Science Diliman (July-December 2018) 30:2, 70-86

Fuzzy on Ideal Sets and a Fuzzy on Ideal Hahn-Banach Theorem

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ABSTRACT

In set theory, an ideal is a collection of sets that are considered to be small or negligible, such that every subset of an element of the ideal must also be in the ideal and the union of any two elements of the ideal must also be in the ideal. A fuzzy set is a class of objects with grades of membership in the interval [0, 1]. It is used to mathematically represent uncertainty and provide a formal tool to deal with imprecisions present in many problems. We use ideals to define fuzzy on ideal sets, which can be seen as a generalization of the fuzzy sets. We establish some of its basic properties, and we state and prove a Hahn-Banach Theorem with the fuzzy on ideal sets, which can be seen as a generalization of a fuzzy Hahn-Banach Theorem, which in turn, is a fuzzified generalization of an analytic form of the classical Hahn-Banach Theorem.

Mathematics Subject Classification (2010): 62B86

Keywords: Fuzzy, ideal, Hahn-Banach theorem

INTRODUCTION

The concept of ideal spaces was first studied by Kuratowski (1966) and Vaidyanathaswamy (1945). Formally, given a set X, an ideal $I(X)$ is a nonempty collection of subsets of X that satisfies:

i. $A \in I(X)$ and $B \subseteq A$ implies $B \in I(X)$; and

ii. $A \in I(X)$ and $B \in I(X)$ implies $A \cup B \in I(X)$.

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ISSN 0115-7809 Print / ISSN 2012-0818 Online

Ideal spaces were then imported to topology. For instance, Jankovic and Hamlett (1990) investigated the notion of topological spaces with ideals. Thereafter, ideal spaces found their way to other concepts in topology.

In 1965, fuzzy sets were introduced by Zadeh (1965) and Klaua (1965) as extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms: an element either belongs or does not belong to the set. That is, a set A on a universal set X can be identified as the characteristic function of A having only the values 0 or 1. On the other hand, a fuzzy set on X is formally defined as a mapping from X into the unit interval $[0, 1]$. Then in 1978, Zadeh introduced possibility theory as an extension of his theory of fuzzy sets and fuzzy logic (Zadeh 1978). Possibility theory should not be confused with probability theory. It is an uncertainty theory trying to make sense of incomplete information and is viewed as a complement to probability theory. Similar to a probability distribution, the theory uses a possibility distribution. A possibility distribution is a mapping π_x from a set of states to a totally ordered set such as the unit interval $[0, 1]$. As one may easily notice, a possibility distribution can be used as an interpretation of the fuzzy sets. We provide the example given by Zadeh (1978) to better understand a possibility distribution and differentiate it from a probability distribution.

Suppose we have the statement "Hans ate x eggs for breakfast," with x taking values in $U = \{1,2,3,\ldots\}$. We may associate a possibility distribution with x by interpreting $\pi_x(u)$ as the "degree of ease with which Hans can eat u eggs". We may also associate a probability distribution with x by interpreting $P_x(u)$ as the probability of "Hans eating u eggs for breakfast". The values of $\pi_x(u)$ and $P_x(u)$ may look like as shown in the following table.

u	1	2	3	4	5	6	7	8
$\pi_x(u)$	1	1	1	1	0.7	0.5	0.4	0.2
$P_x(u)$	0.2	0.7	0.1	0	0	0	0	0

One may easily notice that the sum of all values for $P_x(u)$ is equal to 1, but for $\pi_x(u)$, it is not. We may also observe that the possibility that Hans may eat 3 eggs for breakfast is 1 but the probability that he may do so is quite small. Hence, we can say that a high degree of possibility does not imply a high degree of probability, nor does a low degree of probability imply a low degree of possibility. However, if an event is impossible, then it should be improbable.

In this study, we want to capture statements like the following. Suppose we want to complete the statement "Tomorrow ____________," we may have the following choices:

a. will be sunny

b. there will be rain

c. will be cloudy

d. there will be a storm

We may give a value of 0.8 for the possibility that "tomorrow will be sunny", 0.6 for the possibility that "tomorrow there will be rain", 0.4 for the possibility that "tomorrow will be cloudy", and 0.2 for the possibility that "tomorrow there will be a storm". Now, consider the possibility that "tomorrow will be sunny and there will also be rain." This condition is very rare but not impossible. We cannot just give this possibility a value equal to the infimum of the possibility that "tomorrow will be sunny" and the possibility that "tomorrow there will be rain". We know that this possibility should be far less than any of these two possibilities. Hence, we may give it a separate possibility value not dependent on the other two mentioned possibilities, say, 0.01. With this situation in mind, we define a fuzzy on ideal set in the next section.

