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Fixed-point Theorem and the Nishida-Nirenberg Method in Solving Certain Nonlinear Singular Partial Differential Equations

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ABSTRACT

In their 2012 work, Lope, Roque, and Tahara considered singular nonlinear partial differential equations of the form $tu_t = F(t,x,u,u_x)$, where the function F is assumed to be continuous in t and holomorphic in the other variables. They have shown that under some growth conditions on the coefficients of the partial Taylor expansion of F as $t \to 0$, the equation has a unique solution $u(t,x)$ with the same growth order as that of $F(t,x,0,0)$. Koike considered systems of partial differential equations using the Banach fixed point theorem and the iterative method of Nishida and Nirenberg (1995). In this paper, we prove the result obtained by Lope and others using the method of Koike, thereby avoiding the repetitive step of differentiating a recursive equation with respect to x as was done by the aforementioned authors.

Keywords: Singular partial differential equations, nonlinear, fixed point

INTRODUCTION

In 1856, Briot and Bouquet established well-known results on the ordinary differential equation

$$t\frac{du}{dt} = f(t,u), \quad f(0,0) = 0,$$

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from which Gérard and Tahara (1996a, 1996b) modeled the nonlinear singular partial differential equation

$$t\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x}), \qquad (1)$$

where $\partial u/\partial x = (\partial u/\partial x_1,\ldots,\partial u/\partial x_n)$. The function $F(t,x,u,v)$ is a function on $\Omega = [0,T]\times D_R \times B_\rho \times D_\rho$, with $B_r = \{x\in \mathrm{C} : |x|\le r\}$ and $D_r = \{x\in \mathrm{C}^n : |x|\le r\}$, where $|x| = \max_{1\le i\le n}|x_i|$ and C is the set of complex numbers. Assuming that $F(t,x,u,v)$ is holomorphic with respect to all the variables, Gérard and Tahara have proven the unique existence of a holomorphic as well as a type of singular solution to (1). They also provided extensions of their results to higher order nonlinear singular equations.

Lope and others (2012), considering (1) with the weaker assumption that F is only continuous in t and holomorphic in the other variables, showed that under some growth conditions on the coefficients of the partial Taylor expansion of F, the equation has a unique solution $u(t,x)$ with the same growth order as that of $F(t,x,0,0)$. In their proof, they needed a modified version of Nagumo's lemma and had to differentiate recursive equations. In this paper, we wish to give an alternative proof, based on Koike's method (1995) and direct application of Nagumo's lemma, thereby avoiding the repetitive differentiation.

Remark 1.1 While preparing this paper, the authors were also working on extending the result of Lope and others (2012) to the class of ultradifferentiable maps – a class smaller than the class of continuous maps but strictly larger than the class of analytic maps. They had seen that the method presented in this paper is working more readily than the method used by Lope and others.

THE MAIN RESULT AND PRELIMINARIES

Let $T > 0$, $0 < R < 1$, $\rho > 0$, and $r > 0$. We will consider the same problem considered by Lope and others that is, singular nonlinear partial differential equations of the form (1), where the function $F(t,x,u,v)$ is assumed to be continuous on Ω and is holomorphic in the variables (x,u,v) for any fixed t.

We first define what is, as named by Tahara, a *weight function* (1998). We say that $\mu(t)$ is a *weight function* on $[0, T]$ if it is a continuous, nonnegative, increasing function on $(0, T)$ such that $\mu(t)/t$ is integrable on $(0, T)$. Note that such a function must satisfy $\lim_{t\to 0}\mu(t) = 0$. Examples of weight functions are t^δ, $1/(-\log t)^{1+\delta}$ and $1/(-\log t)(\log(-\log t))^\delta$ for any positive δ.

For a weight function $\mu(t)$, the function $\varphi(t) = \int_0^t \mu(\tau)d\tau/\tau$ is well-defined on $[0, T]$. For any $r > 0$, the region W_r is defined as

$$W_r = \{(t,x); 0 \le t \le T \text{ and } \varphi(t)/r + |x| < R\}. \tag{2}$$

We note that W_r also depends on T, but we will not explicitly indicate this in our notation for the sake of simplicity. We define two spaces on W_r: the space $X_0(W_r)$, composed of all functions in $C^0(W_r)$ that are holomorphic in x for any fixed t, and the space $X_1(W_r)$, composed of all functions in $C^1(W_r \cap \{t > 0\}) \cap X_0(W_r)$. Observe that if $r_1 < r_2$, then $W_{r_1} \subset W_{r_2}$ and $X_j(W_{r_2}) \subset X_j(W_{r_1})$ for $j = 0,1$.

We will now give the assumed conditions on (1) as given in Lope and others (2012). Let $\mu(t)$ be any weight function and $\alpha \in [0,1]$. Set $a(t,x) = F(t,x,0,0)$ and $\lambda(t,x) = F_u(t,x,0,0)$. We work on (1) under the following assumptions:

(A1) $a(t,x)$ and $a_{x_i}(t,x)$ for $1 \le i \le n$ are both bounded by $A\mu(t)^\alpha$ on $[0, T] \times D_R$;

(A2) $F_{v_i}(t,x,0,0) = O(\mu(t))$ (as $t \to 0$) for $1 \le i \le n$ uniformly on D_R;

(A3) Re $\lambda(t,x) \le -c$ on $[0, T] \times D_R$ for some $c > 0$;

(A4) For all $1 \le i, j \le n$, $F_{uv_i}(t,x,u,v)$ and $F_{v_i v_j}(t,x,u,v)$ are of order $O(\mu(t)^{1-\alpha})$ (as $t \to 0$) uniformly on $D_R \times B_\rho \times D_\rho$.

