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Research Paper

The Relationship of Multiset, Stirling Number, Bell Number, and Catalan Number

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Abstract
Catalan numbers is not as famous as Fibonacci numbers, however this number has own its beauty and arts. Catalan numbers was discovered by Ming Antu in 1730, however, this numbers is credited to Eugene Catalan when he was studying parentheses in 1838. Catalan numbers mostly occurs in counting or enumeration problems. The Catalan numbers can be defined in more than one forms, and the most famous form is $C_n = \frac{1}{n+1}\binom{2n}{n}$. In this study we will discuss the multiset construction and the relationship of the results of Multiset with Stirling, Bell, and Catalan numbers.

Keywords
Counting, Enumeration, Multiset, Catalan Numbers, Stirling Numbers

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1. INTRODUCTION

In mathematics, there are many types of sequence of numbers. Each sequence has its own definition, uniqueness and benefits. For example, a very famous number sequence, namely the Fibonacci number sequence, which has a uniqueness, namely the Golden Ratio and its benefits have also been widely applied in life. Catalan numbers are not as popular as the Fibonacci numbers, but this sequence of numbers also has many benefits. Catalan numbers are named after the Belgian scientist Eugene C. Catalan based on his work while studying parentheses sequences in 1838. The parentheses in question are sequences that are well formed parentheses (Pak, 2014). In addition to brackets, other forms related to Catalan numbers are the triangular forms of the convex polygon (Cayley, 1890).

Actually in 1730 Ming Antu had discovered the Catalan Numbers based on the geometrical models constructed (Koshy, 2008). However, because the results were in China and not known in the Western world, Catalan was bettern known, and the numbers are called as Catalan Numbers. This number is unique in that it can be defined in several forms and the most famous form is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The following is the first nine terms of Catalan number: 1, 2, 5, 14, 42, 132, 429, 1430, 4862,$\cdots$. Not as Fibonacci sequence where we can easily predict the n^{th} term if given the $(n\text{-}1)^{th}$ and $(n\text{-}2)^{th}$ terms using its recursive formula $F_n = F_{(n-1)}+F_{(n-2)}$, in Catalan sequence to predict the n^{th} term by using recursive formula $C_{(n+1)} = C_0C_n+C_1C_{(n-1)}+C_2C_{(n-2)}+\cdots+C_nC_0$ is a little bit harder, because we have to know almost all values of the terms. For Catalan sequence it is easier to count the nth term by using the explisit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. The specialty of Catalan numbers is that they often appear in different problems with different solutions, but have the same final solution, namely Catalan numbers.

Shapiro (1976) investigated lattice paths in the first quadrant and derive a similar triangle to Pascal's triangle and called it Catalan's triangle because it involves the Catalan numbers. Other geometric shapes that turn out to form Catalan numbers include the combination of the Laticce Path based on Catalan numbers (Saračević et al., 2018), binary tress and triangulation of a convex polygons (Koshy, 2008; Stojadinovic, 2015). Lee and Oh (2018) showed that any binomial coefficient can be written as weighted sums along rows of the Catalan triangle. Ceballos and D'León (2018) investigated a generalization of the Catalan objects indexed by a composition of the form s = (s(1), s(2),...,s(a)), which we call a signature, and the object is combinatoric of planar rooted tree.

Catalan numbers are also used to solve the pill problem (Bayer and Brandt, 2014). Boyadzhiev (2021) explored a formula for generating various series, including Catalan, Bernoulli, Harmonic, and Stirling numbers. A new conservative matrix derived by Catalan numbers was investigated by İlkhan (2020). So, actually there are a lot of benefits from the Catalan numbers themselves. In addition, Catalan numbers have also been widely applied in various fields, such as engineering, in computational geometry, geographic information systems, cryptographic geodesy, and medicine (Selim and Saračević, 2019). For data security, Saračević et al. (2018) investigated the application of Catalan numbers and lattice path for cryptography; and Saračević et al. (2017) used Catalan keys based on dynamic programming for steganography. Moreover, in 2019, Saračević et al. (2019) continued exploring Catalan numbers and Dyck word for steganography. (Ndagijimana, 2016) investigated the properties and application of Catalan number in the RNA (Ribonucleic Acid) secondary structure. For more comprhehensive discussion about Catalan number can be found in Koshy (2008) and Stanley (2015).

One of the problems whose solution is Catalan numbers is the problem of Catalan numbers in Algebra, namely the Parker permutation problem (Guy, 1993). In the book Catalan numbers and their applications written by T. Koshy it was informed that this problem had been answered by Ira M. Gessel of Brandies University, Waltham, Massachusetts in 1993, by providing a solution in determining the number of multisets with n-elements $a_1, a_2, a_3, \cdots, a_n$ are members of $a_i \in Z_n$ such that $a_1 + a_2 + a_3 + \cdots + a_n =$ is the identity of the sum Z_n (Koshy, 2008). In this study we will discuss the multisets of Group Additive Z_n with $1 \leq n \leq 10$, and shows the ralations of those multisets with Bell numbers, Stirling numbers, and Catalan numbers.

2. THE METHOD

2.1 Multiset

Let S be a non-empty set. A multiset M from a set S is an ordered pair: $M = \{(s_i, n_i) | s_i \in S, n_i \in \mathbb{Z}^+, z_i \neq s_j \text{ for } i \neq j\}$, where $\mathbb{Z}^+ = \{1, 2, \ldots\}$, n_i is called as the multiplicity of element s_i in M. If the underlying set is finite, the multiset is called finite. The size of a finite multiset M is defined as the sum of the multiplicities of all of its elements (Roman, 2005). For example, $M = \{(p,3), (q,1), (r,2)\}$ is a multiset of underlying set $S = \{p, q, r\}$ in which element p has multiplicity 3, q has multiplicity 1 and r has multiplicity 2. Another way to write this multiset is by writing out all elements according the multiplicities, for above example write M as $M = \{p, p, p, q, r, r\}$.

If X is a set of elements, a multiset A from the set X is represented by a function C_A is $C_A: M \rightarrow N$, where N is the set of non-negative integers. For every $x \in X$, $C_A(x)$ is the characteristic value of x in A and shows how many times the element x appears in A. A multiset A is a set if $C_A(x) = 0$ or 1 for all x (Tripathy et al., 2018).

There are a lot of possible permutation of multiset, for example, Albert et al. (2001) investigated a permutation of a multiset which do not contain certain ordered patterns of length 3.
The following table shows the example of possible multiset with n elements of additive group $\mathbb{Z}_{n+1}$, $1 \leq n \leq 4$ (Koshy, 2008).

Table 1 is the table that shows the number of multisets of group $\mathbb{Z}_{n+1}$, $1 \leq n \leq 4$ where the number of multisets constitute three Catalan numbers which are 1, 2 and 5.