As an application of this new concept, we consider the Hahn-Banach Theorem. The theorem is no doubt an important and powerful result in functional analysis. It was generalized in many directions. One of it is by fuzzy sets, like what Rhie and Hwang (1999) did. In a similar way, we generalize the theorem with fuzzy on ideal sets.

FUZZY ON IDEAL SETS

We now formally define a fuzzy on ideal set.

Definition 2.1. Let X be a nonempty set $I(X)$ an ideal on X, and I be the unit interval $[0, 1]$. A function $\mu : I(X) \rightarrow I$ called a fuzzy on ideal set provided:

i. $\mu(\emptyset) = 0$ and

ii. for nonempty sets $A, B \in I(X)$, with $A \subseteq B$, we have $\mu(B) \leq \mu(A)$.

We denote by $I^{I(X)}$ the set of all such μ.

It is important to note here that the "reverse inequality", $\mu(B) \leq \mu(A)$, of Definition 2.1 encapsulates our preceding idea that the possibility that "tomorrow will be sunny and there will also be rain" should not just be equal to the infimum of the possibility that "tomorrow will be sunny" and the possibility that "tomorrow there will be rain" as it can be far less.

Remark 2.2. We may think of a fuzzy on ideal set as a generalization of a fuzzy set in X in the sense that the set of all fuzzy sets I^X can be embedded in $I^{P(X)}$, where $P(X)$ is the power set of X− the largest ideal on X. To see this, let $\mu : X \rightarrow I$ be a fuzzy set in X. We can identify μ with $\widetilde{\mu} \in I^{P(X)}$ defined by $\widetilde{\mu}(A) = \begin{cases} \mu(x), & \text{if } A=\{x\}, x \in X; \\ 0, & \text{otherwise.} \end{cases}$

Example 2.3. Let $X = \{a,b\}$ and $\mu = \{(a,1), (b,0.5)\}$ be on X. We may identify μ with the fuzzy on ideal set $\widetilde{\mu}$ defined as $\widetilde{\mu} = \{(\{a\},1), (\{b\},0.5), (\{a,b\},0), (\emptyset,0)\}$.

Example 2.4. Let X be a nonempty set and $\mu : X \rightarrow I$ be a fuzzy set. We can define a fuzzy on ideal set $\pi : P(X) \rightarrow I$ as $\pi(A) = inf_{x\in A}\, \mu(x)$ and $\pi(\emptyset) = 0$. This is called a guaranteed possibility in Dubois and Prade (2000).

Remark 2.5. It is important to note that Definition 2.1 does not define a measure. For $A \subseteq B$, a measure m should have $m(A) \leq m(B)$, not the reverse inequality, as in our definition.

Next, we define some relational operators between fuzzy on ideal sets.

Definition 2.6. Let X be a nonempty set and $\mu_1, \mu_2 \in I^{I(X)}$. We say that $\mu_1 \leq \mu_2$, $\mu_1 \geq \mu_2$ or $\mu_1 = \mu_2$, provided that, for every $A \in I(X)$, we have $\mu_1(A) \leq \mu_2(A)$, $\mu_1(A) \geq \mu_2(A)$, or $\mu_1(A) = \mu_2(A)$, respectively.

Definition 2.7. Let X be a nonempty set and $\mu \in I^{I(X)}$. The complement of μ denoted by $\mu^c : I(X) \rightarrow I$ is defined by $\mu^c(\emptyset) = 0$, and for $\emptyset \neq A \in I(X)$, $\mu^c(A) = \inf_{x\in A} \{1 - \mu(\{x\})\}$.

Remark 2.8. For $A = \{x\}$, the preceding definition coincides with the definition of the complement of a fuzzy set.

Definition 2.9. Let X be a nonempty set and $I(X)$ be an ideal on X. If $\{\mu_j \mid j \in J\}$ is a collection of fuzzy on ideal sets, then the union and the intersection of the μ_j's are given by:

i. $(\vee_{j\in J}\mu_j)(A) = \sup\{\mu_j(A) | j \in J\}$; and

ii. $(\wedge_{j\in J}\mu_j)(A) = \inf\{\mu_j(A) | j \in J\}$, respectively, for every $A \in I(X)$.

Next, we show that the complement, union, and intersection of fuzzy on ideal sets are also fuzzy on ideal sets.

Proposition 2.10. Let X be a nonempty set and $I(X)$ be an ideal on X. If $\{\mu_j \mid j \in J\}$ is a collection of fuzzy on ideal sets, then μ_j^c, $\vee_{j\in J}\mu_j$, and $\wedge_{j\in J}\mu_j$ are fuzzy on ideal sets.

