ and as x→0.

Proposition 1: The limit of the WEP density function as $x \to \infty$ is 0 and as $x \to 0$ is;

$$\lim_{x\to 0} g(x) = \begin{cases} 0 & , \quad \alpha\theta > 1 \\ \frac{\lambda}{k} & , \quad \alpha\theta = 1 \\ \infty & , \quad \alpha\theta < 1 \end{cases}$$

Proof:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left[\begin{array}{c} \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha-1} \\ \exp\left(-\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha}\right) \end{array}\right] = 0$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha-1} \exp\left(-\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha}\right)$$

The limit of $\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)$ as x→0 is 1,

We now have a situation where as x→0 when θ>1, $\alpha\theta$>1 it becomes 0, when θ<1, $\alpha\theta$<1 it becomes ∞ ; when α=θ=1 we have $\lim_{x\to 0}\frac{\lambda}{k}\exp(-\{\lambda(\frac{x}{k})\})$ which reduces to constant $\frac{\lambda}{k}$. This completes the proof.

Proposition 2: The limit of WEP cdf as $x \to \infty$ is 1 and as $x \to 0$ is 0

Proof:

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty}\left[\ 1 - \exp\left(-\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha}\right)\ \right] = 1 - 0 = 1$$

$$\lim_{x\to 0} G(x) = \lim_{x\to 0}\left[\ 1 - \exp\left(-\{\lambda\left(\frac{x}{k}\right)^{\theta}\}^{\alpha}\right)\ \right] = 1 - 1 = 0$$

Proposition 3: The limit of the WEP hazard rate function as x→∞ is ∞ and as →0 is given by

$$\begin{cases} 0 & , \quad \alpha\theta > 1 \\ \frac{\lambda}{k} & , \quad \alpha\theta = 1 \\ \infty & , \quad \alpha\theta < 1 \end{cases}$$

Proof: The hazard rate function is defined as,

$$h(x) = \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\ \lambda\left(\frac{x}{k}\right)^{\theta}\ \right\}^{\alpha-1}$$

$\lim_{x\to\infty} h(x) = \frac{\infty}{0} = \infty$ when $\alpha\theta > 1$; $\lim_{x\to\infty} h(x) = \frac{\infty}{0} = \infty$ when $\alpha\theta < 1$. The asymptotic limits when $\alpha\theta = 1$ is also $\lim_{x\to\infty} h(x) = \frac{\lambda}{k}\left(\frac{1}{0}\right) = \infty$.

Table 1. Sub-Models of WEP Distribution

α	λ	θ	k	Reduced Model	Author/References
α	$\frac{1}{2\beta}$	2	k	Weibull-Rayleigh	Ahmad et al. (2017)
α	$\frac{1}{p}$	2	1	Weibull-Rayleigh	Akarawak et al. (2017)
1	λ	θ	k	Exponential Pareto	Al Kadim and Boshi (2013)
2	$\frac{1}{8\sigma^2}$	2	β	Rayleigh Rayleigh	Ateeq et al. (2019)
1	$\frac{\lambda *}{2}$	2	k	Exponential Rayleigh	New
1	1	λ	k	Exponential Exponential	New
1	$\frac{1}{2}$	2	k	Rayleigh	Rayleigh (1896)
1	$\frac{1}{2}$	θ	k	Weibull	Weibull (1951)

Asymptotes of hazard rate as x→0

$$\lim_{x\to 0} h(x) = \frac{0}{1} = 0 \text{ when } \alpha\theta > 1; \lim_{x\to 0} h(x) = \frac{\infty}{1} = \infty$$

when $\alpha\theta < 1$. The asymptotic limits when $\alpha\theta = 1$ is also

$\lim_{x\to 0} h(x) = \frac{\lambda}{k}$, the proof is completed.

Proposition 4: Let g(x) be the PDF and h(x) the hazard rate function of WEP distribution then, as x→0 we have g(0)=h(0). Proof: Combining results from propositions (1) and (3) established the proof and is given by,

$$\lim_{x\to 0} h(x) = \begin{cases} 0 & , \quad \alpha\theta > 1 \\ \frac{\lambda}{k} & , \quad \alpha\theta = 1 \\ \infty & , \quad \alpha\theta < 1 \end{cases} = \lim_{x\to 0} g(x)$$

2.4.3 Quantile Function, Simulation, and Median

Let U be a uniform random variable on the interval (0, 1); the quantile function is defined by Q(u)=G^{-1} (x), and the WEP distribution has the quantile function derived and presented as;

$$Q(u) = k\left\{ \ \tfrac{1}{\lambda}\left[-\log(1-u)^{\frac{1}{\alpha}}\right] \ \right\}^{\frac{1}{\theta}} \tag{13}$$

Let X be a random variable from the Weibull Exponential Pareto distribution, simulation can be done through the inverse transformation of the variable using uniform interval U (0, 1) and the random variable X taking as,

$$X = k\left\{ \ \tfrac{1}{\lambda}\left[-\log(1-u)^{\frac{1}{\alpha}}\right] \ \right\}^{\frac{1}{\theta}} \tag{14}$$

The median of the WEP can be derived by substituting u=0.5 in Equation (14) to get

$$\text{median} = k\left\{ \ \tfrac{1}{\lambda}\left[-\log(0.5)^{\frac{1}{\alpha}}\right] \ \right\}^{\frac{1}{\theta}} \tag{15}$$

2.4.4 Moments of WEP Distribution

Theorem 1: Let X be a continuous random variable from the WEP distribution with density function g(x), then the r^{th} moment about the origin is given by;

$$\mu^{(r)} = k^r\left(\frac{1}{\lambda}\right)^{r/\theta}\Gamma\left(\frac{r}{\alpha\theta}+1\right) \tag{16}$$

Proof :
The moment of X is defined as $E(x^r)= \int_{-\infty}^{\infty} x^r\, g(x)dx$

$$E(x^r) = \int_0^\infty x^r\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)dx \tag{17}$$