Table 1. The Number of Multisets With n Elements of Group $\mathbb{Z}_{n+1}$, $1 \leq n \leq 4$

n	$\mathbb{Z}_{n+1}$	Multiset with n-elements	The number of multisets
1	$\{\bar{0}, \bar{1}\}$	$\bar{0}$	1
2	$\{\bar{0}, \bar{1}, \bar{2}\}$	$\{\bar{0}, \bar{0}\}$ $\{\bar{1}, \bar{2}\}$	2
3	$\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$	$\{\bar{0}, \bar{0}, \bar{0}\}$ $\{\bar{0}, \bar{1}, \bar{3}\}$ $\{\bar{0}, \bar{2}, \bar{2}\}$ $\{\bar{1}, \bar{1}, \bar{2}\}$ $\{\bar{2}, \bar{3}, \bar{3}\}$	5

2.2 Catalan Numbers

There are some definitions regarding Catalan numbers. However, the most famous way of defining Catalan numbers is

$$C_n = \frac{1}{n+1}\binom{2n}{n} \text{where } n \geq 0, n \in \mathbb{Z}^+ \quad (1)$$

Singmaster (1978) gave a mathematical proof that Equation (1) can also be written as

$$C_n = \frac{\binom{2n+1}{n}}{2n+1} \quad (2)$$

Table 2. Second-kind Stirling Number, $s(n, k)$ and Bell number $B_n = \sum_{k=0}^{n} S(n,k)$

n\k	1	2	3	4	5	6	7	8	9	Bell's number
1	1									1
2	1	1								2
3	1	3	1							5
4	1	7	6	1						15
5	1	15	25	10	1					52
6	1	31	90	65	15	1				203
7	1	63	301	350	140	21	1			877
8	1	127	966	1701	1050	266	28	1		4140
9	1	255	3025	7770	6951	2646	462	36	1	21147

2.3 Bell Numbers

A partition of a set S is a collection of non-empty subsets $A_i \subseteq S$, $1 \leq i \leq k$, such that $\cup_{i=1}^{k} A_i = S$ and for every $i \neq j$ $A_i \cap A_j = \emptyset$. Bell number B_n is the number of partitions of an n-element set and is defined as: $B_n = \sum_{k=0}^{n} S(n,k)$. Since $B_n = 1$, then $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. For example, $B_2 = 2$ because the 2-element set $\{a, b\}$ can be partitioned in 2 distinct ways: $\{a, b\}$, and $\{\{a\}, \{b\}\}$, and $b_3 = 5$, because there are 5 different partitions of 3-element set $\{a, b, c\}$ which are: $\{\{a\}, \{b\}, \{c\}\}$, $\{\{a,b\}, \{c\}\}$, $\{\{a,c\}, \{b\}$ $\{\{b, c\}, \{c\}$, and $\{a, b, c\}$.

2.4 Striling Numbers

Stirling numbers appear in a variety of combinatorial situations. Stirling numbers are classified into two kinds: Stirling numbers of the first kind and Stirling numbers of the second kind.