Proof. Let $\varnothing \neq A, B \in I(X)$, such that $A \subseteq B$. Then, $\{\mu(\{x\}) : x\in A\} \subseteq \{\mu(\{x\}) : x \in B\}$, and so, $\{1-\mu(\{x\}) : x \in A\} \subseteq \{1-\mu(\{x\}) : x \in B\}$. Thus, for $j \in J$, $\mu_j^c(A) = \inf_{x\in A}\{1-\mu_j(\{x\})\} \geq \inf_{x\in B}\{1-\mu_j(\{x\})\} = \mu_j^c(B)$.

By Definition 2.7, $\mu_j^c(\varnothing) = 0$ for each $j\in J$, and hence, μ_j^c is a fuzzy on ideal set. Now, since $A \subseteq B$ and $\mu_j \in I^{I(X)}$, $\mu_j(A) \geq \mu_j(B)$ where $j\in J$. It follows that $\sup\{\mu_j(A) | j \in J\} \geq \sup\{\mu_j(B) | j \in J\}$, and so, $(\vee_{j\in J}\mu_j)(A) \geq (\vee_{j\in J}\mu_j)(B)$. Note that $(\vee_{j\in J}\mu_i)(\varnothing) = \sup\{\mu_j(\varnothing) \mid j \in J\} = 0$, and hence, $\vee_{j\in J}\mu_j$ is a fuzzy on ideal set. Similarly, we can show that $\wedge_{j\in J}\mu_j$ is a fuzzy on ideal set. ■

MAPPINGS

In this section, we show that, given only a mapping between nonempty sets (not a mapping between fuzzy on ideals sets), we can define the image and pre-image of fuzzy on ideal sets. The following are preparatory definitions and results.

Definition 3.1. Let X and Y be nonempty sets, and let $f: X \to Y$ be a mapping. If $I(X)$ and $I(Y)$ are ideals on X and Y, respectively, we define $f(I(X)) = \{f(A) : A \in I(X)\}$ and $f^{-1}(I(Y)) = \{A : A \subseteq f^{-1}(B),\ B \in I(Y)\}$, where $f(A)$ and $f^{-1}(B)$ are the usual image and pre-image of $A \subseteq X$ and $B \subseteq Y$, respectively.

Theorem 3.2. Let X and Y be nonempty sets and let $f: X \to Y$ be a mapping. If $I(X)$ and $I(Y)$ are ideals on X and Y, respectively, then $f(I(X))$ and $f^{-1}(I(Y))$ are ideals on Y and X, respectively.

Proof. Since $I(X)$ is an ideal on X, $I(X) \neq \varnothing$. It follows that $f(I(X)) \neq \varnothing$. Let $B_2 \in f(I(X))$. Then, there exists $A_2 \in I(X)$, such that $f(A_2) = B_2$. Let $B_1 \subseteq B_2$. Now, let $A_1 = A_2 \cap f^{-1}(B_1)$. Then, $A_1 \subseteq A_2$ and $f(A_1) = B_1$. Note that $A_1 \in I(X)$, since it is a subset of $A_2 \in I(X)$. Thus,

$B_1 \in f(I(X))$. Next, suppose that $B_1, B_2 \in f(I(X))$, then there exist $A_1, A_2 \in I(X)$, such that $f(A_1) = B_1$ and $f(A_2) = B_2$. Now, $A_1 \cup A_2 \in I(X)$, since $I(X)$ is an ideal. Thus, $B_1 \cup B_2 = f(A_1) \cup f(A_2) = f(A_1 \cup A_2) \in f(I(X))$. Therefore, $f(I(X))$ is an ideal on Y. Since $I(Y)$ is an ideal on Y, $I(Y) \neq \varnothing$. It follows that $f^{-1}(I(Y)) \neq \varnothing$. Let $A_2 \in f^{-1}(I(Y))$. Then, there exists $B_2 \in I(Y)$ such that $A_2 \subseteq f^{-1}(B_2)$. Let $A_1 \subseteq A_2$. Then, we also have $A_1 \subseteq f^{-1}(B_2)$, and thus, $A_1 \in f^{-1}(I(Y))$. Next, suppose that $A_1, A_2 \in f^{-1}(I(Y))$, then there exist $B_1, B_2 \in I(Y)$, such that $A_1 \subseteq f^{-1}(B_1)$ and $A_1 \subseteq f^{-1}(B_2)$. Since $B_1 \cup B_2 \in I(Y)$, $I(Y)$ being an ideal on Y, we have $A_1 \cup A_2 \subseteq f^{-1}(B_1 \cup B_2)$, then $A_1 \cup A_2 \subseteq f^{-1}(I(Y))$. Thus, $f^{-1}(I(Y))$ is an ideal on X. ■

The next definition defines the image and pre-image of fuzzy on ideal sets out of an ordinary mapping, and the next proposition proves that it is well-defined.