We now restate the main theorem as stated in Lope and others (2012). We use the notation $|\partial g/\partial x|$ for $\max_{1 \le i \le n} |\partial g/\partial x_i|$.

Theorem 2.1 (Main Theorem) *Suppose* $(A_1) - (A_4)$ *hold. If* $\alpha \in (0,1]$ *and* T *is sufficiently small, or if* $\alpha = 0$ *and both* T *and* A *are sufficiently small, then there exists an* $r > 0$ *such that (1) has a unique solution* $u(t,x) \in X_1(W_r)$ *that satisfies*

$$|u(t,x)| \le \frac{4A}{c}\mu(t)^\alpha \quad \text{and} \quad \left|\frac{\partial u}{\partial x}(t,x)\right| \le \rho\left(\frac{\mu(t)}{\mu(T)}\right)^\alpha \quad \text{on } W_r. \tag{3}$$

Using the partial Taylor expansion of $F(t,x,u,v)$, we rewrite (1) as

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right)u = a(t,x) + \Phi[u] + f(t,x,u), \tag{4}$$

where $\Phi[u]$ is defined by

$$\Phi[u] = \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + G\left(t,x,u,\frac{\partial u}{\partial x}\right), \tag{5}$$

$f(t,x,u) = \sum_{m=2}^{\infty}(\partial^m F/\partial u^m)(t,x,0,0)\, u^m/m!$ is holomorphic in (x,u) and $G(t,x,u,v)$ is the sum of the remaining terms in the partial Taylor expansion of $F(t,x,u,v)$ whose degree with respect to (u,v) is at least 2. Note that $G(t,x,u,0) = 0$, since each term of G has at least one $\partial u/\partial x_i$ as a factor.

From assumptions (A_1) to (A_4) and the holomorphy of $F(t,x,u,v)$ with respect to (x,u,v), it follows that there exist positive constants B, Λ, and $B_{1,1}$ and $B_{0,2}$, such that the following estimates hold:

- $\max_{1\le i\le n}\{|b_i(t,x)|\} \le B\mu(t)$ on $[0,T]\times D_R$;
- $|\frac{\partial \lambda}{\partial x}(t,x)| \le \Lambda$ on $[0,T]\times D_R$;
- $|\frac{\partial^2 F}{\partial u \partial v_i}(t,x,0,0)| \le B_{1,1}\mu(t)^{1-\alpha}$ and $|\frac{\partial^2 F}{\partial v_i \partial v_j}(t,x,0,0)| \le B_{0,2}\mu(t)^{1-\alpha}$

  for $1 \le i, j \le n$ on Ω.

Remark 2.2 Let $u \in X_0(W_r)$ such that $|u(t,x)| \le C\mu(t)^\alpha$ for some constant $C > 0$, a weight function $\mu(t)$ and $\alpha \in [0,1]$. We observe that, $f = u^2\tilde{f}$ and $f_u = u\bar{f}$, where $\tilde{f}$ and $\bar{f}$ are also holomorphic in (x,u) and that $|\tilde{f}| \le |\bar{f}|$ on W_r. Let K_1 and K_2 be the bound for $\bar{f}$ and $\tilde{f}_{x_i}$ ($i = 1,\dots,n$) respectively, on the closed domain $\omega = \{(t,x,w)\in \mathbb{R}\times\mathbb{C}^n\times\mathbb{C};\, 0\le t\le T, |x|\le R, |w|\le \rho\}$. We have the following estimates on W_r:

$$|f(t,x,u(t,x))| \le K_1(C\mu(t)^\alpha)^2 \tag{6}$$

$$|f_u(t,x,u)| \le K_1 C\mu(t)^\alpha \tag{7}$$

$$|f_{x_i}(t,x,u)| \le K_2(C\mu(t)^\alpha)^2, \tag{8}$$

where the derivative on the third inequality is only on the space variable x_i and none on $u(t,x)$.

We next present two important lemmas that appeared in Lope and others but are also important in our proof. The first lemma states some elementary results on linear Fuchsian equations while the second is called Nagumo's lemma. For the sake of completeness, we will reproduce the proof of the first as found in Lope and others (2012). The proof of the second can be seen in Hörmander (1963) (Lemma 5.1.3).

Lemma 2.3 *Suppose* (A_3) *holds. For any* $g(t,x)\in X_0(W_r)$, *the equation*

$$(t\frac{\partial}{\partial t}-\lambda(t,x))\,w=g(t,x) \tag{9}$$

has a unique solution $w(t,x)\in X_1(W_r)$, and it is given by

$$w(t,x)=\int_0^t \exp\,(\int_\tau^t\lambda(s,x)\frac{ds}{s})\,g(\tau,x)\frac{d\tau}{\tau}. \tag{10}$$

Moreover, the following estimates hold on W_r given any nondecreasing, nonnegative function $\psi(t)$:

a. If $|g(t,x)|\le M\psi(t)$, then $|w(t,x)|\le \dfrac{M}{c}\psi(t)$.

b. If $|g(t,x)|\le \dfrac{M\mu(t)\psi(t)}{(R-|x|-\varphi(t)/r)^2}$, then $|w(t,x)|\le \dfrac{Mr\,\psi(t)}{R-|x|-\varphi(t)/r}$.

c. If $|g(t,x)|\le \dfrac{M\mu(t)\psi(t)(R-|x|)}{(R-|x|-\varphi(t)/r)^2}$, then $|w(t,x)|\le \dfrac{M\varphi(t)\psi(t)}{R-|x|-\varphi(t)/r}$.