Table 3. The Multiset Group $\mathbb{Z}_{n+1}$ with $1 \leq n \leq 7$

n	$\mathbb{Z}_{n+1}$	Multiset with n-elements	The number of Multiset
1	$\{\bar0,\bar0\}$	$\{\bar0\}$	1
2	$\{\bar0,\bar1,\bar2\}$	$\{\bar0,\bar0\}\ \{\bar1,\bar2\}$	2
3	$\{\bar0,\bar1,\bar2,\bar3\}$	$\{\bar0,\bar0,\bar0\}\ \{\bar0,\bar1,\bar3\}\ \{\bar0,\bar2,\bar2\}\ \{\bar1,\bar1,\bar2\}\ \{\bar2,\bar3,\bar3\}$	5
4	$\{\bar0,\bar1,\bar2,\bar3,\bar4\}$	$\{\bar0,\bar0,\bar1,\bar4\}\ \{\bar0,\bar0,\bar2,\bar3\}\ \{\bar0,\bar1,\bar1,\bar3\}\ \{\bar0,\bar1,\bar2,\bar2\}\ \{\bar0,\bar2,\bar4,\bar4\}\ \{\bar0,\bar3,\bar3,\bar4\}$ $\{\bar1,\bar1,\bar1,\bar2\}\ \{\bar1,\bar1,\bar4,\bar4\}\ \{\bar1,\bar2,\bar3,\bar4\}\ \{\bar1,\bar3,\bar3,\bar3\}\ \{\bar2,\bar2,\bar2,\bar4\}\ \{\bar2,\bar2,\bar3,\bar3\}$ $\{\bar3,\bar4,\bar4,\bar4\}$	14
5	$\{\bar0,\bar1,\bar2,\bar3,\bar4,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar0\}\ \{\bar0,\bar0,\bar0,\bar0,\bar5\}\ \{\bar0,\bar0,\bar0,\bar2,\bar4\}\ \{\bar0,\bar0,\bar0,\bar3,\bar3\}\ \{\bar0,\bar0,\bar1,\bar1,\bar4\}$ $\{\bar0,\bar0,\bar1,\bar2,\bar3\}\ \{\bar0,\bar0,\bar2,\bar2,\bar2\}\ \{\bar0,\bar1,\bar1,\bar1,\bar3\}\ \{\bar0,\bar0,\bar2,\bar5,\bar5\}\ \{\bar0,\bar0,\bar3,\bar4,\bar5\}$ $\{\bar0,\bar0,\bar4,\bar4,\bar4\}\ \{\bar0,\bar1,\bar1,\bar2,\bar2\}\ \{\bar0,\bar1,\bar1,\bar5,\bar5\}\ \{\bar0,\bar1,\bar3,\bar3,\bar5\}\ \{\bar0,\bar1,\bar3,\bar4,\bar4\}$ $\{\bar0,\bar2,\bar2,\bar4,\bar4\}\ \{\bar0,\bar2,\bar2,\bar3,\bar5\}\ \{\bar0,\bar2,\bar3,\bar3,\bar4\}\ \{\bar0,\bar3,\bar3,\bar3,\bar3\}\ \{\bar1,\bar1,\bar1,\bar1,\bar2\}$ $\{\bar1,\bar1,\bar2,\bar4,\bar5\}\ \{\bar1,\bar1,\bar2,\bar3,\bar5\}\ \{\bar1,\bar1,\bar2,\bar4,\bar4\}\ \{\bar1,\bar1,\bar3,\bar3,\bar4\}\ \{\bar1,\bar2,\bar2,\bar2,\bar5\}$ $\{\bar1,\bar2,\bar2,\bar3,\bar4\}\ \{\bar1,\bar2,\bar3,\bar3,\bar3\}\ \{\bar1,\bar2,\bar5,\bar5,\bar5\}\ \{\bar1,\bar3,\bar4,\bar5,\bar5\}\ \{\bar1,\bar4,\bar4,\bar4,\bar5\}$ $\{\bar0,\bar2,\bar2,\bar2,\bar4\}\ \{\bar2,\bar2,\bar2,\bar3,\bar3\}\ \{\bar2,\bar2,\bar4,\bar5,\bar5\}\ \{\bar2,\bar3,\bar3,\bar5,\bar5\}\ \{\bar2,\bar3,\bar4,\bar4,\bar5\}$ $\{\bar2,\bar4,\bar4,\bar4,\bar4\}\ \{\bar3,\bar3,\bar3,\bar4,\bar5\}\ \{\bar3,\bar3,\bar4,\bar4,\bar4\}\ \{\bar4,\bar5,\bar5,\bar5,\bar5\}$	42
6	$\{\bar0,\bar1,\bar2,\bar3,\bar4,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar0,\bar0\}\ \{\bar0,\bar0,\bar0,\bar0,\bar1,\bar6\}\ \{\bar0,\bar0,\bar0,\bar0,\bar2,\bar5\}\ \{\bar0,\bar0,\bar0,\bar0,\bar3,\bar4\}$ $\{\bar0,\bar0,\bar0,\bar1,\bar1,\bar5\}\ \{\bar0,\bar0,\bar0,\bar1,\bar3,\bar3\}\ \{\bar0,\bar0,\bar0,\bar2,\bar2,\bar3\}\ \{\bar0,\bar0,\bar0,\bar2,\bar6,\bar6\}$ $\{\bar0,\bar0,\bar0,\bar3,\bar5,\bar6\}\ \{\bar0,\bar0,\bar0,\bar4,\bar4,\bar6\}\ \{\bar0,\bar0,\bar0,\bar4,\bar5,\bar5\}\ \{\bar0,\bar0,\bar1,\bar1,\bar1,\bar4\}$ $\{\bar0,\bar0,\bar1,\bar1,\bar2,\bar3\}\ \{\bar0,\bar0,\bar1,\bar2,\bar2,\bar2\}\ \{\bar0,\bar0,\bar1,\bar1,\bar6,\bar6\}\ \{\bar0,\bar0,\bar1,\bar2,\bar5,\bar6\}$ $\{\bar0,\bar0,\bar1,\bar3,\bar4,\bar6\}\ \{\bar0,\bar0,\bar1,\bar3,\bar5,\bar5\}\ \{\bar0,\bar0,\bar1,\bar4,\bar4,\bar5\}\ \{\bar0,\bar0,\bar2,\bar2,\bar4,\bar6\}$ $\{\bar0,\bar0,\bar2,\bar2,\bar5,\bar5\}\ \{\bar0,\bar0,\bar2,\bar3,\bar3,\bar6\}\ \{\bar0,\bar0,\bar4,\bar5,\bar6,\bar6\}\ 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		$\{\bar0,\bar0,\bar0,\bar0,\bar0,\bar0,\bar0\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar0,\bar1,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar0,\bar2,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar0,\bar3,\bar5\}$	
		$\{\bar0,\bar0,\bar0,\bar0,\bar1,\bar2,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar1,\bar3,\bar4\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar2,\bar2,\bar4\}$	$\{\bar0,\bar0,\bar0,\bar0,\bar2,\bar3,\bar3\}$	
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		$\{\bar0,\bar0,\bar0,\bar0,\bar5,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar1,\bar1,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar1,\bar2,\bar4\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar1,\bar3,\bar3\}$	
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		$\{\bar0,\bar0,\bar0,\bar1,\bar3,\bar5,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar3,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar4,\bar4,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar4,\bar5,\bar6\}$	
		$\{\bar0,\bar0,\bar0,\bar1,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar4,\bar4,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar1,\bar4,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar2,\bar3,\bar4,\bar7\}$	
		$\{\bar0,\bar0,\bar0,\bar2,\bar3,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar2,\bar4,\bar4,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar2,\bar4,\bar5,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar3,\bar3,\bar3,\bar7\}$	
		$\{\bar0,\bar0,\bar0,\bar3,\bar3,\bar4,\bar6\}$	$\{\bar0,\bar0,\bar0,\bar3,\bar3,\bar5,\bar5\}$	$\{\bar0,\bar0,\bar0,\bar4,\bar4,\bar4,\bar4\}$	$\{\bar0,\bar0,\bar0,\bar4,\bar6,\bar7,\bar7\}$	
		$\{\bar0,\bar0,\bar0,\bar5,\bar5,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar5,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar0,\bar6,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar1,\bar1,\bar4\}$	
		$\{\bar0,\bar0,\bar1,\bar1,\bar1,\bar2,\bar3\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar2,\bar2,\bar2\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar1,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar2,\bar5,\bar7\}$	
		$\{\bar0,\bar0,\bar1,\bar1,\bar2,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar3,\bar4,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar3,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar1,\bar1,\bar4,\bar4,\bar6\}$	
		$\{\bar0,\bar0,\bar1,\bar1,\bar4,\bar5,\bar5\}$	$\{\bar0,\bar0,\bar1,\bar2,\bar2,\bar4,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar2,\bar2,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar1,\bar2,\bar3,\bar3,\bar7\}$	
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		$\{\bar0,\bar0,\bar1,\bar3,\bar3,\bar4,\bar5\}$	$\{\bar0,\bar0,\bar1,\bar3,\bar4,\bar4,\bar4\}$	$\{\bar0,\bar0,\bar1,\bar3,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar4,\bar5,\bar7,\bar7\}$	
		$\{\bar0,\bar0,\bar1,\bar4,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar5,\bar5,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar5,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar1,\bar2,\bar7,\bar7,\bar7\}$	
		$\{\bar0,\bar0,\bar1,\bar3,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar4,\bar5,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar4,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar1,\bar5,\bar5,\bar6,\bar7\}$	
		$\{\bar0,\bar0,\bar1,\bar5,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar2,\bar3,\bar7\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar2,\bar4,\bar6\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar2,\bar5,\bar5\}$	
		$\{\bar0,\bar0,\bar2,\bar2,\bar3,\bar3,\bar6\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar3,\bar4,\bar5\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar4,\bar4,\bar4\}$	$\{\bar0,\bar0,\bar2,\bar2,\bar6,\bar7,\bar7\}$	
		$\{\bar0,\bar0,\bar2,\bar3,\bar3,\bar3,\bar5\}$	$\{\bar0,\bar0,\bar2,\bar3,\bar3,\bar4,\bar4\}$	$\{\bar0,\bar0,\bar2,\bar3,\bar5,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar2,\bar3,\bar6,\bar6,\bar7\}$	
		$\{\bar0,\bar0,\bar2,\bar4,\bar4,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar2,\bar4,\bar5,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar2,\bar4,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar2,\bar5,\bar5,\bar5,\bar7\}$	
