Thermodynamics of a One-Dimensional Gravitational Gas in a Uniform External Field

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ABSTRACT

We derive exact expressions for the partition function, equation of state, mean internal energy, and heat capacity at constant volume of a one-dimensional gravitational gas (1DGG) in a uniform external field.

INTRODUCTION

There has been much interest in the study of many-body system with gravitational interactions. The primary motivation for these studies has been the desire to understand the large-scale structure of the universe (Saslaw, 1985). Another motivation for these studies is that these systems offer an opportunity to re-study basic aspects of statistical mechanics and thermodynamics.

Various model systems have been investigated in order to understand many aspects of the behavior of large-scale structures using many different approaches. Examples of model gravitational systems that have been studied include three-dimensional systems with a softened potential (Follana & Laliena, 2000; Sommer-Larsen et al., 1998), spherically symmetric three-dimensional systems (Henriksen & Widrow, 1997; Davies & Widrow, 1997; Perez & Aly, 1996; Perez et al., 1996; Henriksen & Widrow, 1995), two-dimensional systems (El-Zant, 1998; Medvedev, 2000; Gromov, 1995) both with cylindrical and without cylindrical symmetry and one-dimensional systems (Tsuchiya et al., 1996; 1998; Aurell et al., 1999).

One problematic aspect of the physics of many-body systems with gravitational interactions has to do with their equilibrium thermodynamics. Miller (1974) and Rybicki (1971) pointed out that the thermodynamics of self-gravitating may be counterintuitive. A great part of our intuition about the behavior of many-body systems has been acquired from what is known about systems with short-range interactions. Because of this it is not unreasonable to ask how the thermodynamics of systems with long-range interactions is different.

Studies on the thermodynamics of the one dimensional gravitational gas (1DGG), i.e., a system of very large parallel mass sheets, have been done by Salzberg (1965) and by Muriel et al. (1994). Salzberg (1965) calculated the equation of state, entropy, enthalpy and the Gibbs free energy for a 1DGG in a box with one movable wall. Later, Muriel et al. (1994) obtained the exact partition function, equation of state, mean internal energy, and heat capacities at constant pressure and constant volume of a 1DGG with an arbitrary number of particles in a box.

In this paper, we study the thermodynamics of a 1DGG in a uniform external field by adopting the methods used in Muriel et al. (1994). The paper is organized as follows: In Section 2, we write the expression for the total energy of the 1DGG in a uniform external field. In Section 3, we derive an analytical expression for the partition function of the system. In Section 4, we obtain expressions for the equation of state, mean internal energy, and heat capacity at constant volume. We conclude the paper in Section 5.

THE ONE-DIMENSIONAL GRAVITATIONAL GAS

The gravitational potential energy of a pair of very large parallel sheets, which we shall label i and j, is given by $Gm_im_j|x_i - x_j|$ where G is a gravitational constant, m_i and m_j are their 'masses', and $|x_i - x_j|$ is the distance between the particles. (Note that we will use the terms 'sheet' and 'particle' interchangeably.) The total energy of the N-particle 1DGG under the influence of an external field is

$$E_{TOTAL} = \sum_{i=1}^{N} p_i^2/2m + Gm^2 \sum_{j>i}^{i=1..N-1} |x_i - x_j| + mg\sum_{i=1}^{N} x_i \quad (1)$$

where m is the mass of each particle, g is the strength of the external field, x_i and p_i are the ith particle's position and momentum,respectively.

If $x_1 < x_2 < x_3 ... < x_{N-1} < x_N$,the total energy can be rewritten as

$$E_{TOTAL} = \sum_{i=1}^{N} p_i^2/2m + \sum_{i=1}^{N}\left[-Gm^2(N+1-2i) + mg\right]x_i \quad (2)$$

We emphasize that although eq. 2 is true only if $x_1 < x_2 < x_3 ... < x_{N-1} < x_N$, nothing prevents us from re-indexing the particles in the system.

EVALUATION OF THE PARTITION FUNCTION

The partition function is evaluated using

$$Z = \frac{1}{N!h^N} \int \cdots \int \prod_{i=1}^{N} (dx_i\, dp_i)\, e^{-\beta E_{TOTAL}} \quad (3)$$

where $\beta = 1/kT$ is the inverse temperature, and h is Planck's constant. Planck's constant will not appear in the final expressions for the equation of state, mean internal energy, and constant volume heat capacity but is being introduced merely to make the expression for the partition function consistent with its quantum mechanical counterpart. After performing the momentum integrations, substituting the expression for the total energy given by (2) into the expression for the partition function (3), noting that for an N-particle system, there are N! possible orderings, and separating the integrals we can express the partition function as

$$Z = \left(\frac{2\pi m}{\beta h^2}\right)^{N/2} \int_0^L dx_N e^{\beta(\gamma(1-N-\alpha)x_N} \int_0^L dx_{N-1} e^{\beta(\gamma(3-N-\alpha)x_N}$$

$$\cdots \int_0^{x_3} dx_2 e^{\beta\gamma(N-3-\alpha)x_2} \int_0^{x_2} dx_1 e^{\beta\gamma(N-1-\alpha)x_1} \quad (4)$$

where $\gamma = Gm^2$, $mg = \alpha\gamma$, and L is the 'volume' of the system.