Proof. We note first that by (A_3),

$$\left|\exp\,(\int_\tau^t\lambda(s,x)\frac{ds}{s})\right|\le\left(\frac{\tau}{t}\right)^c.$$

The integral representation (10) of the solution is easily verified, and the estimate in (a) follows from it. To prove (b), we use the fact that $\varphi'(t)=\mu(t)/t$ and estimate as follows:

$$\begin{aligned}|w(t,x)| &\le \int_0^t(\frac{\tau}{t})^c\,\frac{M\mu(\tau)\psi(\tau)}{(R-|x|-\varphi(\tau)/r)^2}\frac{d\tau}{\tau}\\ &\le M\psi(t)\int_0^t\frac{\varphi'(\tau)}{(R-|x|-\varphi(\tau)/r)^2}d\tau\\ &= Mr\,\psi(t)(\frac{1}{R-|x|-\varphi(t)/r}-\frac{1}{R-|x|})\\ &\le \frac{Mr\,\psi(t)}{R-|x|-\varphi(t)/r},\end{aligned}$$

after simply ignoring the nonnegative subtrahend. As for (c), we estimate as in (b) but instead of dropping the subtrahend, we make use of the presence of $(R-|x|)$ to cancel the unwanted term in the denominator. ■

Lemma 2.4 *Let* $f(x)$ *be a holomorphic function on* B_R. *If*

$$|f(x)|<=\frac{C}{(R-|x|)^a}\ on\ B_R$$

for some $C \geq 0$ and $a \geq 0$, then

$$\left|\frac{\partial f}{\partial x}(x)\right| \leq \frac{\gamma_a C}{(R-|x|)^{a+1}} \text{ on } B_R,$$

where $\gamma_0 = 1$ and $\gamma_a = (1+a)(1+1/a)^a$ for $a > 0$.

Given $u, w \in X_0(W_r)$, we define

$$\Phi[u,w](t,x) = \sum_{i=1}^{n} b_i(t,x)\frac{\partial w}{\partial x_i} + G(t,x,u,\frac{\partial w}{\partial x}).$$

We used the same Greek letter as in (5) but this should not cause any ambiguity. Using this notation, we see that $\Phi[u,u]$ is just the quantity $\Phi[u]$ defined in (5). In view of (B_1) and (B_3), we have the following lemma that gives the estimate for the modulus of the difference $\Phi[u_1,w_1]-\Phi[u_2,w_2]$. This lemma and Lemma 2.1 in Lope and others are analogous (2012).

Lemma 2.5 *Let* u_1, w_1, u_2, w_2 *be in* $X_0(W_r)$. *Suppose that for some constant* $C > 0$, *a weight function* $\mu(t)$ *and* $\alpha \in [0,1]$, $|u_1(t,x)|$, $\left|\frac{\partial w_1}{\partial x}(t,x)\right|$, $|u_2(t,x)|$ *and* $\left|\frac{\partial w_2}{\partial x}(t,x)\right|$ *are all bounded by* $C\mu(t)^\alpha$ *on* W_r. *Then on* W_r, *we have*

$$|\Phi[u_1,w_1]-\Phi[u_2,w_2]| \leq \sum_{i=1}^{n} B\mu(t)\left|\frac{\partial w_1}{\partial x_i}-\frac{\partial w_2}{\partial x_i}\right| + nCB_{1,1}\mu(t)|u_1-u_2|$$
$$+\sum_{i=1}^{n}(B_{1,1}+nB_{0,2})C\mu(t)\left|\frac{\partial w_1}{\partial x_i}-\frac{\partial w_2}{\partial x_i}\right|.$$

PROOF OF MAIN THEOREM

3.1 Existence of a solution

We use the Banach fixed point theorem or the contraction mapping principle as in Walter (1985) and follow the construction of a complete metric space by Bacani and Tahara (2012). Equation (4) is equivalent, by Lemma 2.3, to

$$u(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[a(\tau,x)+\Phi[u](\tau,x)+f(\tau,x,u)]\frac{d\tau}{\tau}. \quad (11)$$

If the operator $\psi[\cdot,w]$, for a fixed w, defined by

$$\psi[u,w](t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[a(\tau,x)+\Phi[u,w](\tau,x)+f(\tau,x,u)]\frac{d\tau}{\tau}$$

is a contraction mapping from a suitable function space E (which must be a complete metric space) into itself, then we have a unique $u \in E$ such that $u = \psi[u, w]$, that is, a unique fixed point. We denote this u by $S[w]$. Therefore, we have our solution to (4) if we can find a $u \in E$ such that $u = S[u]$.

Let us now define the function space E. For $r > 0$ and $T > 0$, we denote by $B(W_r)$ the set of all functions $u \in X_0(W_r)$ such that for some $C > 0$, $|u(t,x)| \leq C\mu(t)^\alpha$ on W_r. We define a norm $\|u\|_{B,r}$ of $u \in B(W_r)$ by

$$\| u \|_{B,r} = \sup_{(t,x) \in W_r, t>0} \frac{|u(t,x)|}{\mu(t)^\alpha}.$$

We note that this norm is simpler than the one defined in Bacani and Tahara (2012). It is easy to see that $(B(W_r), \| \cdot \|_{B,r})$ is a Banach space.

In the following, we will also use the notation $\| u(x) \|_t$ for $\sup_{0<\tau\leq t} |u(\tau, x)|$.