		$\{\bar0,\bar0,\bar2,\bar5,\bar5,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar3,\bar3,\bar3,\bar3,\bar4\}$	$\{\bar0,\bar0,\bar3,\bar3,\bar4,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar3,\bar3,\bar5,\bar6,\bar7\}$	
		$\{\bar0,\bar0,\bar3,\bar3,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar3,\bar4,\bar4,\bar6,\bar7\}$	$\{\bar0,\bar0,\bar3,\bar4,\bar5,\bar5,\bar7\}$	$\{\bar0,\bar0,\bar3,\bar4,\bar5,\bar6,\bar6\}$	
		$\{\bar0,\bar0,\bar3,\bar5,\bar5,\bar5,\bar6\}$	$\{\bar0,\bar0,\bar4,\bar4,\bar4,\bar5,\bar7\}$	$\{\bar0,\bar0,\bar4,\bar4,\bar4,\bar6,\bar6\}$	$\{\bar0,\bar0,\bar4,\bar4,\bar5,\bar5,\bar6\}$	
		$\{\bar0,\bar0,\bar4,\bar5,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar0,\bar4,\bar7,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar5,\bar6,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar0,\bar6,\bar6,\bar6,\bar7,\bar7\}$	
		$\{\bar0,\bar1,\bar1,\bar1,\bar1,\bar1,\bar3\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar1,\bar2,\bar2\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar1,\bar5,\bar7\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar1,\bar6,\bar6\}$	
		$\{\bar0,\bar1,\bar1,\bar1,\bar2,\bar4,\bar7\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar2,\bar5,\bar6\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar3,\bar3,\bar7\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar3,\bar4,\bar6\}$	
7	$\{\bar0,\bar1,\bar2,\bar3,\bar4,\bar5,\bar6,\bar7\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar3,\bar5,\bar5\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar4,\bar4,\bar5\}$	$\{\bar0,\bar1,\bar1,\bar1,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar1,\bar1,\bar2,\bar2,\bar3,\bar7\}$	429
		$\{\bar0,\bar1,\bar1,\bar2,\bar2,\bar4,\bar6\}$	$\{\bar0,\bar1,\bar1,\bar2,\bar2,\bar5,\bar5\}$	$\{\bar0,\bar1,\bar1,\bar2,\bar3,\bar3,\bar6\}$	$\{\bar0,\bar1,\bar1,\bar2,\bar3,\bar4,\bar5\}$	
		$\{\bar0,\bar1,\bar1,\bar2,\bar4,\bar4,\bar4\}$	$\{\bar0,\bar1,\bar1,\bar3,\bar3,\bar3,\bar5\}$	$\{\bar0,\bar1,\bar1,\bar3,\bar3,\bar4,\bar4\}$	$\{\bar0,\bar1,\bar2,\bar2,\bar2,\bar2,\bar7\}$	
		$\{\bar0,\bar1,\bar2,\bar2,\bar2,\bar3,\bar6\}$	$\{\bar0,\bar1,\bar2,\bar2,\bar2,\bar4,\bar5\}$	$\{\bar0,\bar1,\bar2,\bar2,\bar3,\bar3,\bar5\}$	$\{\bar0,\bar1,\bar2,\bar2,\bar3,\bar4,\bar4\}$	
		$\{\bar0,\bar1,\bar2,\bar3,\bar3,\bar3,\bar4\}$	$\{\bar0,\bar1,\bar3,\bar3,\bar3,\bar3,\bar3\}$	$\{\bar0,\bar1,\bar3,\bar3,\bar3,\bar7,\bar7\}$	$\{\bar0,\bar1,\bar3,\bar3,\bar4,\bar6,\bar7\}$	
		$\{\bar0,\bar1,\bar3,\bar3,\bar5,\bar5,\bar7\}$	$\{\bar0,\bar1,\bar3,\bar3,\bar5,\bar6,\bar6\}$	$\{\bar0,\bar1,\bar3,\bar4,\bar4,\bar5,\bar7\}$	$\{\bar0,\bar1,\bar3,\bar4,\bar4,\bar6,\bar6\}$	
		$\{\bar0,\bar1,\bar3,\bar4,\bar5,\bar5,\bar6\}$	$\{\bar0,\bar1,\bar3,\bar5,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar1,\bar3,\bar7,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar1,\bar4,\bar4,\bar4,\bar4,\bar7\}$	
		$\{\bar0,\bar1,\bar4,\bar4,\bar4,\bar5,\bar6\}$	$\{\bar0,\bar1,\bar4,\bar4,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar1,\bar4,\bar6,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar1,\bar5,\bar5,\bar7,\bar7,\bar7\}$	
		$\{\bar0,\bar1,\bar5,\bar6,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar1,\bar6,\bar6,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar2,\bar2,\bar2,\bar2,\bar2,\bar6\}$	$\{\bar0,\bar2,\bar2,\bar2,\bar2,\bar3,\bar5\}$	
		$\{\bar0,\bar2,\bar2,\bar2,\bar2,\bar4,\bar4\}$	$\{\bar0,\bar2,\bar2,\bar2,\bar3,\bar3,\bar4\}$	$\{\bar0,\bar2,\bar2,\bar2,\bar4,\bar7,\bar7\}$	$\{\bar0,\bar2,\bar2,\bar2,\bar5,\bar6,\bar7\}$	
		$\{\bar0,\bar2,\bar2,\bar2,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar2,\bar2,\bar3,\bar3,\bar3,\bar3\}$	$\{\bar0,\bar2,\bar2,\bar3,\bar3,\bar7,\bar7\}$	$\{\bar0,\bar2,\bar2,\bar3,\bar4,\bar6,\bar7\}$	
		$\{\bar0,\bar2,\bar2,\bar3,\bar5,\bar5,\bar7\}$	$\{\bar0,\bar2,\bar2,\bar3,\bar5,\bar6,\bar6\}$	$\{\bar0,\bar2,\bar2,\bar4,\bar4,\bar5,\bar7\}$	$\{\bar0,\bar2,\bar2,\bar4,\bar4,\bar6,\bar6\}$	
		$\{\bar0,\bar2,\bar2,\bar4,\bar5,\bar5,\bar6\}$	$\{\bar0,\bar2,\bar2,\bar5,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar2,\bar3,\bar3,\bar3,\bar6,\bar7\}$	$\{\bar0,\bar2,\bar3,\bar3,\bar4,\bar5,\bar7\}$	
		$\{\bar0,\bar2,\bar3,\bar3,\bar4,\bar6,\bar6\}$	$\{\bar0,\bar2,\bar3,\bar3,\bar5,\bar5,\bar6\}$	$\{\bar0,\bar2,\bar3,\bar4,\bar4,\bar4,\bar7\}$	$\{\bar0,\bar2,\bar3,\bar4,\bar4,\bar5,\bar6\}$	
		$\{\bar0,\bar2,\bar3,\bar4,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar2,\bar3,\bar6,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar2,\bar4,\bar4,\bar4,\bar4,\bar7\}$	$\{\bar0,\bar2,\bar4,\bar4,\bar4,\bar5,\bar6\}$	
		$\{\bar0,\bar2,\bar4,\bar4,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar2,\bar4,\bar5,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar2,\bar4,\bar6,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar2,\bar5,\bar5,\bar6,\bar7,\bar7\}$	
		$\{\bar0,\bar2,\bar5,\bar6,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar2,\bar6,\bar6,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar3,\bar3,\bar3,\bar3,\bar5,\bar7\}$	$\{\bar0,\bar3,\bar3,\bar3,\bar3,\bar6,\bar6\}$	
		$\{\bar0,\bar3,\bar3,\bar3,\bar4,\bar4,\bar7\}$	$\{\bar0,\bar3,\bar3,\bar3,\bar4,\bar5,\bar6\}$	$\{\bar0,\bar3,\bar3,\bar3,\bar5,\bar5,\bar5\}$	$\{\bar0,\bar3,\bar3,\bar4,\bar4,\bar4,\bar6\}$	
		$\{\bar0,\bar3,\bar3,\bar4,\bar4,\bar5,\bar5\}$	$\{\bar0,\bar3,\bar3,\bar5,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar3,\bar3,\bar6,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar3,\bar4,\bar4,\bar4,\bar4,\bar5\}$	
		$\{\bar0,\bar3,\bar4,\bar4,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar3,\bar4,\bar5,\bar6,\bar7,\bar7\}$	$\{\bar0,\bar3,\bar4,\bar6,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar3,\bar5,\bar5,\bar5,\bar7,\bar7\}$	
		$\{\bar0,\bar3,\bar5,\bar5,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar3,\bar5,\bar6,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar4,\bar4,\bar4,\bar4,\bar4,\bar4\}$	$\{\bar0,\bar4,\bar4,\bar4,\bar6,\bar7,\bar7\}$	
		$\{\bar0,\bar4,\bar4,\bar5,\bar5,\bar7,\bar7\}$	$\{\bar0,\bar4,\bar4,\bar5,\bar6,\bar6,\bar7\}$	$\{\bar0,\bar4,\bar4,\bar6,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar4,\bar5,\bar5,\bar5,\bar6,\bar7\}$	
		$\{\bar0,\bar4,\bar5,\bar5,\bar6,\bar6,\bar6\}$	$\{\bar0,\bar5,\bar5,\bar5,\bar5,\bar5,\bar7\}$	$\{\bar0,\bar5,\bar5,\bar5,\bar5,\bar6,\bar7\}$	$\{\bar0,\bar5,\bar5,\bar5,\bar6,\bar6,\bar6\}$	
		$\{\bar0,\bar5,\bar7,\bar7,\bar7,\bar7,\bar7\}$	$\{\bar0,\bar6,\bar6,\bar7,\bar7,\bar7,\bar7\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar1,\bar2,\bar2\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar1,\bar4,\bar7\}$	
		$\{\bar1,\bar1,\bar1,\bar1,\bar1,\bar5,\bar6\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar2,\bar3,\bar7\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar2,\bar4,\bar6\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar2,\bar5,\bar5\}$	
		$\{\bar1,\bar1,\bar1,\bar1,\bar3,\bar3,\bar6\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar3,\bar4,\bar5\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar4,\bar4,\bar4\}$	$\{\bar1,\bar1,\bar1,\bar1,\bar6,\bar7,\bar7\}$	
		$\{\bar1,\bar1,\bar1,\bar2,\bar2,\bar2,\bar7\}$	$\{\bar1,\bar1,\bar1,\bar2,\bar2,\bar3,\bar6\}$	$\{\bar1,\bar1,\bar1,\bar2,\bar2,\bar4,\bar5\}$	$\{\bar1,\bar1,\bar1,\bar2,\bar3,\bar3,\bar5\}$	
		$\{\bar1,\bar1,\bar1,\bar2,\bar3,\bar4,\bar4\}$	$\{\bar1,\bar1,\bar1,\bar3,\bar3,\bar3,\bar4\}$	$\{\bar1,\bar1,\bar1,\bar2,\bar5,\bar7,\bar7\}$	$\{\bar1,\bar1,\bar1,\bar2,\bar6,\bar6,\bar7\}$	