In evaluating the above multiple integrals, it is useful to define the following auxiliary variables:

$$I = f_{1-N}(L) = \int_0^L dx_N e^{\beta(\gamma(1-N-\alpha)x_N} f_{-(N-3)}(x_N)$$

$$f_{3-N)}(x_N) = \int_0^{x_N} dx_{N-1} e^{\beta(\gamma(3-N-\alpha)x_{N-1}} f_{-(N-5)}(x_{N-1})$$

$$\vdots$$

$$f_{N-1}(x_2) = \int_0^{x_2} dx_1 e^{\beta(\gamma(N-1-\alpha)x_1} f_{(N+1)}(x_1)$$

$$f_{N+1}(x_1) = 1 \quad (5)$$

Differentiating each of the above espressions (5), taking their corresponding Laplace transforms, and dividing both sides of the resulting equations by s gives

$$g_{1-N}(s) = \frac{1}{s} g_{3-N}(s + \beta\gamma(N-1+\alpha))$$

$$g_{3-N}(s) = \frac{1}{s} g_{5-N}(s + \beta\gamma(N-3+\alpha))$$

$$\vdots$$

$$g_{N-1}(s) = \frac{1}{s} g_{N+1}(s + \beta\gamma(N-1+\alpha))$$

$$g_{N+1}(s) = \frac{1}{s} \quad (6)$$

Note that the g_j's in (6) are the Laplace transforms of the f_j's in (5). By evaluating the recursion relations in (6) sequentially, and simplifying the sums of arithmetic progressions that arise in the process, we obtain the following expression for $g_{1\text{-}N}(s)$:

$$g_{1\text{-}N}(s) = \prod_{j=0}^{N}\left[\frac{1}{s+\beta\gamma j(N-j-\alpha)}\right] \tag{7}$$

The partition function can now be written as

$$Z = \left(\frac{2\pi m}{\beta h^2}\right)^{N/2} \Lambda^{-1}\left[\prod_{j=0}^{N}\left[\frac{1}{s+\beta\gamma j(N-j-\alpha)}\right]\right] \tag{8}$$

where Λ^{-1} is the inverse Laplace transform, which can be evaluated using the Bromwich integral formula: $\Lambda^{-1}(g(s)) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} ds\, g(s) e^{sL}$, where a is chosen so that all poles of $g(s)$ in the complex plane are to the left of the line $Re(z) = a$.

For all irrational values of α and almost all rational values, the expression for the partition function can be written more explicitly as:

$$Z = \left(\frac{2\pi m}{h^2}\right)^{N/2} \beta^{-\frac{5N}{2}} \gamma^{-2N} \sum_{j=0}^{N} e^{-\beta\gamma L_j(N-j-\alpha)} h(j;N,\alpha) \tag{9}$$

where $h\,(j, N, \alpha)$ is

$$\prod_{k\neq j}^{k=0...N} \frac{1}{(j-k)(J+k-N+\alpha)} \tag{10}$$

This expression for the partition function will have to be modified for some rational values of α to account for the residues of second or higher order poles that result when ordinarily distinct poles coincide. We see no reason to expect qualitative changes in the thermodynamics of the classical 1DGG under the influence of an external field for these special rational values of α (as it is always possible to approximate a rational number with a sequence of irrational numbers and we do not expect any "quantization effects" in our system).

THERMODYNAMIC QUANTITIES AND RELATIONS

By using the final form of the partition function (9), we obtain the following:

Equation of state: The pressure p, 'volume' L, and inverse temperature β of the system are related as follows

$$p = \frac{1}{\beta Z}\frac{\partial Z}{\partial L} = -\gamma\,\frac{\Theta}{\Phi} \tag{11}$$

where

$$\Theta = \sum_{j=0}^{N} j(N-j-\alpha)h(j;N,\alpha)e^{-\beta\gamma Lj[N-j-\alpha]} \tag{12}$$

and

$$\Phi = \sum_{j=0}^{N} h(j;N,\alpha)e^{-\beta\gamma Lj[N-j-\alpha]} \tag{13}$$

Mean internal energy:

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \frac{3N}{2\beta} + Lp = \frac{2}{3}NkT - \gamma\, L\frac{\Theta}{\Phi} \tag{14}$$

Constant 'volume' heat capacity:

$$C_v = \left.\frac{\partial \bar{E}}{\partial T}\right|_L$$

$$= \frac{3Nk}{2} - \gamma L\left[\frac{\sum_{j=1}^{N} h(j;N,\alpha)e^{-\beta\gamma Lj[N-j-\alpha]}\left(\frac{\partial\Theta}{\partial T} - j(n-j-\alpha)\frac{\partial\Phi}{\partial T}\right)}{\Phi^2}\right] \tag{15}$$

CONCLUSION

In this paper, we presented exact analytical results on the statistical thermodynamics of the one-dimensional gravitational gas in a uniform external field. This is an addition to the small roster of classical gas models with exactly solvable statistical thermodynamics.

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