Let $u = \lambda\left(\frac{x}{k}\right)^{\theta}$ which implies that $x = k\left(\frac{u}{\lambda}\right)^{1/\theta}$

then substitute new variables so that

$$\frac{du}{dx} = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \text{ and } du = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}dx$$

$$E(x^r) = \int_0^\infty\left(k\left(\frac{u}{\lambda}\right)^{1/\theta}\right)^r\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}u^{\alpha-1}\exp(-u^\alpha) \frac{1}{\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}}du,$$

$$= \alpha k^r\left(\frac{1}{\lambda}\right)^{r/\theta}\int_0^\infty u^{\frac{r}{\theta}+\alpha-1}(-u^\alpha)du,$$

let $y = u^\alpha$, so that $u = y^{1/\alpha}$ hence $\frac{dy}{du} = \alpha u^{\alpha-1} = \alpha(y^{1/\alpha})^{\alpha-1}$

$$E(x^r) = \alpha k^r \left(\frac{1}{\lambda}\right)^{r/\theta} \int_0^\infty (y^{1/\alpha})^{\frac{r}{\theta}+\alpha-1} \exp(-y) \frac{1}{\alpha (y^{1/\alpha})^{\alpha-1}} dy$$

$$E(x^r) = k^r \left(\frac{1}{\lambda}\right)^{r/\theta} \int_0^\infty y^{\frac{r}{\alpha\theta}} \exp(-y) dy;$$

Using $\int_0^\infty y^m exp(-y)dy = \Gamma(m+1)$completes the proof.

The first four raw moments can be obtained from Equation (16) as follows

$$\mu_1' = k \left(\frac{1}{\alpha}\right)^{1/\theta} \Gamma\left(\frac{1}{\alpha\theta}+1\right);$$

$$\mu_2' = k^2 \left(\frac{1}{\alpha}\right)^{2/\theta} \Gamma\left(\frac{2}{\alpha\theta}+1\right);$$

$$\mu_3' = k^3 \left(\frac{1}{\alpha}\right)^{3/\theta} \Gamma\left(\frac{3}{\alpha\theta}+1\right);$$

$$\mu_4' = k^4 \left(\frac{1}{\alpha}\right)^{4/\theta} \Gamma\left(\frac{4}{\alpha\theta}+1\right);$$

2.4.5 The Mean, Variance, Skewness, and Kurtosis of WEP Distribution

The mean is the first moment about the origin when $r = 1$, and is derived as

$$E(X) = \mu_1' = k \left(\frac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{18}$$

The variance is obtained as

$$\text{Variance}(X) = \mu_2' - [\mu_1']^2$$
$$= k^2 \left\{ \left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) - \left(\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right)\right)^2 \right\} \tag{19}$$

The measures of skewness and kurtosis denoted by *Skew* and *Kurt* respectively are obtained using the first four raw moments in sub-section 2.4.4 as follows;

$$Skew = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}$$

$$= \frac{\left(k^3 \left(\tfrac{1}{\lambda}\right)^{3/\theta} \Gamma\left(\tfrac{3}{\alpha\theta}+1\right) \; -3k^2 \left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) k \left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right) + 2\left[k\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)\right]^3 \right)}{\left(k^2 \left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) - \left[k\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)^2\right]^{3/2} \right)} \tag{20}$$

$$Kurt = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2\mu^2 - 3\mu^4}{(\mu_2' - \mu^2)^2}$$

$$= \frac{\left(k^4 \left(\tfrac{1}{\lambda}\right)^{4/\theta} \Gamma\left(\tfrac{4}{\alpha\theta}+1\right) - 4k^3 \left(\tfrac{1}{\lambda}\right)^{3/\theta} \Gamma\left(\tfrac{3}{\alpha\theta}+1\right) k \left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right) + 6k^2 \left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) \left[k\left(\tfrac{1}{\alpha\theta}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)\right]^2 - 3\left[k\left(\tfrac{1}{\alpha\theta}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)\right]^4 \right)}{\left(k^2 \left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) - \left[k\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)^2\right]^{3/2} \right)} \tag{21}$$

Proposition 5: Let X be a random variable from the WEP distribution, the coefficient of variance denoted CV is given by

$$CV = \frac{\sqrt{\left\{\left(\tfrac{1}{\lambda}\right)^{2/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right) - \left(\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{2}{\alpha\theta}+1\right)\right)^2\right\}}}{\left(\tfrac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\tfrac{1}{\alpha\theta}+1\right)} \tag{22}$$

Proof The coefficient of variance is defined as

$$CV = \frac{\sqrt[2]{\text{variance}}}{E(x)}$$

By using Equations (18) and (19), the result is obtained.

2.4.6 Moment Generating Function of WEP Distribution

Theorem 2: Let X be a Weibull Exponential Pareto random variable with probability density function g(x), the moment generating function of X denoted M_x (t) is given by

$$\sum_{i=0}^{\infty} \frac{t^i}{i!} k^i \left(\frac{1}{\lambda}\right)^{i/\theta} \Gamma\left(\frac{i}{\alpha\theta}+1\right) \tag{23}$$

Proof: The moment generating function for a continuous random variable is defined Alzaatreh et al. (2013a) as;

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} g(x) dx$$

$$\int_0^\infty e^{tx} \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \left\{\lambda\left(\frac{x}{k}\right)^\theta\right\}^{\alpha-1} \exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^\theta\right\}^\alpha\right) dx$$

$$e^{tx} = \sum_{i=0}^{\infty} \left(1 + \frac{t^i x^i}{i!}\right) = \sum_{i=0}^{\infty} \frac{t^i x^i}{i!}$$

$$E(e^{tx}) == \sum_{i=0}^{\infty} \frac{t^i}{i!} E(x^i)$$

By substituting E(x^i) in Equation (16), we shall obtain

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(x^i) = \sum_{i=0}^{\infty} \frac{t^i}{i!} k^i \left(\frac{1}{\lambda}\right)^{i/\theta} \Gamma\left(\frac{i}{\alpha\theta} + 1\right)$$

2.4.7 Relationship between WEP and the Weibull-Pareto (WPD) Distribution

The relationship between the WEP and WPD is established as follows:

Theorem 3: Let Y be a random variable that follows the Weibull-Pareto distribution with parameter (α,λ,k) defined and studied by (Alzaatreh et al., 2013b), then the random variable x=$k(\log(\frac{y}{k}))^{1/\theta}$ follows the WEP distribution with parameters (α,λ,θ,k).