$\{\bar{1},\bar{1},\bar{1},\bar{3},\bar{4},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{1},\bar{3},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{1},\bar{3},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{1},\bar{4},\bar{4},\bar{6},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{1},\bar{4},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{1},\bar{4},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{1},\bar{5},\bar{5},\bar{5},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{2},\bar{6}\}$
$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{3},\bar{5}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{4},\bar{4}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{3},\bar{3},\bar{4}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{3},\bar{3},\bar{3},\bar{3}\}$
$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{4},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{2},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{3},\bar{3},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{2},\bar{3},\bar{4},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{3},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{3},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{4},\bar{4},\bar{5},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{2},\bar{4},\bar{4},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{4},\bar{5},\bar{5},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{5},\bar{5},\bar{5},\bar{5}\}$	$\{\bar{1},\bar{1},\bar{2},\bar{7},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{3},\bar{3},\bar{3},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{3},\bar{4},\bar{5},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{3},\bar{4},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{3},\bar{5},\bar{5},\bar{6}\}$
$\{\bar{1},\bar{1},\bar{3},\bar{4},\bar{4},\bar{4},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{4},\bar{4},\bar{5},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{4},\bar{5},\bar{5},\bar{5}\}$	$\{\bar{1},\bar{1},\bar{3},\bar{6},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{4},\bar{4},\bar{4},\bar{4},\bar{6}\}$	$\{\bar{1},\bar{1},\bar{4},\bar{4},\bar{4},\bar{5},\bar{5}\}$	$\{\bar{1},\bar{1},\bar{4},\bar{5},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{4},\bar{6},\bar{6},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{1},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{5},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{1},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{2},\bar{2},\bar{2},\bar{2},\bar{3},\bar{4}\}$
$\{\bar{1},\bar{2},\bar{2},\bar{2},\bar{3},\bar{3},\bar{3}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{3},\bar{3},\bar{4},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{3},\bar{3},\bar{5},\bar{6}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{3},\bar{4},\bar{4},\bar{6}\}$
$\{\bar{1},\bar{3},\bar{3},\bar{3},\bar{4},\bar{5},\bar{5}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{4},\bar{4},\bar{4},\bar{5}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{4},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{3},\bar{5},\bar{6},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{3},\bar{3},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{4},\bar{4},\bar{4},\bar{4},\bar{4}\}$	$\{\bar{1},\bar{3},\bar{4},\bar{4},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{4},\bar{5},\bar{5},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{3},\bar{4},\bar{5},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{4},\bar{6},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{3},\bar{5},\bar{5},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{3},\bar{5},\bar{5},\bar{6},\bar{6},\bar{6}\}$
$\{\bar{1},\bar{4},\bar{4},\bar{4},\bar{5},\bar{7},\bar{7}\}$	$\{\bar{1},\bar{4},\bar{4},\bar{4},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{4},\bar{4},\bar{5},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{1},\bar{4},\bar{4},\bar{5},\bar{6},\bar{6},\bar{6}\}$
$\{\bar{1},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{1},\bar{4},\bar{5},\bar{5},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{1},\bar{5},\bar{5},\bar{5},\bar{5},\bar{5},\bar{6}\}$	$\{\bar{1},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{1},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{2},\bar{2},\bar{4}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{2},\bar{3},\bar{3}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{2},\bar{7},\bar{7}\}$
$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{3},\bar{6},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{4},\bar{5},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{4},\bar{6},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{2},\bar{5},\bar{5},\bar{6}\}$
$\{\bar{2},\bar{2},\bar{2},\bar{3},\bar{3},\bar{6},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{3},\bar{4},\bar{4},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{3},\bar{5},\bar{5},\bar{5}\}$
$\{\bar{2},\bar{2},\bar{2},\bar{4},\bar{4},\bar{4},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{4},\bar{4},\bar{5},\bar{5}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{5},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{2},\bar{6},\bar{6},\bar{7},\bar{7}\}$
$\{\bar{2},\bar{2},\bar{3},\bar{3},\bar{3},\bar{4},\bar{7}\}$	$\{\bar{2},\bar{2},\bar{3},\bar{3},\bar{3},\bar{5},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{3},\bar{3},\bar{4},\bar{4},\bar{6}\}$	$\{\bar{2},\bar{2},\bar{3},\bar{3},\bar{4},\bar{5},\bar{5}\}$
$\{\bar{2},\bar{2},\bar{3},\bar{4},\bar{4},\bar{4},\bar{5}\}$	$\{\bar{2},\bar{2},\bar{4},\bar{4},\bar{4},\bar{4},\bar{4}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{3},\bar{3},\bar{3},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{3},\bar{3},\bar{4},\bar{6}\}$
$\{\bar{2},\bar{3},\bar{3},\bar{3},\bar{3},\bar{5},\bar{5}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{3},\bar{4},\bar{4},\bar{5}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{4},\bar{4},\bar{4},\bar{4}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{3},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{2},\bar{3},\bar{3},\bar{4},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{5},\bar{5},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{5},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{3},\bar{6},\bar{6},\bar{6},\bar{6}\}$
$\{\bar{2},\bar{3},\bar{4},\bar{4},\bar{5},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{4},\bar{4},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{4},\bar{5},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{6},\bar{6}\}$
$\{\bar{2},\bar{3},\bar{5},\bar{5},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{2},\bar{3},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{4},\bar{4},\bar{4},\bar{4},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{4},\bar{4},\bar{4},\bar{5},\bar{6},\bar{7}\}$
$\{\bar{2},\bar{4},\bar{4},\bar{4},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{2},\bar{4},\bar{4},\bar{5},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{2},\bar{4},\bar{4},\bar{5},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{2},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5},\bar{6}\}$
$\{\bar{2},\bar{4},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{5},\bar{5},\bar{5},\bar{5},\bar{5},\bar{5}\}$	$\{\bar{2},\bar{5},\bar{5},\bar{7},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{2},\bar{5},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{2},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{3},\bar{3},\bar{3},\bar{6}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{3},\bar{3},\bar{4},\bar{5}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{3},\bar{4},\bar{4},\bar{4}\}$
$\{\bar{3},\bar{3},\bar{3},\bar{3},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{4},\bar{5},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{4},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{3},\bar{5},\bar{5},\bar{6},\bar{7}\}$
$\{\bar{3},\bar{3},\bar{3},\bar{5},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{4},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{5},\bar{6},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{6},\bar{6},\bar{6}\}$
$\{\bar{3},\bar{3},\bar{4},\bar{5},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{3},\bar{3},\bar{4},\bar{5},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{3},\bar{3},\bar{5},\bar{5},\bar{5},\bar{5},\bar{6}\}$	$\{\bar{3},\bar{3},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{4},\bar{6},\bar{7}\}$	$\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{7}\}$	$\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{5},\bar{6},\bar{6}\}$	$\{\bar{3},\bar{4},\bar{4},\bar{5},\bar{5},\bar{5},\bar{6}\}$
$\{\bar{3},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5},\bar{5}\}$	$\{\bar{3},\bar{4},\bar{5},\bar{7},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{4},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{3},\bar{5},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{3},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{4},\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{6}\}$	$\{\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5}\}$
$\{\bar{4},\bar{4},\bar{4},\bar{7},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{4},\bar{4},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}\}$	$\{\bar{4},\bar{4},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{4},\bar{5},\bar{5},\bar{5},\bar{7},\bar{7},\bar{7}\}$
$\{\bar{4},\bar{5},\bar{5},\bar{6},\bar{6},\bar{7},\bar{7}\}$	$\{\bar{4},\bar{5},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{4},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6}\}$	$\{\bar{5},\bar{5},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7}\}$
	$\{\bar{5},\bar{5},\bar{5},\bar{6},\bar{6},\bar{6},\bar{7}\}$	$\{\bar{5},\bar{5},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6}\}$	