Definition 3.3. Let X and Y be nonempty sets, and let $f: X \to Y$ be a mapping. Let $\mu \in I^{I(X)}$ and $\sigma \in I^{I(Y)}$ for some ideals $I(X)$ and $I(Y)$ of X and Y, respectively. Define the image of μ, denoted by $f[\mu]$, and the pre-image of σ, denoted by $f^{-1}[\sigma]$, as follows:

i. $f[\mu]: f(I(X)) \to I$, where for $B \in f(I(X))$, $f[\mu](B) = \sup_{A \in S_B} \mu(A)$, where $S_B = \{A \in I(X)) : f(A) = B\}$; and

ii. $f^{-1}[\sigma]: f^{-1}(I(Y)) \to I$, where for $A \in f^{-1}(I(Y))$, $f^{-1}[\sigma](A) = (\sigma \circ f)(A)$.

Proposition 3.4. Let X and Y be nonempty sets, and let $f: X \to Y$ be a mapping. Let $\mu \in I^{I(X)}$ and $\sigma \in I^{I(Y)}$. Then, $f[\mu]$ and $f^{-1}[\sigma]$ are fuzzy on ideal sets on Y and X, respectively.

Proof. We first note that, by Theorem 3.2, $f(I(X))$ and $f^{-1}(I(Y))$ are ideals on Y and X respectively. If $B \neq \varnothing$, then $S_B = \{\varnothing\}$, and so, $f[\mu]\varnothing = \sup_{A \in S_B} \mu(A) = 0$. Also, for $\varnothing = A \in f^{-1}(I(Y))$, we have $f(A) = \varnothing$, and so, $f^{-1}[\sigma]:f^{-1}(\varnothing) = \sigma(f(\mathrm{A})) = 0$. Now, let $\varnothing \neq B_1, B_2 \in f(I(X))$, such that $B_1 \subseteq B_2$. Then, for $A_1, A_2 \in I(X)$, such that $f(A_1) = B_1$, $f(A_2) = B_2$, and $A_1 \subseteq A_2$, we have $\mu(A_2) \leq \mu(A_1)$, since μ is a fuzzy on ideal set. Hence, $\sup_{A \in S_{B_2}} \mu(A) \leq \sup_{A \in S_{B_1}} \mu(A)$, and so, $f[\mu](B_2) \leq f[\mu](B_1)$. Now, for $\varnothing \neq A_1, A_2 \in f^{-1}(I(Y))$, such that $A_1 \subseteq A_2$, we have $f(A_1) \subseteq (A_2)$. Hence, $\sigma f(A_2)) \leq \sigma\ f(A_1))$, since σ is a fuzzy on ideal set. Consequently, $f^{-1}[\sigma](A_2) \leq f^{-1}[\sigma](A_1)$. Therefore, $f[\mu]$ and $f^{-1}[\sigma]$ are fuzzy on ideal sets on Y and X, respectively. ■

FUZZY ON IDEAL HAHN-BANACH THEOREM

Rhie and Hwang (1999) fuzzified the analytic form of the Hahn-Banach Theorem. They built the idea from the works of Katsaras (1981, 1984) and Katsaras and Liu (1977) on fuzzy vector spaces and fuzzy seminorm, and the work of Krishna and Sarma (1991) on the generation of the fuzzy vector topology from an ordinary vector topology. We state and prove the analytic form of the Hahn-Banach theorem in the fuzzy on ideal setting. This can be seen as a generalization of the fuzzy Hanh-Banach Theorem by Rhie and Hwang (1999). We follow the ideas and the flow of proof by Rhie and Hwang (1999).

We recall first that for a vector space X over $\mathbb{R}$ and $A, B \subseteq X$, we have $A+B=\{a+b : a\in A$ and $b\in B\}$ and for $t \in \mathbb{R}$, $tA = \{ta : a\in A\}$.

Let $I(X)$ be an ideal on X. We define an associated set X_0 by $X_0 = \{a : \{a\} \in I(X)\}$.

That is, X_0 is the set out of the singleton subsets of $I(X)$. Note that $X_0 \subseteq X$. For $\varnothing \neq A\in I(X)$, by the first property of an ideal, all singleton subsets of A are also in $I(X)$, and so, $A \subseteq X_0$. We want next that $I(X)$ be closed under finite addition and scalar multiplication. The next proposition shows that it is enough to assume that X_0 is a linear subspace of X, such that $X_0\in I(X)$.