Proof: Given that Y follows the Weibull-Pareto distribution, then Y~(α,λ,k), required to show that X~(α,λ,θ,k). The CDF is given by Alzaatreh et al. (2013b) as

$$1 - \exp\left(-\left\{\lambda \log\left(\frac{y}{k}\right)\right\}^{\alpha}\right) \tag{24}$$

Required to show that X~(α,λ,θ,k),

$$X = k\left(\log\left(\frac{Y}{k}\right)\right)^{1/\theta}$$

By transformation of variable,

$(\log(\frac{Y}{k}))=(\frac{X}{k})^{\theta}$, this implies that $(\frac{Y}{k}) = e^{(\frac{X}{k})^{\theta}}$ and the random variable $Y = ke^{(\frac{X}{k})^{\theta}}$

By substituting Y into Equation (24), the CDF of WEP distribution in Equation (9) is obtained.

2.4.8 Shannon Entropy of Weibull Exponential Pareto Distribution

Shannon (2001) defined entropy as the measure of the level of variation of uncertainty associated with a random variable. If a random variable T follows the Weibull distribution with parameters c and γ, the Shannon entropy for the Weibull distribution by Song (2001) is given by

$$\eta_T = \vartheta\left(1 - \frac{1}{c}\right) - \log\left(\frac{c}{\gamma}\right) + 1 \tag{25}$$

The Shannon entropy for the Weibull-X family of distribution by Alzaatreh et al. (2013b) is given by

$$\eta_x = -\{\log f(F^{-1}(1 - e^{-T}))\} - \gamma\Gamma\left(1 + \frac{1}{c}\right) + \vartheta\left(1 - \frac{1}{c}\right) - \log\left(\frac{c}{\gamma}\right) + 1 \tag{26}$$

Where ϑ the Euler's constant in both equations. The mean of Weibull distribution for the random variable T is

$$\mu_T = \gamma\Gamma\left(1 + \frac{1}{c}\right) \tag{27}$$

Theorem 4: Let X be a random variable from the WEP distribution, then the Shannon entropy $\eta_x$for the WEP distribution is given by

$$k\left(\frac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\frac{1}{\alpha\theta} + 1\right) - \Gamma\left(1 + \frac{1}{\alpha}\right) + \vartheta\left(1 - \frac{1}{\alpha}\right) - \log(\alpha) + 1 \tag{28}$$

Proof: By definition, Shannon entropy ηX is the expectation of the negative logarithm of the density function *g(x)*. The expectation of the PDF of the Weibull-X family in Equation (4) is given by

$$E[-\log\{g(x)\}] = E\left[-\log\left(\frac{f(x)}{1 - F(x)} r\{-\log(1 - F(x))\}\right)\right]$$

$$E[-\log\{f(x)\} + \log(1 - F(x)) - \log[r\{-\log(1 - F(x))\}]]$$

$$E[-\log(f(x))] + E[\log(1 - F(x))] + E[-\log[r\{-\log(1 - F(x))\}]]$$

By variable transformation, if T= -log (1-F(x)), then (1-F(x))=e^{-T},

$$\eta_x = E[-\log\{g(x)\}] = E[-\log(x)] - E[T] + E[-\log[r(t)]] = \mu_x - \mu_T + \eta_T$$

Where
μ_x = The mean of WEP distribution
μ_T = The mean of Weibull distribution
η_T = The Shannon entropy of Weibull distribution
η_X = The Shannon entropy of the Weibull-X distribution

By substituting for μ_x which is the mean of WEP in Equation (18) and also the values of η_T and μ_T in Equations (19) and (27) respectively, the Shannon entropy η_X of the WEP distribution is obtained as

$$\eta_x = k\left(\frac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\frac{1}{\alpha\theta} + 1\right) - \Gamma\left(1 + \frac{1}{\alpha}\right) + \vartheta\left(1 - \frac{1}{\alpha}\right) - \log(\alpha) + 1$$

2.4.9 Renyi Entropy of Weibull Exponential Pareto Distribution

The Renyi entropy of a continuous random variable X with PDF g(x) is defined by Rényi (1961) as the measure of uncertainty associated with X, and the Renyi entropy of X is defined by

$$\|_R(X) = \frac{1}{1-\delta}\log[\|(\delta)] \tag{29}$$

Where

$$\|_{(\delta)} = \int_{-\infty}^{\infty} .f^{\delta}(x)dx,\ \delta > 0 \text{ and } \delta \neq 1 \tag{30}$$

The Renyi entropy of a random variable X that follows the WEP distribution is derived by substituting the PDF of WEP into Equation (30)

$$\|_{(\delta)} = \int_{\infty}^{\infty} \left(\frac{\alpha\lambda\theta}{k}\right)^{\delta} \left(\frac{x}{k}\right)^{\delta(\theta-1)} \left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\delta(\alpha-1)} \exp\left(-\left\{\delta\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)dx$$

Let $u = \delta\lambda\left(\frac{x}{k}\right)^{\theta}$ and obtain $x = k\left(\frac{u}{\delta\lambda}\right)^{1/\theta}$

$$\frac{du}{dx} = \frac{\delta\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} du = \frac{\delta\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} dx$$

$$\|_{(\delta)} = \int_{\infty}^{0} x^{\theta\delta\alpha-\delta}(\alpha\lambda\theta)^{\delta}\frac{\lambda^{\delta\alpha-\delta}(\frac{1}{k})^{\theta\delta\alpha}\exp(-u^{\alpha})}{\frac{\delta\lambda\theta}{k}(\frac{1}{k})^{\theta-1}\left(k(\frac{u}{\delta\lambda})^{1/\theta}\right)^{\theta-1}}du$$

$$= \int_{\infty}^{0}\left(\frac{1}{\lambda}\right)^{-\frac{\delta}{\theta}+\frac{1}{\theta}}\left(\frac{u}{\delta}\right)^{\delta\alpha-\frac{\delta}{\theta}+\frac{1}{\theta}}\alpha^{\delta}\left(\frac{\theta}{k}\right)^{\delta-1}\exp(-u^{\alpha})(u)^{-1}du$$