Table 4. Multiset of Additive Group $\mathbb{Z}_{n+1}$ with $1 \leq n \leq 7$ which is the Partition of $\sum_{k=0}^{n-1} p(k(n+1); n, n)$

n	$\mathbb{Z}_{n+1}$	k	Multiset	The number of Multisets	Total
1	$\{\bar0,\bar1\}$	0	$\{\bar0\}$	1	1
2	$\{\bar0,\bar1,\bar2\}$	0	$\{\bar0,\bar0\}$	1	2
		1	$\{\bar1,\bar2\}$	1	
3	$\{\bar0,\bar1,\bar2,\bar3\}$	0	$\{\bar0,\bar0,\bar0\}$	1	
		1	$\{\bar0,\bar1,\bar3\}$ $\{\bar0,\bar2,\bar2\}$ $\{\bar1,\bar1,\bar2\}$	3	5
		2	$\{\bar2,\bar3,\bar3\}$	1	
4	$\{\bar0,\bar1,\bar2,\bar3,\bar4\}$	0	$\{\bar0,\bar0,\bar0,\bar0\}$	1	14
		1	$\{\bar0,\bar0,\bar1,\bar4\}$ $\{\bar0,\bar0,\bar2,\bar3\}$ $\{\bar0,\bar1,\bar1,\bar3\}$ $\{\bar0,\bar1,\bar2,\bar3\}$ $\{\bar1,\bar1,\bar1,\bar2\}$	5	
		2	$\{\bar0,\bar2,\bar4,\bar4\}$ $\{\bar0,\bar3,\bar3,\bar4\}$ $\{\bar1,\bar1,\bar4,\bar4\}$ $\{\bar1,\bar2,\bar3,\bar4\}$ $\{\bar1,\bar3,\bar3,\bar3\}$ $\{\bar2,\bar2,\bar2,\bar4\}$ $\{\bar2,\bar2,\bar3,\bar3\}$	7	
		3	$\{\bar3,\bar4,\bar4,\bar4\}$	1	
5	$\{\bar0,\bar1,\bar2,\bar3,\bar4,\bar5\}$	0	$\{\bar0,\bar0,\bar0,\bar0,\bar0\}$	1	181
		1	$\{\bar0,\bar0,\bar0,\bar1,\bar5\}$ $\{\bar0,\bar0,\bar0,\bar2,\bar4\}$ $\{\bar0,\bar0,\bar0,\bar3,\bar3\}$ $\{\bar0,\bar0,\bar1,\bar1,\bar4\}$ $\{\bar0,\bar0,\bar1,\bar2,\bar3\}$ $\{\bar0,\bar0,\bar2,\bar2,\bar2\}$ $\{\bar0,\bar1,\bar1,\bar1,\bar3\}$ $\{\bar0,\bar1,\bar1,\bar2,\bar2\}$ $\{\bar1,\bar1,\bar1,\bar1,\bar2\}$	9	
		2	$\{\bar0,\bar0,\bar2,\bar5,\bar5\}$ $\{\bar0,\bar0,\bar3,\bar4,\bar5\}$ $\{\bar0,\bar0,\bar4,\bar4,\bar4\}$ $\{\bar0,\bar1,\bar1,\bar5,\bar5\}$ $\{\bar0,\bar1,\bar2,\bar4,\bar5\}$ $\{\bar0,\bar1,\bar3,\bar3,\bar5\}$ $\{\bar0,\bar1,\bar3,\bar4,\bar4\}$ $\{\bar0,\bar2,\bar2,\bar4,\bar4\}$ $\{\bar0,\bar2,\bar2,\bar3,\bar5\}$ $\{\bar0,\bar2,\bar3,\bar3,\bar4\}$ $\{\bar0,\bar3,\bar3,\bar3,\bar3\}$ $\{\bar1,\bar1,\bar1,\bar4,\bar5\}$ $\{\bar1,\bar1,\bar2,\bar3,\bar5\}$ $\{\bar1,\bar1,\bar2,\bar4,\bar4\}$ $\{\bar1,\bar1,\bar3,\bar3,\bar4\}$ $\{\bar1,\bar2,\bar2,\bar2,\bar5\}$ $\{\bar1,\bar2,\bar2,\bar3,\bar4\}$ $\{\bar1,\bar2,\bar3,\bar3,\bar3\}$ $\{\bar2,\bar2,\bar2,\bar2,\bar4\}$ $\{\bar2,\bar2,\bar2,\bar3,\bar3\}$	20	
		3	$\{\bar0,\bar3,\bar5,\bar5,\bar5\}$ $\{\bar0,\bar4,\bar4,\bar5,\bar5\}$ $\{\bar1,\bar2,\bar5,\bar5,\bar5\}$ $\{\bar1,\bar3,\bar4,\bar5,\bar5\}$ $\{\bar1,\bar4,\bar4,\bar4,\bar5\}$ $\{\bar2,\bar2,\bar4,\bar5,\bar5\}$ $\{\bar2,\bar3,\bar3,\bar5,\bar5\}$ $\{\bar2,\bar3,\bar4,\bar4,\bar5\}$ $\{\bar2,\bar4,\bar4,\bar4,\bar4\}$ $\{\bar3,\bar3,\bar3,\bar4,\bar5\}$ $\{\bar3,\bar3,\bar4,\bar4,\bar4\}$	11	
		4	$\{\bar4,\bar5,\bar5,\bar5,\bar5\}$	1	
		4	$\{\bar0,\bar0,\bar4,\bar7,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar0,\bar5,\bar6,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar0,\bar6,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar1,\bar3,\bar7,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar1,\bar4,\bar6,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar1,\bar5,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar1,\bar5,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar1,\bar6,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar0,\bar2,\bar2,\bar7,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar2,\bar3,\bar6,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar2,\bar4,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar2,\bar4,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar2,\bar5,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar2,\bar5,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar0,\bar2,\bar6,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar0,\bar3,\bar3,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar3,\bar3,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar3,\bar4,\bar4,\bar7,\bar7,\bar7\}$ $\{\bar0,\bar3,\bar4,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar3,\bar4,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar0,\bar3,\bar5,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar0,\bar3,\bar5,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar0,\bar3,\bar5,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar0,\bar4,\bar4,\bar4,\bar6,\bar7,\bar7\}$ $\{\bar0,\bar4,\bar4,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar0,\bar4,\bar4,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar0,\bar4,\bar4,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar0,\bar4,\bar5,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar0,\bar4,\bar5,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar0,\bar5,\bar5,\bar5,\bar5,\bar5,\bar7\}$ $\{\bar0,\bar5,\bar5,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar0,\bar5,\bar5,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar1,\bar1,\bar2,\bar7,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar1,\bar3,\bar6,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar1,\bar4,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar1,\bar4,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar1,\bar5,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar1,\bar6,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar2,\bar2,\bar6,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar3,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar3,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar4,\bar4,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar4,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar4,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar2,\bar5,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar1,\bar2,\bar5,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar2,\bar5,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar1,\bar3,\bar3,\bar4,\bar7,\bar7,\bar7\}$ $\{\bar1,\bar3,\bar3,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar3,\bar3,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar3,\bar4,\bar4,\bar6,\bar7,\bar7\}$ $\{\bar1,\bar3,\bar4,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar1,\bar3,\bar4,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar3,\bar4,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar1,\bar3,\bar5,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar1,\bar3,\bar5,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar1,\bar4,\bar4,\bar4,\bar5,\bar7,\bar7\}$ $\{\bar1,\bar4,\bar4,\bar4,\bar6,\bar6,\bar7\}$ $\{\bar1,\bar4,\bar4,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar1,\bar4,\bar4,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar1,\bar4,\bar5,\bar5,\bar5,\bar5,\bar7\}$ $\{\bar1,\bar4,\bar5,\bar5,\bar5,\bar6,\bar6\}$ $\{\bar1,\bar5,\bar5,\bar5,\bar5,\bar5,\bar6\}$ $\{\bar2,\bar2,\bar2,\bar5,\bar7,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar2,\bar6,\bar6,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar3,\bar4,\bar7,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar3,\bar5,\bar6,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar3,\bar6,\bar6,\bar6,\bar7\}$ $\{\bar2,\bar2,\bar4,\bar4,\bar6,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar4,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar2,\bar2,\bar4,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar2,\bar2,\bar4,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar2,\bar2,\bar5,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar2,\bar2,\bar5,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar2,\bar3,\bar3,\bar3,\bar7,\bar7,\bar7\}$ $\{\bar2,\bar3,\bar3,\bar4,\bar6,\bar7,\bar7\}$ $\{\bar2,\bar3,\bar3,\bar5,\bar5,\bar7,\bar7\}$ $\{\bar2,\bar3,\bar3,\bar5,\bar6,\bar6,\bar7\}$ $\{\bar2,\bar3,\bar3,\bar6,\bar6,\bar6,\bar6\}$ $\{\bar2,\bar3,\bar4,\bar4,\bar5,\bar7,\bar7\}$ $\{\bar2,\bar3,\bar4,\bar4,\bar6,\bar6,\bar7\}$ $\{\bar2,\bar3,\bar4,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar2,\bar3,\bar4,\bar5,\bar6,\bar6,\bar6\}$ $\{\bar2,\bar3,\bar5,\bar5,\bar5,\bar5,\bar7\}$ $\{\bar2,\bar4,\bar4,\bar4,\bar4,\bar7,\bar7\}$ $\{\bar2,\bar4,\bar4,\bar4,\bar5,\bar6,\bar7\}$ $\{\bar2,\bar4,\bar4,\bar4,\bar6,\bar6,\bar6\}$ $\{\bar2,\bar4,\bar4,\bar5,\bar5,\bar5,\bar7\}$ $\{\bar2,\bar4,\bar4,\bar5,\bar5,\bar6,\bar6\}$ $\{\bar2,\bar4,\bar5,\bar5,\bar5,\bar5,\bar6\}$ $\{\bar2,\bar5,\bar5,\bar5,\bar5,\bar5,\bar5\}$ $\{\bar3,\bar3,\bar3,\bar3,\bar6,\bar7,\bar7\}$ $\{\bar3,\bar3,\bar3,\bar4,\bar5,\bar7,\bar7\}$ $\{\bar3,\bar3,\bar3,\bar4,\bar6,\bar6,\bar7\}$ $\{\bar3,\bar3,\bar3,\bar5,\bar5,\bar6,\bar7\}$ $\{\bar3,\bar3,\bar3,\bar5,\bar6,\bar6,\bar6\}$	112	