Remark 3.1 The following estimate will come handy in the proof of the next proposition. For $0 < t \leq T$, we have

$$\sup_{(t,x)\in W_r, t>0} \frac{\| u(x) \|_t}{\mu(t)^\alpha} \leq \sup_{(t,x)\in W_r, t>0} \left(\sup_{0<\tau\leq t} \frac{|u(\tau,x)|}{\mu(\tau)^\alpha} \right) = \sup_{(t,x)\in W_r, t>0} \frac{|u(t,x)|}{\mu(t)^\alpha}.$$

For $M > 0$, we set $B_M(W_r) = \{u \in B(W_r) : \| u \|_{B,r} \leq M\}$. This is a closed subset of $B(W_r)$ and so it is a complete metric space. $B_M(W_r)$ is our function space E, for a chosen M. We note here that it is necessary, for our operator $\psi[\cdot, w]$ to be defined, that w and its partial derivatives with respect to x_i be also in E.

In the following, we will choose M to be of the form $\frac{8A}{c}\left(1 + \frac{2\Lambda}{c}\right)$. Note that M depends on A. This choice of M will be clear in Proposition 3.3. We also define

$$C_1 = nB + 2nMB_{1,1} + n^2MB_{0,2}.$$

Proposition 3.2 *Let $r>0$. Then, there exist $M, T>0$ such that for every fixed $w \in B_M(W_r)$, such that $\partial w/\partial x_i$ $(i = 1, \dots, n)$ is also in $B_M(W_r)$, the following are true:*

(a) $\psi[\cdot, w]$ is a mapping from $B_M(W_r)$ to itself.
(b) $\psi[\cdot, w]$ is a contraction map.

Proof. We choose M and T so that the following inequalities hold:

$$C_1 \frac{8}{c}\left(1+\frac{2\Lambda}{c}\right)\mu(T)+K_1 M\,\frac{8}{c}\left(1+\frac{2\Lambda}{c}\right)\mu(T)^\alpha \le 3, \tag{12}$$

$$\frac{nB_{1,1}M\mu(T)+K_1 M\mu(T)^\alpha}{c}<1. \tag{13}$$

We can satisfy the two inequalities above by choosing a sufficiently small T if $\alpha \neq 0$. But if $\alpha = 0$, we need to choose T and also a sufficiently small A (and hence, a small M).

Let $u_j, w, \partial w/\partial x_i$ ($j = 1,2$ and $i = 1,\ldots, n$) be in $B_M(W_r)$. Then, $|u_j(t,x)|$, $|w(t,x)|$, $|\partial w(t,x)/\partial x_i|$ are all bounded by $M\mu(t)^\alpha$ on W_r. From the definition of ψ, we have for any $(t,x)\in W_r$:

$$|\psi[u_1,w](t,x)| \le \int_0^t \left(\frac{\tau}{t}\right)^c |a(\tau,x)+\Phi[u_1,w]+f(\tau,x,u_1)|\frac{d\tau}{\tau}.$$

By (A_1), Lemma 2.5 (noting that $\Phi[0,0]=0$), and the estimate in (6), we see that

$$|a(\tau,x)+\Phi[u_1,w]+f(\tau,x,u_1)| \le A\mu(\tau)^\alpha + \left(C_1 M\mu(\tau)+K_1 M^2\mu(\tau)^\alpha\right)\mu(\tau)^\alpha.$$

Thus, by (12) we obtain

$$|\psi[u_1,w](t,x)| \le \int_0^t \left(\frac{\tau}{t}\right)^c 4A\mu(\tau)^\alpha \frac{d\tau}{\tau} \le \frac{4A\mu(t)^\alpha}{c}, \tag{14}$$

which is the same as the estimate in the main theorem. It follows that $\|\psi[u_1,w]\|_{B,r} \le \dfrac{4A}{c} < M$, proving (a).

Let us now prove that ψ is a contraction. In order to estimate the difference $\psi[u_1,w]-\psi[u_2,w]$, we first note that

$$f(\tau,x,u_1)-f(\tau,x,u_2)=(u_1-u_2)\int_0^1 \frac{\partial f}{\partial u}(\tau,x,u_2+s(u_1-u_2))ds. \tag{15}$$

Thus with Lemma 2.5 and (7), we have

$$|\Phi[u_1,w]-\Phi[u_2,w]|+|f(u_1)-f(u_2)| \le \left(nB_{1,1}M\mu(\tau)+K_1 M\mu(\tau)^\alpha\right)|u_1-u_2|.$$

Therefore,

$$|(\psi[u_1,w]-\psi[u_2,w])(t,x)| \le \int_0^t \left(\frac{\tau}{t}\right)^c |((\Phi[u_1,w]-\Phi[u_2,w])(\tau,x)+f(\tau,x,u_1)-f(\tau,x,u_2))|\frac{d\tau}{\tau}$$

$$\le \int_0^t \left(\frac{\tau}{t}\right)^c (nB_{1,1}M\mu(\tau)+K_1 M\mu(\tau)^\alpha)|u_1-u_2|\frac{d\tau}{\tau}$$

$$\leq (nB_{1,1}M\mu(t)+K_1M\mu(t)^{\alpha})\,\|(u_1-u_2)(x)\|_r\cdot\int_0^t\left(\frac{\tau}{t}\right)^c\frac{d\tau}{\tau}$$
$$=\frac{\left(nB_{1,1}M\mu(t)+K_1M\mu(t)^{\alpha}\right)}{c}\|(u_1-u_2)(x)\|_r.$$

Dividing both sides by $\mu(t)^{\alpha}$ and then taking the supremum over W_r with $t>0$, considering also Remark 3.1, we have

$$\|\psi[u_1,w]-\psi[u_2,w]\|_{B,r}\leq\frac{\left(nB_{1,1}M\mu(T)+K_1M\mu(T)^{\alpha}\right)}{c}\|u_1-u_2\|_{B,r}.$$