Let $y = u^{\alpha}$, $\frac{dy}{du} = \alpha u^{\alpha-1} = \alpha(y^{1/\alpha})^{\alpha-1}$

$$\|_{(\delta)} = \int_{\infty}^{0}\left(\frac{1}{\lambda}\right)^{-\frac{\delta}{\theta}+\frac{1}{\theta}}\left(\frac{y^{1/\lambda}}{\delta}\right)^{\delta\alpha-\frac{\delta}{\theta}+\frac{1}{\theta}}\alpha^{\delta}\left(\frac{\theta}{k}\right)^{\delta-1}\frac{\exp(-y)(y^{1/\alpha})^{-1}}{\alpha(y^{1/\alpha})^{\alpha-1}}dy$$

$$= \int_{\infty}^{0}(\lambda)^{\frac{1-\delta}{\theta}}\left(\frac{1}{\delta}\right)^{\delta\alpha-\frac{\delta}{\theta}+\frac{1}{\theta}}\left(\frac{\alpha\theta}{k}\right)^{\delta-1}\exp(-y)y^{\delta-\frac{\delta}{\alpha\delta}+\frac{1}{\alpha\theta}-1}dy$$

$$\|(\delta) = (\lambda)^{\frac{1-\delta}{\theta}}\left(\frac{1}{\delta}\right)^{\frac{\delta\alpha\theta-\delta+1}{\theta}}\left(\frac{\alpha\theta}{k}\right)^{\delta-1}\Gamma\left(\frac{\delta\alpha\theta-\delta+1}{\alpha\theta}\right) \tag{31}$$

Substitute the quantity in Equation (31) into Equation (29) to get the desired result given by

$$\|_R(X) = \frac{1}{1-\delta}\log[\|(\delta)]$$
$$= \frac{1}{1-\delta}\log\left[(\lambda)^{\frac{1-\delta}{\theta}}\left(\frac{1}{\delta}\right)^{\frac{\delta\alpha\theta-\delta+1}{\theta}}\left(\frac{\alpha\theta}{k}\right)^{\delta-1}\Gamma\left(\frac{\delta\alpha\theta-\delta+1}{\alpha\theta}\right)\right]$$

2.4.10 Distribution of the Order Statistics of WEP Distribution

Let $X_1, X_2, \ldots, x_n$ be a random sample of size n from the WEP distribution with the CDF and PDF given as G(x) and g(x) respectively, if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is the order statistics of the random sample, the PDF of the r^{th} order statistics defined by David and Nagaraja (2003) is given by

$$fx_{r:n}(x) = \frac{n}{(r-1)(n-r)!}G(x)^{r-1}(1-G(x))^{n-r}G'(x) \tag{32}$$

Substitute the CDF and PDF of the Weibull Exponential Pareto distribution defined in Equations (9) and (10) into Equation (32) to get

$$fx_{r;n}(x) = \begin{cases} \frac{n}{(r-1)(n-r)!}\left[1-\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{r-1} \\ \left[\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n-r+1} \\ X\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \end{cases} \tag{33}$$

The first-order statistics is obtained from Equation (33) when r=1 and is given by

$$fx_{1:n}(x) = \left[\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n}\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \tag{34}$$

The maximum order statistics is derived from Equation (33) when r=n and is given by

$$fx_{n;n}(x) = \begin{cases} \left[1-\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right]^{n-1} \\ \left[\exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right] \\ X\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \end{cases} \tag{35}$$

2.4.11 Moments, Mean, and Variance of Order Statistics of WEP Distribution

The moment of order statistics including the mean and variance of order statistics are derived here

Theorem 5: Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics of the random sample from the WEP distribution of the density function $fx_{r;n}(x)$, derived in Equation (33), then the s^{th} moment of the r^{th} order statistics is given by

$$\mu^s_{r.n} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^s \left(\frac{1}{\lambda}\right)^{s/\theta} \\ X \left(\frac{1}{m}\right)^{\left(\frac{s}{\alpha\theta}+1\right)} \Gamma\left(\frac{s}{\alpha\theta}+1\right) \end{cases} \quad (36)$$

Proof: The s^{th} moment of $X_{(r:n)}$ is defined as

$$E(X^s_{r:n}) = \int_{-\infty}^{\infty} x^s f x_{r;n}(x) dx \quad (37)$$

Equation (33) can be expressed by series expansion as

$$\begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \\ X \left\{\lambda \left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \exp\left(-\left\{m\lambda \left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right) \end{cases} \quad (38)$$

Where m=n-r+i+1, and $C_{r:n} = \frac{n}{(r-1)!(n-r)!}$
Substitute Equation (38) into (37) to obtain

$$\begin{cases} C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^{\infty} x^s \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \\ X \left\{\lambda \left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha-1} \exp\left(-\left\{m\lambda \left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right) dx \end{cases} \quad (39)$$

By following the steps in subsection 2.4.4, Theorem 1, the solution to the integral in Equation (39) is obtained as

$$k^s \left(\frac{1}{\lambda}\right)^{s/\theta} \left(\frac{1}{m}\right)^{\left(\frac{s}{\alpha\theta}+1\right)} \Gamma\left(\frac{s}{\alpha\theta}+1\right) \quad (40)$$

And the s^{th} moment of the r^{th} order statistics of WEP distribution is given by

$$C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^s \left(\frac{1}{\lambda}\right)^{s/\theta} \left(\frac{1}{m}\right)^{\left(\frac{s}{\alpha\theta}+1\right)} \Gamma\left(\frac{s}{\alpha\theta}+1\right)$$

Corollary 1: The mean of order statistics of the WEP distribution is obtained as

$$\mu_{r;nWEP} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k \left(\frac{1}{\lambda}\right)^{1/\theta} \\ X \left(\frac{1}{m}\right)^{\left(\frac{1}{\alpha\theta}+1\right)} \Gamma\left(\frac{1}{\alpha\theta}+1\right) \end{cases} \quad (41)$$

Corollary 2: Let $\sigma^s{}_{r:n(WEP)}$ denote the variance of order statistics of the WEP distribution, then the explicit expression for the variance is given by

$$\sigma^s_{r:nWEP} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^2 \\ \left(\frac{1}{\lambda}\right)^{2/\theta} \left(\frac{1}{m}\right)^{\left(\frac{2}{\alpha\theta}+1\right)} \Gamma\left(\frac{2}{\alpha\theta}+1\right) \\ -\left[\frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k \right. \\ \left. \left(\frac{1}{\lambda}\right)^{1/\theta} \left(\frac{1}{m}\right)^{\left(\frac{1}{\alpha\theta}+1\right)} \Gamma\left(\frac{1}{\alpha\theta}+1\right)\right]^2 \end{cases} \quad (42)$$