	$\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{4},\bar{7},\bar{7}\}$ $\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{5},\bar{6},\bar{7}\}$ $\{\bar{3},\bar{3},\bar{4},\bar{4},\bar{6},\bar{6},\bar{6}\}$ $\{\bar{3},\bar{3},\bar{4},\bar{5},\bar{5},\bar{5},\bar{7}\}$	
	$\{\bar{3},\bar{3},\bar{4},\bar{5},\bar{5},\bar{6},\bar{6}\}$ $\{\bar{3},\bar{3},\bar{5},\bar{5},\bar{5},\bar{5},\bar{6}\}$ $\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{4},\bar{6},\bar{7}\}$ $\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{7}\}$	
	$\{\bar{3},\bar{4},\bar{4},\bar{4},\bar{5},\bar{6},\bar{6}\}$ $\{\bar{3},\bar{4},\bar{4},\bar{5},\bar{5},\bar{5},\bar{6}\}$ $\{\bar{3},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5},\bar{5}\}$ $\{\bar{4},\bar{4},\bar{4},\bar{4},\bar{4},\bar{5},\bar{7}\}$	
	$\{\bar{4},\bar{4},\bar{4},\bar{4},\bar{4},\bar{6},\bar{6}\}$ $\{\bar{4},\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{6}\}$ $\{\bar{4},\bar{4},\bar{4},\bar{5},\bar{5},\bar{5},\bar{5}\}$	
	$\{\bar{0},\bar{5},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{0},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{1},\bar{4},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{1},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$	
	$\{\bar{1},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{2},\bar{3},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{2},\bar{4},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{2},\bar{5},\bar{5},\bar{7},\bar{7},\bar{7},\bar{7}\}$	
	$\{\bar{2},\bar{5},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{2},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$ $\{\bar{3},\bar{3},\bar{6},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{3},\bar{4},\bar{5},\bar{7},\bar{7},\bar{7},\bar{7}\}$	
5	$\{\bar{3},\bar{4},\bar{6},\bar{6},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{3},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{3},\bar{5},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7}\}$ $\{\bar{3},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$	26
	$\{\bar{4},\bar{4},\bar{4},\bar{7},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{4},\bar{4},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}\}$ $\{\bar{4},\bar{4},\bar{6},\bar{6},\bar{6},\bar{7},\bar{7}\}$ $\{\bar{4},\bar{5},\bar{5},\bar{5},\bar{7},\bar{7},\bar{7}\}$	
	$\{\bar{4},\bar{5},\bar{5},\bar{6},\bar{6},\bar{7},\bar{7}\}$ $\{\bar{4},\bar{5},\bar{6},\bar{6},\bar{6},\bar{6},\bar{7}\}$ $\{\bar{4},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6}\}$ $\{\bar{5},\bar{5},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7}\}$	
	$\{\bar{5},\bar{5},\bar{5},\bar{6},\bar{6},\bar{6},\bar{7}\}$ $\{\bar{5},\bar{5},\bar{6},\bar{6},\bar{6},\bar{6},\bar{6}\}$	
6	$\{\bar{6},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7},\bar{7}\}$	1