In view of (13), $\psi[\cdot,w]$ is indeed a contraction. ■

We will now construct a sequence of approximate solutions to our equation. To aid us in the construction, we further want the following inequalities to hold:

$$M\mu(T)^{\alpha}\leq\rho, \tag{16}$$

$$\max\{K_1,K_2\}M\mu(T)^{\alpha}\leq\frac{c}{4}, \tag{17}$$

$$K:=1-\left(\frac{nMB_{1,1}\mu(T)}{c}+\frac{4AK_1\mu(T)^{\alpha}}{c^2}\right)>0. \tag{18}$$

Inequality (16) ensures us that any solution will be in the domain of definition of $F(t,x,u,v)$. The use of the other two inequalities will be seen later.

As in the preceding proposition, we can satisfy all the above inequalities by choosing a sufficiently small T if $\alpha\neq 0$. But, if $\alpha=0$, we need to choose T and also a sufficiently small A.

We now fix T and A satisfying inequalities (12), (13), and (16) – (18).

As remarked at the beginning of Section 3.1, given $w\in B_M(W_r)$ such that $\partial w/\partial x_i$ is also in $B_M(W_r)$, the map $\psi[\cdot,w]:B_M\to B_M$ has a unique fixed point u, which we denote by $S[w]$. To find a solution to (11), we wish to find an $r>0$ and construct $u\in B_M(W_r)$ such that $u=S[u]$. To this end, we define approximate solutions as follows:

$$u_0=0$$

and for $k\geq 1$,

$$u_k=S[u_{k-1}].$$

Our u would then be the $\lim_{k\to\infty}u_k$. To prove the convergence of this limit, we use the method of Nirenberg and Nishida. We thus need to define a decreasing sequence of positive numbers, $\{r_k\}_{k\geq 1}$, tending to a positive limit r_∞. We define the sequence as follows:

$$r_1 < \frac{1}{2C},$$

$$r_k = r_{k-1}(1 - (2Cr_1)^{k-1}),$$

where

$$C = \frac{4C_1}{K^2}\left(1 + \frac{2\Lambda}{c}\right). \tag{19}$$

By our choice of r_1, the series $\sum_{k\geq 1}(2Cr_1)^k$ is convergent, and so r_∞ is well-defined and positive. We will show that r_∞ satisfies the positive number required in the theorem, that is, there exists a unique $u \in B_M(W_{r_\infty})$ such that $u = S[u]$.

Clearly, $\frac{\partial u_0}{\partial x_i} = 0 \in B_M(W_{r_1})$, for all $i = 1, \ldots, n$. Hence, there exists a unique

$u_1 = S[u_0] = \psi[u_1, u_0] \in B_M(W_{r_1})$, and by the estimate in (14),

$$|u_1(t,x)| \leq \frac{4A}{c}\mu(t)^\alpha \text{ on } W_{r_1}. \tag{20}$$

Now,

$$u_1(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[a(\tau,x) + \Phi[u_1,u_0] + f(\tau,x,u_1)]\frac{d\tau}{\tau}$$

$$= \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[a(\tau,x) + f(\tau,x,u_1)]\frac{d\tau}{\tau}, \tag{21}$$

since $\Phi[u_1, 0] = 0$. Equation (21) can be expressed as the partial differential equation

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right)u_1 = a(t,x) + f(t,x,u_1). \tag{22}$$

Differentiating (22) with respect to x_i, we see that

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x) - \frac{\partial f}{\partial u}(t,x,u_1)\right)\frac{\partial u_1}{\partial x_i} = \frac{\partial a}{\partial x_i} + \frac{\partial \lambda}{\partial x_i}u_1 + \frac{\partial f}{\partial x_i}(t,x,u_1). \tag{23}$$

In view of (20) and (17), we see that $\mathrm{Re}(\lambda(t,x) + f_u(t,x,u_0)) \leq -c/2 < 0$, and that

$$\left|\frac{\partial f}{\partial x_i}(t,x,u_0)\right| \leq \frac{c}{4}\cdot\frac{4A}{c}\mu(t)^\alpha = A\mu(t)^\alpha.$$

Applying Lemma 2.3(a) to (23), we obtain, for all $i = 1,\ldots, n$,

$$\left|\frac{\partial u_1}{\partial x_i}(t,x)\right| \le \frac{A + 4A\Lambda/c + A}{c/2}\mu(t)^\alpha = \left(1+\frac{2\Lambda}{c}\right)\frac{4A}{c}\mu(t)^\alpha \quad \text{on } W_{r_1}. \tag{24}$$

We now proceed by induction. We note that by proving the following proposition we show that the sequence $\{u_k\}$ converges and thus the solution $u = \lim_{k\to\infty} u_k$ exists and satisfies the estimates in the Main Theorem because each u_k does. We also note that the last estimate in (c) is the reason for our form of M, and, that estimate with (16) implies the second estimate in the main theorem.