Proof: The variance is proved using the relation;

$$\sigma^s_{r:nWEP} = \mu^2_{r:n} - (\mu_{r:nWEP})^2$$

Using Equation (36), $\mu^2{}_{r:n}$ is obtained as

$$\mu^2_{r:n} = \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^2 \left(\frac{1}{\lambda}\right)^{2/\theta} \left(\frac{1}{m}\right)^{\left(\frac{2}{\alpha\theta}+1\right)} \Gamma\left(\frac{2}{\alpha\theta}+1\right)$$

Using Equation (41), $(\mu_{r;nWEP})^2$ is obtained as

$$(\mu_{r:nWEP})^2 = \left[\frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k \left(\frac{1}{\lambda}\right)^{1/\theta} \left(\frac{1}{m}\right)^{\left(\frac{1}{\alpha\theta}+1\right)} \Gamma\left(\frac{1}{\alpha\theta}+1\right)\right]^2$$

3. RESULTS AND DISCUSSION

3.1 Generalization of Properties of Order Statistics for Some Lifetime Distribution

The mean, variance, skewness, kurtosis, and some other important statistical properties can be derived for some lifetime distributions using the theory and application of order statistics. This sub-section generalized some few properties of order statistics for some lifetime distribution as follows:

3.1.1 Exponential Pareto Distribution (Al Kadim and Boshi, 2013)

Corollary 3: If α=1, in Equation (36) the result for sth moment of the rth order statistics of the EP distribution is given by

$$\mu^{s}_{r:nEP} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} r-1 \\ i \end{pmatrix} k^s \left(\frac{1}{\lambda}\right)^{s/\theta} \\ X \left(\frac{1}{m}\right)^{\left(\frac{s}{\theta}+1\right)} \Gamma\left(\frac{s}{\theta}+1\right) \end{cases} \tag{43}$$

The mean of order statistics of EP distribution is derived and given by

$$\mu_{r:nEP} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} r-1 \\ i \end{pmatrix} k \left(\frac{1}{\lambda}\right)^{1/\theta} \\ X \left(\frac{1}{m}\right)^{\left(\frac{1}{\theta}+1\right)} \Gamma\left(\frac{1}{\theta}+1\right) \end{cases} \tag{44}$$

The mean of the minimum $X_{(1:n)}$ order statistics of EP distribution is given by

$$\mu_{1:nEP} = nk \left(\frac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\frac{1}{\theta}+1\right) \tag{45}$$

The mean of the maximum $X_{(n:n)}$ order statistics of EP distribution is given by

$$\mu_{n:nEP} = \sum_{i=0}^{n-1} (-1)^i \begin{pmatrix} n-1 \\ i \end{pmatrix} nk \left(\frac{1}{\lambda}\right)^{1/\theta} \left(\frac{1}{i+1}\right)^{\left(\frac{1}{\theta}+1\right)} \Gamma\left(\frac{1}{\theta}+1\right) \tag{46}$$

Corollary 4: If r=n=1 and α=1, the result for a central moment about the origin and the mean for the EP distribution Al Kadim and Boshi (2013) is obtained from this study as

$$\mu'_s = k^s \left(\frac{1}{\lambda}\right)^{s/\theta} \Gamma\left(\frac{s}{\theta}+1\right) \tag{47}$$

The mean of a random variable X from the EP distribution is

$$E(X) = k \left(\frac{1}{\lambda}\right)^{1/\theta} \Gamma\left(\frac{1}{\theta}+1\right) \tag{48}$$

3.1.2 Weibull-Rayleigh Distribution (Ahmad et al., 2017)

Corollary 5. If θ=2 and $\lambda = \frac{1}{2\beta}$, in Equation (36) the sth moment of the rth order statistics of the WR distribution is given by

$$\mu^{s}_{r:nWR} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} r-1 \\ i \end{pmatrix} k^s \\ X (2\beta)^{s/2} \left(\frac{1}{m}\right)^{\left(\frac{s}{2\alpha}+1\right)} \Gamma\left(\frac{s}{2\alpha}+1\right) \end{cases} \tag{49}$$

The mean of order statistics of WR distribution is derived and given by

$$\mu_{r:nWR} = \begin{cases} \frac{n}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \begin{pmatrix} r-1 \\ i \end{pmatrix} k \\ X (2\beta)^{1/2} \left(\frac{1}{m}\right)^{\left(\frac{1}{2\alpha}+1\right)} \Gamma\left(\frac{1}{2\alpha}+1\right) \end{cases} \tag{50}$$

The mean of the minimum $X_{(1:n)}$ order statistics of WR distribution is given by

$$\mu_{1:nWR} = nk(2\beta)^{1/2}\Gamma\left(\frac{1}{2\alpha}+1\right) \tag{51}$$

The mean of the maximum $X_{(n:n)}$ order statistics of WR distribution is given by

$$\mu_{n:nWR} = \begin{cases} \sum_{i=0}^{n-1} (-1)^i \begin{pmatrix} n-1 \\ i \end{pmatrix} nk(2\beta)^{1/2} \\ X \left(\frac{1}{i+1}\right)^{\left(\frac{1}{2\alpha}+1\right)} \Gamma\left(\frac{1}{2\alpha}+1\right) \end{cases} \tag{52}$$

Corollary 6. The sth moment of the WR distribution developed by Ahmad et al. (2017) is given by

$$\mu'_s = (2\beta)^{s/2} k^s \Gamma\left(\frac{s}{2\alpha}+1\right) \tag{53}$$

Proof: If r=n=1 and θ=2 and λ=1/2β in Equation (36) the desired result is obtained The mean of a random variable X from the WR distribution is

$$E(X) = (2\beta)^{1/2} k \Gamma\left(\frac{1}{2\alpha}+1\right) \tag{54}$$

Remark: Similar results can be deduced for the Rayleigh-Rayleigh distribution developed by Ateeq et al. (2019) and for other models presented in Table 1.

3.2 Parameter Estimation and Simulation

This section is for determining the estimates of parameters of WEP distribution using the method of maximum likelihood estimation (MLE). A Simulation study is also conducted to assess the performance of the procedure.