Proposition 4.1. Let X be a vector space over $\mathbb{R}$, and let X_0 be a linear subspace of X. If $I(X)$ is an ideal of X, such that $X_0\in I(X)$, then for every $A, B \subseteq X_0$ and every $t \in \mathbb{R}$, we have $tA, A+B\in I(X)$.

Proof. Let $A, B \subseteq X_0$ and $t \in \mathbb{R}$. Since X_0 is a linear subspace of X, it follows that $tA = \{ta : a\in A\} \subseteq X_0$ and $A+B = \{a+b : a\in A$ and $b\in B\} \subseteq X_0$. Since $X_0 \in I(X)$, then any subset of X_0 is in $I(X)$. That is, tA and $A+B$ are in $I(X)$. ■

To move forward we need to define finite addition and scalar multiplication of fuzzy on ideal sets, such that the result is also a fuzzy on ideal set.

Definition 4.2. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. For any $\mu, v \in I^{I(X)}$, we define $\mu+v$ as follows: $(\mu+v)(\varnothing)=0$ and for $\varnothing \neq A\in I(X)$,

$$(\mu+v)(A)=\begin{cases} \sup\limits_{x_1+x_2=x} \{\mu(\{x_1\})\wedge v(\{x_2\}) : x_1, x_2 \in X_0\}, & \textit{if } A=\{x\}; \quad \textit{(or A is a singleton)} \\ \inf\{(\mu+v)(\{x\}) : x\in A\}, & \textit{otherwise}. \quad \textit{(or A is not a singleton)} \end{cases}$$

In the case $A = \{x\}$ in Definition 4.2, we note that, since X_0 is a linear subspace of X, we can always express x as $x = 0 + x$, and thus, the set under the sup is never empty.

Definition 4.3. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. For a scalar $t \in \mathbb{R}$ and $\in I^{I(X)}$, we define $t\mu$ as follows: $(t\mu)(\varnothing) = 0$ and for $\varnothing \neq A \in I(X)$,

$$(t\mu)(A) = \begin{cases} \mu\left(t^{-1}A\right), & if\ t \neq 0; \\ 0, & if\ t=0\ and\ A \neq \{0\}; \\ sup\left\{\mu(\{y\}) : y \in X_0\right\}, & if\ t=0\ and\ A=\{0\}. \end{cases}$$

Proposition 4.4. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. If $t \in \mathbb{R}$ and $\mu, v \in I^{I(X)}$, then $t\mu$ and $\mu+v$ are in $I^{I(X)}$.

Proof. We note that, by Definition 4.3, we have $(t\mu)(\varnothing) = 0$. Now, let $\varnothing \neq A_1$, $A_2 \in I(X)$, such that $A_1 \subseteq A_2$ and $t \in \mathbb{R}$. Let $t \neq 0$. Then, and $t^{-1} A_1 \subseteq t^{-1}A_2$. Note that $t^{-1}A_1$, $A_2 \in I(X)$ by Proposition 4.1. It follows that $\mu(t^{-1}A_1) \geq \mu(t^{-1}A_2)$ since $\mu \in I^{I(X)}$. Suppose $t = 0$ and $A_1, A_2 \neq \{0\}$, we have $(t\mu)(A_1) = 0 = (t\mu)(A_2)$. Now, for $t = 0$ and $A_1, A_2 = \{0\}$, we have $(t\mu)(A_1) = (t\mu)(A_2)$. If $t = 0$, $A_1 = \{0\}$ and $A_2 \neq \{0\}$, then $(t\mu)(A_2) = 0 \leq (t\mu)(A_1)$. Thus, $t\mu \in I^{I(X)}$.

Next, we show that $\mu + v \in I^{I(X)}$. By Definition 4.2, $(\mu + v)(\varnothing)=0$. Let $\varnothing \neq A_1, A_2 \in I(X)$, such that $A_1 \subseteq A_2$. Suppose A_2 is a singleton, then A_1 must be a singleton. Then, $A_1 = \{x\} = A_2$. Notice that, $(\mu + v)(A_1) = (\mu + v)(A_2)$. Now, if A_2 is not a singleton, then A_1 is either a singleton or not. In any case, since $A_1 \subseteq A_2$, we have $\{(\mu+v)(\{x\})$: $x \in A_1\} \subseteq \{(\mu+v)(\{x\})$: $x \in A_2\}$.

Thus, $\inf\{(\mu+v)(\{x\})$: $x \in A_1\} \geq \inf\{(\mu+v)(\{x\})$: $x \in A_2\}$. Consequently, $(\mu+v)(A_1) \geq (\mu+v)(A_2)$. Therefore, $\mu + v \in I^{I(X)}$.■

Next, we need to define a fuzzy on ideal seminorm. We begin by defining its properties.