Proposition 3.3 *For $k \ge 2$, the following hold:*

(a) There exists a unique $u_k \in B_M(W_{r_{k-1}})$ such that $u_k = S[u_{k-1}]$ and

$$|u_k(t,x)| \le \frac{4A}{c}\mu(t)^\alpha.$$

(b) On $W_{r_{k-1}}$, we have

$$\{|(u_k - u_{k-1})(t,x)|, \left|\frac{\partial}{\partial x}(u_k - u_{k-1})(t,x)\right|\} \le \frac{4A}{c}\mu(t)^\alpha \frac{C^{k-1}\varphi(t) r_1^{k-2}}{R - |x| - \varphi(t)/r_{k-1}}.$$

(c) On W_{r_k},

$$\{\| u_k - u_{k-1} \|_{B,r_k}, \left\| \frac{\partial}{\partial x}(u_k - u_{k-1}) \right\|_{B,r_k}\} \le \frac{1}{2^{k-1}}\cdot\frac{4A}{c}$$

and thus

$$\left\|\frac{\partial u_k}{\partial x_i}\right\|_{B,r_k} \le \frac{8A}{c}\left(1+\frac{2\Lambda}{c}\right).$$

Proof. Since we have (24), there exists a unique $u_2 \in B_M(W_{r_1})$ such that $u_2 = S[u_1]$ and $|u_2(t,x)| \le \frac{4A}{c}\mu(t)^\alpha$. Note that,

$$(u_2 - u_1)(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[\Phi[u_2,u_1] + f(\tau,x,u_2) - f(\tau,x,u_1)]\frac{d\tau}{\tau}.$$

Hence, on W_{r_1},

$$|u_2 - u_1| \le \int_0^t \left(\frac{\tau}{t}\right)^c \left|\Phi[u_2,u_1] + (u_2-u_1)\int_0^1 \frac{\partial f}{\partial u}(\tau,x,u_1 + s(u_2-u_1))\,ds\right|\frac{d\tau}{\tau},$$

where $|u_1 + s(u_2 - u_1)| = |(1-s)u_1 + su_2| \le \frac{4A}{c}\mu(t)^\alpha$. By Lemma 2.5,

$$|\Phi[u_2,u_1]| = |\Phi[u_2,u_1] - \Phi[u_1,0]| \le C_1\mu(t)\left|\frac{\partial u_1}{\partial x_i}\right| + nMB_{1,1}\mu(t)|u_2 - u_1|.$$

Also,

$$|u_2 - u_1| \int_0^1 \left| \frac{\partial f}{\partial u}(t, x, u_1 + s(u_2 - u_1)) \right| ds \le |u_2 - u_1| K_1 \frac{4A}{c} \mu(t)^\alpha .$$

Thus, from (24) and by the definition of $\varphi(t)$,

$$\begin{aligned}|(u_2 - u_1)(t,x)| &\le \int_0^t \left(\frac{\tau}{t}\right)^c C_1 \frac{4A}{c}\left(1 + \frac{2\Lambda}{c}\right)\mu(\tau)^{1+\alpha}\frac{d\tau}{\tau} + \int_0^t \left(\frac{\tau}{t}\right)^c \left(nMB_{1,1}\mu(\tau) + \frac{4AK_1}{c}\mu(\tau)^\alpha\right)|u_2 - u_1| \frac{d\tau}{\tau} \\ &\le C_1 \frac{4A}{c}\left(1+\frac{2\Lambda}{c}\right)\mu(t)^\alpha \int_0^t \varphi'(\tau)d\tau + (nMB_{1,1}\mu(t) + \frac{4AK_1}{c}\mu(t)^\alpha) \| (u_2 - u_1)(x) \|_t \int_0^t \left(\frac{\tau}{t}\right)^c \frac{d\tau}{\tau} \\ &= C_1 \frac{4A}{c}\left(1 + \frac{2\Lambda}{c}\right)\mu(t)^\alpha \varphi(t) + (nMB_{1,1}\mu(t) + \frac{4AK_1}{c}\mu(t)^\alpha) \| (u_2 - u_1)(x) \|_t \frac{1}{c}.\end{aligned}$$

Hence,

$$\| (u_2 - u_1)(x) \|_t \le C_1 \frac{4A}{c}\left(1 + \frac{2\Lambda}{c}\right)\mu(t)^\alpha \varphi(t) + (nMB_{1,1}\mu(t) + \frac{4AK_1}{c}\mu(t)^\alpha) \| (u_2 - u_1)(x) \|_t \frac{1}{c}.$$

Therefore, using our defined constant K in (18), we have

$$\| (u_2 - u_1)(x) \|_t K \le C_1 \frac{4A}{c}\left(1 + \frac{2\Lambda}{c}\right)\mu(t)^\alpha \varphi(t).$$

In view of (19), and since $R - |x| - \dfrac{\varphi(t)}{r_1} < 1$, we have

$$|(u_2 - u_1)(t,x)| \le \| (u_2 - u_1)(x) \|_t \le \frac{4AC}{c}\mu(t)^\alpha \varphi(t) \tag{25}$$

$$\le \frac{\frac{4AC}{c}\mu(t)^\alpha \varphi(t)}{R - |x| - \frac{\varphi(t)}{r_1}}. \tag{26}$$

Applying Lemma 2.4 on (25), for all $i = 1, \dots, n$, we also have

$$\left| \frac{\partial (u_2 - u_1)}{\partial x_i}(t,x) \right| \le \frac{\frac{4A}{c}C\mu(t)^\alpha \varphi(t)}{R - |x| - \frac{\varphi(t)}{r_1}}. \tag{27}$$

Thus, with our definition of r_k, we have on W_{r_2},

$$\begin{aligned}\left\{ |(u_2 - u_1)(t,x)|, \left| \frac{\partial (u_2 - u_1)}{\partial x_i}(t,x) \right| \right\} &\le \frac{4AC}{c}\mu(t)^\alpha \frac{r_2(R - |x|)}{R - |x| - \frac{r_2(R-|x|)}{r_1}} \\ &= \frac{4AC}{c}\mu(t)^\alpha \frac{r_2}{2Cr_1} \\ &\le \frac{1}{2}\cdot\frac{4A}{c}\mu(t)^\alpha .\end{aligned}$$

Hence, by triangle inequality, for all $i = 1,\ldots, n$,

$$\left|\frac{\partial u_2}{\partial x_i}\right| \le \left|\frac{\partial (u_2 - u_1)}{\partial x_i}\right| + \left|\frac{\partial u_1}{\partial x_i}\right| \le \frac{3}{2}\cdot\frac{4A}{c}\left(1+\frac{2\Lambda}{c}\right)\mu(t)^{\alpha}.$$

We thus have proved $(a)-(c)$ for the case $k=2$.