3.2.1 Maximum Likelihood Estimation

Let X_1, X_2,..., X_n,be independent and identically distributed random sample of size n from the WEP distribution with PDF derived as $g(x)$ in Equation (9) with a set of parameters φ=(α,θ,λ,k). The likelihood function of the distribution is obtained as;

$$Lik[g(x,\varphi)] = \prod_{i=1}^{n}\left[\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{(\theta-1)}\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{(\alpha-1)} \exp\left(-\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\right)\right] \tag{55}$$

The log-likelihood function logLik[g(x,ϕ)] denoted as LL is

$$LL = \begin{cases} n\log\alpha + n\log\lambda + n\log\theta - n\log k + \\ \quad (\theta-1)\sum\log x - n(\theta-1)\log k \\ \\ +n(\alpha-1)\log\lambda - n(\alpha-1)\log k^{\theta} \\ +(\alpha-1)\sum\log x^{\theta} - \sum\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha} \end{cases} \tag{56}$$

The normal Equations are obtained as derivatives of LL for the parameters

$$0 = \frac{dLL}{d\alpha} = \begin{cases} \frac{n}{\alpha} + n\log\lambda - n\log k^{\theta} + \sum log x^{\theta} \\ \\ -\sum\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\log\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right) \end{cases} \tag{57}$$

$$0 = \frac{dLL}{d\lambda} = \frac{n}{\lambda} + \frac{n\alpha}{\lambda} + \frac{\alpha}{\lambda}\sum\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha} \tag{58}$$

$$0 = \frac{dLL}{dk} = \begin{cases} -\frac{n}{k} - \frac{n(\theta-1)}{k} + \frac{n(\alpha-1)}{k} \\ \\ +\frac{\alpha\theta}{k}\sum\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha} \end{cases} \tag{59}$$

$$0 = \frac{dLL}{d\theta} = \begin{cases} \frac{n}{\theta} + \sum\log x - n\log k - n(\alpha-1)\frac{\theta}{k} \\ \\ +(\alpha-1)\sum\frac{\theta}{x} - \alpha\sum\left\{\lambda\left(\frac{x}{k}\right)^{\theta}\right\}^{\alpha}\log\left(\frac{x}{k}\right) \end{cases} \tag{60}$$

A numerical solution to the above equations is adopted for the estimates of the parameters which are easier using the statistical software.

3.2.2 Simulation Study

Assessment of the estimation of parameter procedure is performed by conducting a simulation study using the R-statistical software as follows;

1. Simulated data are generated using Equation (14) given by

$$X = K\left\{\frac{1}{\lambda}\left[-\log(1-u)^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\theta}}$$

2. The sample sizes taken are n = 20, 50, 250, 350, 500
3. Two sets I and II of parameter values are defined as I = (α=0.5, θ=1, λ=2.5, k=1.5) and II = (α=1.0, θ=2, λ=3.5, k=2.0)
4. Replicate the process for each sample size N=10,000 number of times
5. Compute the MSE by using MSE $\varnothing = \frac{1}{N}\sum_{i=1}^{N}(\varnothing_i' - \varnothing)^2$ where $\varnothing'_i$represents WEP parameters
6. Step five is carried our repeatedly for each parameter.

The estimated values are obtained for the Bias, Mean Square Error (MSE), Root Mean Square Error (RMSE), and standard errors. The simulation study shows that the parameters of the distribution are stable in addition; the consistency of the MSE values for the maximum likelihood estimations implies that the estimation procedure is adequate. The results revealed that the MSE and RSME decrease as the sample size increases for both sets of actual values of the parameter. The standard errors also converge to zero as the sample size increases. The results from the simulation studies are presented in Table 2 for the first set of parameters and Table 3 for the second set of parameters.

3.3 Application to Lifetime Datasets

The usefulness of the distribution is demonstrated by analyzing three real-life datasets using the R software (Core Team, 2013). The best-fitted model is usually identified with the smallest values of goodness-of-fit criteria which are the Log-likelihood (LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), the Kolmogorov statistics and P-value. The criteria are defined as follows;

AIC=-2LL+2c; BIC=-2LL+$c$log(n); CAIC=-2LL+$\frac{2c(n-2)}{n-c-2}$; HQIC = 2log (log(n)($c - 2LL$)). The performance of the WEP distribution is compared with some similar families of distributions and some notable models existing in the literature with the density functions defined by

1. Generalized Exponential Weibull (GEW) distribution

$$GEW(x;\alpha,\beta,\theta,k) = \theta(\alpha+\beta kx^{(k-1)}) \exp(-(\alpha x+\beta x^{k}))(1-\exp(\alpha x+\beta x^{k}))^{\theta-1}$$

2. Transmuted New Weibull Pareto (TNWP) distribution

$$TNWP(X;\lambda,\beta,\theta,k) = \frac{\beta\theta}{k}\left(\frac{x}{k}\right)^{(\beta-1)} e^{(-\theta\left(\frac{x}{k}\right)^{\beta}} \left(1-\lambda+2\lambda e^{-\theta\left(\frac{x}{k}\right)^{\beta}}\right)$$

3. Kumaraswamy Exponential Pareto (KEP) distribution

$$KEP(X;\alpha,\beta,\theta,k)=\frac{\alpha\beta\lambda\theta}{k}\left(\frac{x}{k}\right)^{(\theta-1)}e^{-\lambda\left(\frac{x}{k}\right)^{\theta}}$$
$$\left(1-e^{-\lambda\left(\frac{x}{k}\right)^{\theta}}\right)^{\alpha-1}\left(1-\left(1-e^{-\lambda\left(\frac{x}{k}\right)^{\theta}}\right)^{\alpha}\right)^{\beta-1}$$

4. Gompertz Exponential Pareto (GEP) distribution

$$GEP(X;\alpha,\lambda,\beta,\theta,k)=\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{(\theta-1)}$$
$$\left(e^{-\lambda\left(\frac{x}{k}\right)^{\theta}}\right)^{-\beta}e^{\frac{\alpha}{\beta}\left[1-\left(e^{-\lambda\left(\frac{x}{k}\right)^{\theta}}\right)^{-\beta}\right]}$$

5. Beta Exponential Pareto (BEP) distribution

$$BEP(X;\alpha,\lambda,\beta,\theta,k)=\frac{\alpha\theta}{\lambda B(\alpha\beta)}\left(\frac{x}{\lambda}\right)^{(\theta-1)}$$
$$e^{-\alpha\beta\left(\frac{x}{\lambda}\right)^{\theta}}\left[1-e^{-\beta\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{\alpha-1}$$

3.3.1 Application to the Hydrological Data Set

The first data contain 72 exceedances of flood peaks (inm^3/s) of the Wheaton River discharge near Carcross in Yukon Territory, Canada for the years 1958-1984. The data set has been applied by several authors including Aryal (2019) using BEP, Tahir and Akhter (2018) using TNWP, and recently by Adeyemi et al. (2021) for evaluating the performance of the GEP distribution.