Definition 4.5. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. A $\rho \in I^{I(X)}$ is said to be

i. convex if $\rho(tA+(1-t)B) \geq \min\{\rho(A), \rho(B)\}$ for every $t\in [0,1]$ and $\varnothing \neq A, B \in I(X)$;

ii. balanced if $(t\rho)(A) \leq \rho(A)$ for every $t \in \mathbb{R}$ with $|t| \leq 1$ and $\varnothing \neq A \in I(X)$;

iii. absorbing if $sup_{1>0}(t\rho)(A)=1$ for every $\varnothing \neq A \in I(X)$.

Definition 4.6. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. A $\rho \in I^{I(X)}$ is called a fuzzy on ideal seminorm if it is convex, balanced, and absorbing.

Associated with a fuzzy on ideal seminorm, we define below an important mapping, and then we prove that it has the properties of an ordinary seminorm.

Definition 4.7. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Let ρ be a fuzzy on ideal seminorm. For each $\varepsilon \in (0,1)$, we define $P_\varepsilon : I(X) \to [0,+\infty)$ by $P_\varepsilon(\varnothing) = 0$ and for $\varnothing \neq A \in I(X)$, $P_\varepsilon(A)=inf\{t>0:(t\rho)(A)>\varepsilon\}$.

Observe that P_ε is well-defined since ρ is absorbing. Moreover, $1= sup_{1>0}(t\rho)(\{0\}) = sup_{1>0}(\rho)(t^{-1}\{0\})=sup_{1>0}(\rho)(\{0\})$ implies that $\rho(\{0\})=1$.

Remark 4.8. For $0<\varepsilon_1<\varepsilon_2<1$, we have $\{t>0:(t\rho\ (A)\varepsilon_2\}\subseteq\{t>0:(t\rho)\ (A)>\varepsilon_1\}$. Hence, $P_\varepsilon(A)=inf\{t>0: (t\rho)(A)>\varepsilon_1\}\leq inf\{t>0 : (t\rho)(A)>\varepsilon_2\}=P_{\varepsilon_2}(A)$. That is, $\{P_\varepsilon\}$ is increasing in ε.

Theorem 4.9. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. If ρ is a fuzzy on ideal seminorm, then, for each $\varepsilon \in (0,1)$, we have P_ε satisfying the following properties:

i. $P_\varepsilon(\alpha A) = |\alpha| P_\varepsilon(A)$, for all $\alpha \in \mathbb{R}$ and $\varnothing \neq A \in I(X)$;

ii. $P_\varepsilon(A+B) \leq P_\varepsilon(A) + P_\varepsilon(B)$, for $\varnothing \neq A,B \in I(X)$.

Proof.

i. Let $\varepsilon \in (0,1)$ and $\varnothing \neq A \in I(X)$. If $\alpha = 0$, then $\alpha A = \{0\}$. Observe that

$$P_\varepsilon(0 \cdot A) = inf\{t > 0: (t\rho)(\{0\}) > \varepsilon\} = inf\{t > 0: \rho(t^{-1}\{0\}) > \varepsilon\}$$
$$= inf\{t > 0: \rho(\{0\}) > \varepsilon\} = inf\{t > 0: 1 > \varepsilon\} = 0 = |0|P_\varepsilon(A).$$

Let $\alpha \neq 0$. Since ρ is balanced, $(t\rho)(A) \leq \rho(A)$ for $|t| \leq 1$. In particular, $t = -1$ implies $(-\rho)(A) \leq \rho(A)$, and so, $\rho\left(\frac{A}{-1}\right) \leq \rho(A)$. Now,

$$\left(-\frac{t}{\alpha}\rho\right)(A) = \rho\left(\frac{\frac{\alpha A}{t}}{-1}\right) \leq \rho\left(\frac{\alpha A}{t}\right) = \left(\frac{t}{\alpha}\rho\right)(A).$$

Hence, $\left(-\frac{t}{\alpha}\rho\right)(A) \leq \left(\frac{t}{\alpha}\rho\right)(A)$. Next, consider that

$$-\left(-\frac{t}{\alpha}\rho\right)(A) = \rho\left(\frac{\frac{\alpha A}{-t}}{-1}\right) \leq \rho\left(\frac{\alpha A}{-t}\right).$$

It follows that $\rho\left(\frac{\alpha A}{t}\right) \leq \rho\left(\frac{\alpha A}{-t}\right)$. Thus, $\left(\frac{t}{\alpha}\rho\right)(A) \leq \left(-\frac{t}{\alpha}\rho\right)(A)$.