Suppose now that $(a)-(c)$ hold for $k = j \ge 2$. By (c) of the induction hypothesis, for all $i=1,\ldots,n$, $\partial u_j/\partial x_i \in B_M(W_{r_j})$, and thus, there exists a unique $u_{j+1} \in B_M(W_{r_j})$ such that $u_{j+1} = S[u_j] = \psi[u_{j+1}, u_j]$ and $|u_{j+1}| \le \frac{4A}{c}\mu(t)^{\alpha}$ on W_{r_j}, proving (a) for $k = j+1$. Now, on W_{r_j}

$$(u_{j+1}-u_j)(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\frac{ds}{s}\right)[(\Phi[u_{j+1},u_j]-\Phi[u_j,u_{j-1}]+f(u_{j+1})-f(u_j))(\tau,x)]\frac{d\tau}{\tau},$$

where

$$|(\Phi[u_{j+1},u_j]-\Phi[u_j,u_{j-1}])(\tau,x)| \le C_1\mu(\tau)\left|\frac{\partial(u_j-u_{j-1})}{\partial x_i}\right| + nMB_{1,1}\mu(\tau)|u_{j+1}-u_j|$$

and

$$|(f(u_{j+1})-f(u_j))(\tau,x)| \le |u_{j+1}-u_j| \int_0^1 \left|\frac{\partial f}{\partial u}(\tau,x,u_j+s(u_{j+1}-u_j))\right| ds$$
$$\le \frac{4A}{c}K_1\mu(\tau)^{\alpha}\,|(u_{j+1}-u_j)(\tau,x)|.$$

Thus, with $r_j < r_{j-1}$ and the induction hypothesis (b),

$$|(u_{j+1}-u_j)(t,x)| \le \int_0^t \left(\frac{\tau}{t}\right)^c C_1\mu(\tau)\left|\frac{\partial(u_j-u_{j-1})}{\partial x_i}\right|\frac{d\tau}{\tau} + \int_0^t\left(\frac{\tau}{t}\right)^c\cdot\left\{nMB_{1,1}\mu(\tau)+\frac{4AK_1}{c}\mu(\tau)^{\alpha}\right\}|u_{j+1}-u_j|\frac{d\tau}{\tau}$$

$$\le \int_0^t C_1\mu(\tau)C^{j-1}\left(\frac{\frac{4A}{c}\mu(\tau)^{\alpha}\varphi(\tau)r_1^{j-2}}{R-|x|-\frac{\varphi(\tau)}{r_j}}\right)\frac{d\tau}{\tau} + \left(\frac{nMB_{1,1}\mu(t)}{c}+\frac{4AK_1}{c^2}\mu(t)^{\alpha}\right)\|(u_{j+1}-u_j)(x)\|_t$$

$$\le C_1\frac{4A}{c}C^{j-1}\mu(t)^{\alpha}\varphi(t)r_1^{j-2}\int_0^t\frac{\varphi'(\tau)d\tau}{(R-|x|-\varphi(\tau)/r_j)^2} + (1-K)\|(u_{j+1}-u_j)(x)\|_t$$

$$= \frac{C_1\frac{4A}{c}C^{j-1}\mu(t)^{\alpha}\varphi(t)r_1^{j-2}r_j}{R-|x|-\varphi(t)/r_j} + (1-K)\|(u_{j+1}-u_j)(x)\|_t.$$

Hence,

$$|(u_{j+1}-u_j)(t,x)| \le \| (u_{j+1}-u_j)(x) \|_t \le \frac{C_1 \frac{4A}{Kc} C^{j-1} \mu(t)^\alpha \varphi(t) r_1^{j-2} r_j}{R-|x|-\varphi(t)/r_j}$$

$$\le \frac{C^j \frac{4A}{c} \mu(t)^\alpha \varphi(t) r_1^{j-1}}{R-|x|-\varphi(t)/r_j}, \tag{28}$$

where we use (19) for the second inequality. The first integral in the preceding inequalities can also be estimated in the following way,

$$\int_0^t \left(\frac{\tau}{t}\right)^c C_1 \mu(\tau) \left| \frac{\partial (u_j - u_{j-1})}{\partial x_i} \right| \frac{d\tau}{\tau} \le \left(\frac{C_1 \mu(t) \frac{4A}{c} C^{j-1} \mu(t)^\alpha \varphi(t) r_1^{j-2}}{R-|x|-\frac{\varphi(t)}{r_j}} \right) \cdot \int_0^t \left(\frac{\tau}{t}\right)^c \frac{d\tau}{\tau}$$

$$\le \frac{C_1 \mu(t) \frac{4A}{c} C^{j-1} \mu(t)^\alpha \varphi(t) r_1^{j-2}}{R-|x|-\frac{\varphi(t)}{r_j}} \cdot \frac{1}{c}$$