The Stirling number of the first kind $c(n, k)$ is the number of permutations of an n-element set with exactly cycles. Second-kind Stirling number $S(n, k)$ counts the number of ways that n distinct objects can be partitioned into k indistinguishable subsets, with each subset containing at least one object. According to Riordan (2012), Stirling numbers of the first kind is $s(n, k)$, where $s(n, k)$ satisfies the recursion relation $s(n, k) = (n-1) s(n-1, k) + s(n-1, k-1)$, where k and n are integers, $1 \leq n \leq k-1$ with initial conditions $s(n, 0) = 0$, for $n \geq 1$ and $s(n, n) = 1$, for $n \geq 0$.

Table 2 show the relationship of the Second-kind Stirling number, $s(n, k)$ and Bell number B_n. The sum of the Second-kind Stirling number (the sum in every row) is Bell number (the number in the last column).

3. RESULTS AND DISCUSSION

To determine the number of multisets we can enumerate or use source code. For this study we developed a source code to determine a multiset. The following is an algorithm and the source code for determining all multisets of the additive group $\mathbb{Z}_{n+1}$ using Python.

```
#Multiset Grup Aditif ℤ_{n+1}
inisiasi
Input: n, k, n, k integer
Output : Multisets, number of multiset, multiset for every k and
number of multisets for every k.

Implementation
Begin
Read (n, k)
    Set ℤ_{n+1} ← Set {0, 1, 2, ···, n}
    Max k ← (0, 1, 2,···, n-1)

Target (n, k)
    repeat (n+1)*k until Max k

    candidates←set(Combination with raplecement
    (ℤ_{n+1}, Max k));

    targets ← [Target (n, k) for k in max k];

multiset (candidates, targets):
  for candidate in candidates
    if (sum(candidate) in targets)
      then add multiset in candidate
        repeat multisets until Target (n, k)
          multiset← multiset (candidates, targets);

          Stirling(multisets, targets)
        for multiset in multisets
      if (sum(multiset) in targets)
then add stirling in multiset
    repeat stirling until Target
  k_2← target (n, k) for k = k input;

stirling←Stirling (multiset, k_2);

print("ℤ_{n+1} : ", Set ℤ_{n+1})
print("k : ", Max k)
print("multisets : ", multisets)
print("Bilangan Stirling k = ",k,":", stirling)
print("Jumlah Bilangan Stirling k = ",k," : ", len(stirling))
print("Total Bilangan Catalan : ", len(multisets))

End
```

The multisets generated by that source code are grouped and put in Table 3. From the result of Table 3, we do partition $p(m;n,n)$ on the number of multisets, where $p(m;n,n)$ is the number of partitions of m to at most n parts and every element in each part cannot exceed n, with $m = k(n+1)$. Thus, the number of multisets can be written as $\sum_{k=0}^{n-1} p(k(n+1); n, n)$. The result are shown on Table 4.

Based on the computational results where $0 \leq k \leq n-1$, a sequence of numbers is obtained, which is contained in Table 5 below:

Based on Table 5, many possible multisets are formed from the additive group $\mathbb{Z}_{10}$ forming a Catalan number for each n. In addition, if the multiset additive group $\mathbb{Z}_{10}$ is divided into several parts where each part depends on the value of $k(n+1)$ with $0 \leq k \leq n-1$, then the number of multisets formed will form a new second type of the Stirling number. The Stirling number in the left-hand side (in Table 5, the yellow row) will

Table 5. Number Sequences of the Possible Multiset Sums of the Additive Group $\mathbb{Z}_{10}$

k\n	0	1	2	3	4	5	6	7	8	Catalan number
1	1									1
2	1	1								2
3	1	3	1							5
4	1	5	7	1						14
5	1	9	20	11	1					42
6	1	13	48	51	18	1				132
7	1	20	100	169	112	26	1			429
8	1	28	194	461	486	221	38	1		1430
9	1	40	352	1128	1667	1210	411	52	1	4862

be smaller, and the rightmost portion (in red) will be larger than the Stirling number of the second type that is known. Moreover, the last column which sum of the Stirling's number and originally is Bell's, now was replaced by a Catalan number.

4. CONCLUSION

We can conclude that the multiset formed from the additive group $\mathbb{Z}_{10}$ is a Catalan number based on the results. In addition, if the multiset additive group $\mathbb{Z}_{10}$ is divided into several parts where each part depends on the value of $k(n+1)$ with $0 \leq k \leq n-1$, then the number of multisets will form a new Stirling number of the second kind. However, the second kind of Stirling number formed based on the multiset additive group $\mathbb{Z}_{10}$ has a different initial value, namely $S(n,0) = 1$ and the value $S(n,n) = 0$. For $S(n,1)$ to $S(n,n\text{-}2)$ the value is smaller, while for $S(n,n\text{-}1)$ it has a larger value. Moreover, the sum of the Stirling number and originally is Bell number, now is changed to be Catalan number.

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