Consequently, $\left(\frac{t}{\alpha}\rho\right)(A) = \left(-\frac{t}{\alpha}\rho\right)(A)$. Then,

$$P_\varepsilon(\alpha A) = inf\{t > 0: t\rho(\alpha A) > \varepsilon\} = inf\{t > 0: \left(\frac{t}{\alpha}\rho\right)(A) > \varepsilon\}$$
$$= inf\{t > 0: \left(\frac{t}{|\alpha|}\rho\right)(A) > \varepsilon\} \text{ by the preceding result}$$
$$= inf\{|\alpha|t' > 0: (t'\rho)(A) > \varepsilon\}, \text{ where } t' = \frac{t}{|\alpha|}$$
$$= inf\{|\alpha|t' > 0: (t'\rho)(A) > \varepsilon\} = |\alpha|(inf\{t' > 0: (t'\rho)(A) > \varepsilon\})$$
$$= |\alpha|(inf\{t > 0: (t\rho)(A) > \varepsilon\}) = |\alpha|P_\varepsilon(A).$$

ii. Let $\varnothing \neq A, B \in I(X)$, $r \in \{t > 0 : (t\rho)(A) > \varepsilon\}$, and $s \in \{t > 0 : (t\rho)(B) > \varepsilon\}$. Then, $(r\rho)(A) > \varepsilon$ and $(s\rho)(B) > \varepsilon$. Now, by convexity of ρ, we have

$$((r+s)\rho)(A+B) = \rho\left(\frac{A+B}{r+s}\right) = \rho\left(\frac{r}{r+s}\left(\frac{A}{r}\right) + \frac{s}{r+s}\left(\frac{B}{s}\right)\right) \geq min\left\{\rho\left(\frac{A}{r}\right), \rho\left(\frac{B}{s}\right)\right\}.$$

It follows that $((r+s)\rho)(A+B) \geq min\{(r\rho)(A), (s\rho)(B)\} > \varepsilon$, and so, $r + s \in \{t > 0: (t\rho)(A+B) > \varepsilon\}$. Hence, we have

$$\{t > 0: (t\rho)(A) > \varepsilon\} + \{t > 0: (t\rho)(B) > \varepsilon\} \subseteq \{t > 0: (t\rho)(A+B) > \varepsilon\}.$$

Thus,

$$inf\{t > 0: (t\rho)(A + B) > \varepsilon\} \le inf\{t > 0: (t\rho)(A) > \varepsilon\} + inf\{t > 0: (t\rho)(B) > \varepsilon\}.$$

Therefore, $P_\varepsilon(A + B) \le P_\varepsilon(A) + P_\varepsilon(B)$. ■

The next theorem shows that the infimum of the P_ε has properties similar to it.

Theorem 4.10. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Let ρ be a fuzzy on ideal seminorm. Then, the function $P{:}I(X) \to [0,+\infty)$ defined by $P(\emptyset)=0$, and for $\emptyset \ne A \in I(X)$, $P(A) = inf\{P_\varepsilon(A){:}\varepsilon \in (0,1)\}$ satisfies the following properties:

i. $P(\alpha A) = |\alpha| P(A)$, for all $\alpha \in \mathbb{R}$ and $\emptyset \ne A \in I(X)$;

ii. $P(A + B) \le P(A) + P(B)$, for all $\emptyset \ne A, B \in I(X)$.

Proof. The first property follows directly from the first property in Theorem 4.9. Let $\emptyset \ne A,B \in I(X)$. Since $\{P_\varepsilon\}$ is increasing in ε, for every $\emptyset \ne A \in I(X)$, $P(A) = inf\{P_\varepsilon(A){:}\varepsilon \in (0,1)\} = \lim_{\varepsilon\to 0} P_\varepsilon(A)$. Thus, $P(A+B)=inf\{P_\varepsilon(A+B){:}\varepsilon \in (0,1)\} \le inf\{P_\varepsilon(A)+P_\varepsilon(B){:}\varepsilon \in (0,1)\} = \lim_{\varepsilon\to 0}\{P_\varepsilon(A)+P_\varepsilon(B)\} = \lim_{\varepsilon\to 0}\{P_\varepsilon(A)\}+\lim_{\varepsilon\to 0} P_\varepsilon(B)=P(A)+P(B)$. ■

The next two theorems give us the relationship between fuzzy on ideal seminorms and its associated mappings having the ordinary seminorm properties. It is our key to tap on the classical Hahn-Banach Theorem that will be used in the proof of our fuzzy on ideal Hahn-Banach Theorem.

Theorem 4.11. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Furthermore, let ρ_1 and ρ_2 be two fuzzy on ideal seminorms. If for every $A \in I(X)$, $\rho_1(A) \le \rho_2(A)$, then for every $\varepsilon \in (0,1)$, $P^1_\varepsilon(A) \ge P^2_\varepsilon(A)$ for all $A \in I(X)$.