Using this estimate instead in the preceding computation, we have another estimate for the modulus of $u_{j+1} - u_j$:

$$|(u_{j+1}-u_j)(t,x)| \le \| (u_{j+1}-u_j)(x) \|_t \le \frac{C_1 \frac{4A}{Kc^2} C^{j-1} \mu(t)^{1+\alpha} \varphi(t) r_1^{j-2}}{R-|x|-\frac{\varphi(t)}{r_j}}. \tag{29}$$

Now,

$$\frac{\partial (u_{j+1}-u_j)}{\partial x_i}(t,x) = \int_0^t \frac{\partial}{\partial x_i} \left\{ \exp\left(\int_\tau^t \lambda(s,x) \frac{ds}{s}\right) [(\Phi[u_{j+1},u_j] - \Phi[u_j,u_{j-1}] + f(u_{j+1}) - f(u_j))(\tau,x)] \right\} \frac{d\tau}{\tau}.$$

Using (29), we have the following estimate:

$$\left| \exp\left(\int_\tau^t \lambda(s,x) \frac{ds}{s}\right) [(\Phi[u_{j+1},u_j] - \Phi[u_j,u_{j-1}] + f(u_{j+1}) - f(u_j))(\tau,x)] \right|$$

$$\le \left(\frac{\tau}{t}\right)^c \left\{ \frac{C_1 \frac{4A}{c} C^{j-1} \mu(\tau)^{1+\alpha} \varphi(\tau) r_1^{j-2}}{R-|x|-\frac{\varphi(\tau)}{r_j}} + \frac{c(1-K) C_1 C^{j-1} \frac{4A}{c} \mu(\tau)^{1+\alpha} \varphi(\tau) r_1^{j-2} \frac{1}{Kc}}{R-|x|-\frac{\varphi(\tau)}{r_j}} \right\}$$

$$\le \frac{C_1 C^{j-1} \mu(\tau) \frac{4A}{c} \mu(\tau)^\alpha \varphi(\tau) r_1^{j-2}}{R-|x|-\frac{\varphi(\tau)}{r_j}} \left(\frac{1}{K}\right).$$

Thus, by applying Lemma 2.4 to estimate the modulus of the integrand in the preceding equality, we get

$$\left|\frac{\partial(u_{j+1}-u_j)}{\partial x_i}(t,x)\right| \le \int_0^t \frac{4C_1C^{j-1}\mu(\tau)\frac{4A}{Kc}\mu(\tau)^\alpha\varphi(\tau)r_1^{j-2}}{\left(R-|x|-\frac{\varphi(\tau)}{r_j}\right)^2}\frac{d\tau}{\tau}$$

$$\le C^j\frac{4A}{c}\mu(t)^\alpha\varphi(t)r_1^{j-2}\int_0^t\frac{\varphi'(\tau)d\tau}{\left(R-|x|-\frac{\varphi(\tau)}{r_j}\right)^2}$$

$$\le \frac{C^j\frac{4A}{c}\mu(t)^\alpha\varphi(t)r_1^{j-2}r_j}{R-|x|-\varphi(t)/r_j} \le \frac{C^j\frac{4A}{c}\mu(t)^\alpha\varphi(t)r_1^{j-1}}{R-|x|-\varphi(t)/r_j}. \tag{30}$$

In view of (28) and (30), we have proven (b) for $k=j+1$. To prove (c), we have on $W_{r_{j+1}}$,

$$\left\{|(u_{j+1}-u_j)(t,x)|,\left|\frac{\partial(u_{j+1}-u_j)}{\partial x_i}(t,x)\right|\right\} \le \frac{C^j\frac{4A}{c}\mu(t)^\alpha r_1^{j-1}r_{j+1}(R-|x|)}{R-|x|-\frac{r_{j+1}(R-|x|)}{r_j}}$$

$$\le C^j\frac{4A}{c}\mu(t)^\alpha\frac{r_1^j}{(2Cr_1)^j}$$

$$\le \frac{1}{2^j}\cdot\frac{4A}{c}\mu(t)^\alpha.$$

and by triangle inequality, for $i=1,\ldots,n$,

$$\left|\frac{\partial u_{j+1}}{\partial x_i}\right| \le \sum_{m=1}^{j}\left|\frac{\partial(u_{m+1}-u_m)}{\partial x_i}\right| + \left|\frac{\partial u_1}{\partial x_i}\right| \le \frac{8A}{c}\left(1+\frac{2\Lambda}{c}\right)\mu(t)^\alpha.$$

Since $u=\lim_{k\to\infty}u_k$ is of the form (11), we also have $u\in X_1(W_{r_\infty})$. We are left to show uniqueness.

3.2 Uniqueness of the solution

Since $Cr_\infty < Cr_1 < \frac{1}{2}$, the following proposition implies the uniqueness of the solution. We note also that, since $R-|x|-\frac{\varphi(t)}{r_\infty}<1$, the case $k=0$ in the proposition clearly follows from its hypothesis.

Proposition 3.4 *Suppose* u *and* v *are two solutions of (1) in* $X_1(W_{r_\infty})$ *such that* $\{|u|,|v|,|\partial u/\partial x|,|\partial v/\partial x|\} \leq M\mu(t)^\alpha$ *. Then, for* $k = 0,1,2,\ldots$

$$\left\{|(u-v)(t,x)|,\left|\frac{\partial(u-v)}{\partial x}(t,x)\right|\right\} \leq 2M\mu(t)^\alpha \cdot \frac{(Cr_\infty)^k}{R-|x|-\frac{\varphi(t)}{r_\infty}} \quad \text{on } W_{r_\infty}.$$

The proof is similar to that of Proposition 3.3 and thus we omit it here.

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