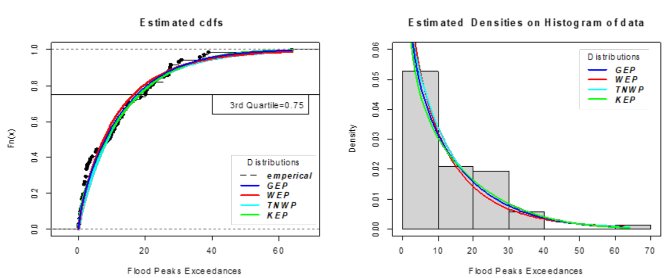

Figure 3. Plots of PDFs and CDFs of Fitted Distributions Fitted to the Hydrological Data

First Data Set: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 0.9, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 1.9, 10.4, 13.0, 10.7, 12.0, 30.0, 9.3, 3.6, 2.5, 27.6, 14.4, 36.4, 1.7, 2.7, 37.6, 64.0, 1.7, 9.7, 0.1, 27.5, 1.1, 2.5, 0.6, 27.0.

The values of the estimated parameters of the competing models and the goodness-of-fit criteria are displayed in Table 4 and Table 5 respectively. The visual result of goodness-of-fit from data analysis in form of the histogram with the estimated densities and the CDFs of the fitted models are displayed in Figure 3. Table 4 revealed that the WEP distribution competes favorably with the GEP model and performs better than TNWP, KEP, and BEP distributions based on the smallest values of goodness-of-fit statistics. Figure 3 supported the selection of the proposed model as more flexible than the other models for the dataset.

3.3.2 Application to Tensile Strength of Polyester Fibers

The second data represents 30 measurements of the tensile strength of polyester fibers available in (Quesenberry and Hales, 1980).

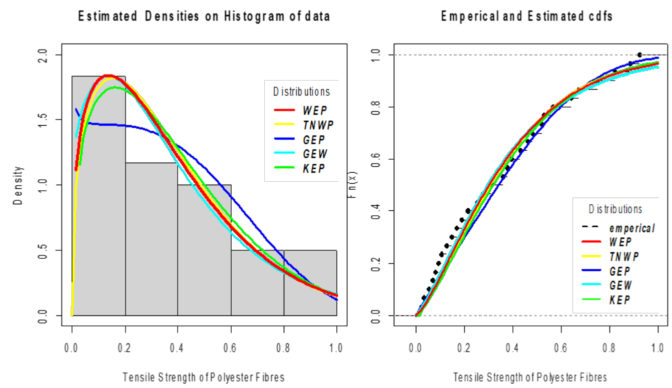

Figure 4. Plots of Histogram with Densities and CDFs of Fitted Distributions to the Tensile Strength Data

Second Data Set: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188,0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519,0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926

The computed results of estimated parameters and the goodness-of-fit criteria are displayed in Table 6 and Table 7 respectively for the models under consideration. Figure 4 is the pictorial display of goodness-of-fit from data analysis in form of the histogram with the estimated densities and CDFs of the fitted models to the tensile strength data set. The fit of the WEP model is compared with TNWP, GEP, GEW, and KEP models. Results displayed in Table 6 revealed that the WEP model has more flexible capabilities for the data than the competitors. The visual results from the estimated densities and CDFs in Figure 4 strengthened the choice of the WEP model.

3.3.3 Application to Lifetimes of Steel Specimens

The data represents the lifetimes (t) of steel specimens tested at stress (s) level of s=38.5 obtained from the 14 different stress levels reported in (Lawless, 2011).

Third Data Set: 60, 51, 83, 140, 109, 106, 119, 76, 68, 67, 111, 57, 69, 75, 122, 128, 95, 87, 82, 132. The lifetime data is fitted to the WEP model and compared with GEP, GEW, and KEP models. Computed results of estimated parameters and the goodness-of-fit criteria are presented in Table 8 and Table 9 respectively. The plots for the densities and CDFs are displayed in Figure 5. Application of the proposed models to the data set

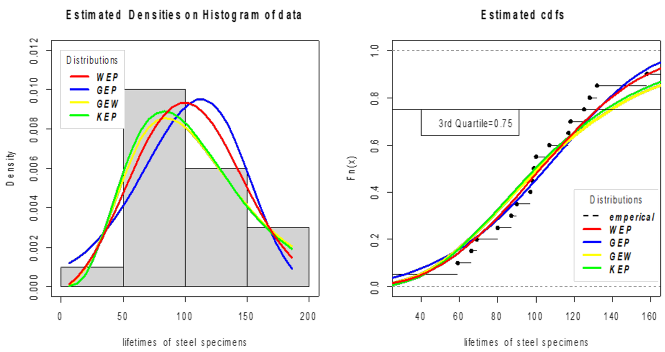

Figure 5. Plots of PDFs and CDFs of Fitted Distributions to the Steel Specimens Data

of steel specimens revealed the suitability of the WEP model as a better lifetime distribution for fitting the data compared to some other existing distributions in the literature. The WEP model provides the smallest AIC, CAIC, and BIC and the plots in Figure 5 substantiate our choice of WEP distribution.

4. CONCLUSION

The WEP model is a new lifetime distribution with adequate potential for analyzing left-skewed, right-skewed, and approximately symmetric phenomena from the field of hydrology, reliability engineering, actuarial and finance, and public health. It is also suitable for fitting real-life datasets in other areas of applications characterized by risky kurtosis. It is confirmed in this study that WEP has superior performance than the competing distributions for modeling the reliability and the hydrological data sets and the lifetime of steel specimens. The discoveries from the theory of order statistics extended in the study established a novel area of study in distribution theory which also contributed to the conclusion that the WEP distribution provides sufficient characterizations for WR, EP, RR distributions, and some other existing lifetime distributions. Other important areas of further research include the estimation of parameters by order statistics and the real-life application of moments of order statistics in predictive modeling.

5. ACKNOWLEDGMENT

The authors are grateful for the contributions and suggestions of the Editors, the Editor-in-Chief and all the anonymous reviewers which improve the original manuscript. Special thanks to Mr. John Anih for the payment of the APC on behalf of the corresponding Author.

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Table 2. Simulation Study for Set of Parameters I=(α=0.5,θ=1,λ=2.5,k=1.5)

		Bias				MSE		
n	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
20	0.6987	-1.2998	0.1999	-0.2995	0.8459	2.0494	0.3998	0.4515
50	0.6990	-1.3005	0.2006	-0.3007	0.6336	1.8353	0.1832	0.2352
250	0.6987	-1.3006	0.2003	-0.2991	0.5183	1.7200	0.0691	0.1185
500	0.7000	-1.3001	0.2001	-0.2989	0.5044	1.7048	0.0544	0.1042
1000	0.6986	-1.3000	0.1998	-0.2992	0.4973	1.6972	0.0471	0.0972
		RMSE				St.Error		
n	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
20	0.9198	1.4316	0.6323	0.6719	0.5981	0.5999	0.5998	0.6015
50	0.7960	1.3547	0.4208	0.4849	0.3793	0.3795	0.3781	0.3804
250	0.7199	1.3115	0.2628	0.3442	0.1695	0.1688	0.1701	0.1704
500	0.7102	1.3057	0.2334	0.3228	0.1199	0.1202	0.1199	0.1196
1000	0.7052	1.3028	0.2171	0.3117	0.0848	0.0846	0.0848	0.0851

Table 3. Simulation Study for Set of Parameters II=(α=1.0,θ=2,λ= 3.5,k=2.0)

		Bias				MSE		
n	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
20	-0.0523	-2.5528	-1.0522	-1.0528	0.0150	6.5292	1.1193	1.1208
50	-0.0526	-2.5526	-1.0525	-1.0527	0.0077	6.5207	1.1126	1.1130
250	-0.0526	-2.5526	-1.0527	-1.0525	0.0038	6.5168	1.1092	1.1088
500	-0.0526	-2.5526	-1.0526	-1.0524	0.0032	6.5161	1.1086	1.1082
1000	-0.0526	-2.5526	-1.0525	-1.0523	0.0030	6.5160	1.1080	1.1081
		RMSE				St.Error		
n	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{a}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
20	0.1225	2.5552	1.0579	1.0587	0.1108	0.1109	0.1107	0.1111
50	0.0875	2.5535	1.0548	1.0549	0.0699	0.0701	0.0701	0.0702
250	0.0612	2.5528	1.0532	1.0530	0.0314	0.0314	0.0313	0.0313
500	0.0571	2.5526	1.0528	1.0526	0.0221	0.0222	0.0221	0.0220
1000	0.0549	2.5526	1.0526	1.0526	0.0156	0.0157	0.0156	0.0156

Table 4. Log-likelihood and Maximum Likelihood Estimates of the Parameters for the Flood Peaks Data

Model	$\hat{a}$	β	$\hat{\lambda}$	θ	$\hat{k}$	-LL
GEP	0.6735	0.2594	0.3099	0.7474	0.8814	499.7330
WEP	0.0786	-	3.1539	11.1779	10.1814	501.1056
TNWP	0.4514	-	0.4092	6.7312	0.8771	502.9130
KEP	0.9245	0.2594	0.3099	0.7624	0.8814	501.1506
BEP	0.0332	0.4984	0.7474	0.2985	0.5482	501.9790

Table 5. Goodness-of-fit Statistics for the Flood Peaks Data

Model	AIC	CAIC	BIC	HQIC	K-S	p-value
WEP	509.1056	509.7026	518.2122	512.7310	0.1067	0.3850
GEP	509.7330	510.6461	521.1164	514.2648	0.1029	0.4310
TNWP	510.9130	511.5100	520.0197	514.5384	0.1069	0.3812
KEP	511.1506	512.0589	522.5340	515.6824	0.1071	0.3802
BEP	511.9590	512,8680	523.3420	516.4910	-	-

Table 6. Log-likelihood and Estimated Parameters for the Tensile Strength Data

Model	$\hat{a}$	β	$\hat{\lambda}$	θ	$\hat{k}$	-LL
WEP	0.0402	-	2.4076	32.9847	0.3891	-1.7503
TNWP	6.5409	-	0.2768	1.7556	1.3982	-1.7377
GEP	15.9217	30.1267	0.3245	0.9161	4.9943	-2.3263
GEW	3.1808	-	2.9046	21.1988	0.1269	-1.2846
KEP	0.1795	13.5155	11.7830	7.4832	4.1216	-2.2521

Table 7. Goodness-of-fit Statistics for the Tensile Strength Data

Model	AIC	CAIC	BIC	HQIC	K-S	p-value
WEP	4.4993	6.0993	10.1041	6.2923	0.0826	0.9760
TNWP	4.5246	6.1246	10.1294	6.3176	0.0853	0.9679
GEP	5.3473	7.8473	12.3533	7.5886	0.0859	0.9661
GEW	5.4308	7.0308	11.0357	7.2238	0.0884	0.9568
KEP	5.4959	7.9959	12.5018	7.7371	0.0917	0.9427

Table 8. Log-likelihood and Estimated Parameters for the Steel Specimen Data

Model	$\hat{a}$	β	$\hat{\lambda}$	θ	$\hat{k}$	-LL
WEP	1.9542	-	33.5981	1.409	9.6908	102.9148
GEP	0.0065	0.7775	0.5749	5.6195	1.1537	102.9122
GEW	0.0232	-	1.0449	10.9496	-0.1876	104.4920
KEP	9.1038	3.5236	0.1109	0.6503	1.4219	105.6481

Table 9. Goodness-of-fit Statistics for the Steel Specimen Data

Model	AIC	CAIC	BIC	HQIC	K-S	p-value
WEP	213.8297	216.4964	217.8126	214.6072	0.1057	0.9615
GEP	214.8244	219.8031	219.8031	215.7963	0.1096	0.9485
GEW	214.8244	219.6506	220.9669	217.7615	0.1387	0.7871
KEP	221.2962	225.5819	226.2748	222.2680	0.1405	0.7744

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