Proof. If for every $A \in I(X)$, $\rho_1(A) \le \rho_2(A)$, then for every $A \in I(X)$ and $t > 0$, $(t\rho_1)(A) = \rho_1(t^{-1}A) \le \rho_2(t^{-1}A) = (t\rho_2)(A)$. Let $\varepsilon \in (0,1)$ and $\emptyset \ne A \in I(X)$. Observe that $\{t>0{:}(t\rho_1)(A) > \varepsilon\}$ is a subset of $\{t > 0{:}(t\rho_2)(A) > \varepsilon\}$. Hence, $inf\{t > 0{:}(t\rho_1)(A) > \varepsilon\} \ge \{t > 0{:}(t\rho_2)(A) > \varepsilon\}$. Thus, $P^1_\varepsilon(A) \ge P^2_\varepsilon(A)$. ■

Remark 4.12. The converse of Theorem 4.11 does not always hold. To see this, let $X = \mathbb{R}$ and $I(X) = P(\mathbb{R})$. Define ρ_1 and ρ_2 as follows:

$$\rho_1(A) = \begin{cases} 1, & if\ A = \{x\}\ and\ x \in [-1,1]; \\ \frac{1}{3}, & if\ A = \{x\}\ and\ x \in (-5,-1) \cup (1,5); \\ 0, & otherwise. \end{cases}$$

$$\rho_2(A) = \begin{cases} 1, & if\ A = \{x\}\ and\ x \in (-1,1); \\ \frac{1}{3}, & if\ A = \{x\}\ and\ x \in [-5,-1] \cup [1,5]; \\ 0, & otherwise. \end{cases}$$

One can check that ρ_1 and ρ_2 are fuzzy on ideal seminorms and $P_\varepsilon^1 = P_\varepsilon^2$ for $\varepsilon \in (0,1)$, but $\rho_1 \nleq \rho_2$ and $\rho_2 \nleq \rho_1$.

The following *-property will give us a sufficient condition for the converse to hold.

Definition 4.13. Let X be a vector space over $\mathbb{R}$, and let $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X. Let $X_0 \in I(X)$. Let ρ be a fuzzy on ideal seminorm. We say that ρ has the *-property if, for every $\emptyset \neq A \in I(X)$, we have $\rho(A) = inf\{\rho(tA): 0 < t < 1\}$.

An example of a fuzzy on ideal seminorm with the *-property will be given later. It is a crucial part of our main theorem. In the meantime, let us prove that, with the *-property, the converse of Theorem 4.11 will hold.

Lemma 4.14. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X. Let $X_0 \in I(X)$. Let ρ be a fuzzy on ideal seminorm with the *-property. If $\emptyset \neq A \in I(X)$ and $\rho(A) < \varepsilon < 1$, then $P_\varepsilon(A) > 1$.

Proof. Let $\emptyset \neq A \in I(X)$ and $\rho(A) < \varepsilon < 1$. Since ρ is balanced being a fuzzy on ideal seminorm, $(t\rho)(A) \leq \rho(A) < \varepsilon$ for $|t| \leq 1$. Thus, $P_\varepsilon(A) = inf\{t > 0: (t\rho)(A) > \varepsilon\} = inf\{t > 1: (t\rho)(A) > \varepsilon\} \geq 1$.

We are left to show that $P_\varepsilon(A) \neq 1$. Suppose $P_\varepsilon(A) = 1$. Then, $(t\rho)(A) > \varepsilon$ for all $t > 1$. Since ρ has the *-property, $\rho(A) = inf\{\rho(tA): 0 < t < 1\} = inf\{(t^{-1}\rho)(A): 0 < t < 1\} = inf\{(t\rho)(A): t > 1\} \geq \varepsilon$. However, this is a contradiction, since $\rho(A) < \varepsilon$. Therefore, $P_\varepsilon(A) > 1$. ■

Theorem 4.15. Let X be a vector space over $\mathbb{R}$, and let $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X. Let $X_0 \in I(X)$. Let ρ_1 and ρ_2 be two fuzzy on ideal seminorms, with ρ_2 having the *-property. If for every $\varepsilon \in (0,1)$, we have $P_\varepsilon^1(A) \geq P_\varepsilon^2(A)$ for every $A \in I(X)$, then $\rho_1(A) \leq \rho_2(A)$ for every $A \in I(X)$.

Proof. If $A = \emptyset$, then by definition, $P_\varepsilon^1(A) = P_\varepsilon^2(A) = \rho_1(A) = \rho_2(A) = 0$. Suppose that for every $\varepsilon \in (0,1)$, $P_\varepsilon^1(A) \geq P_\varepsilon^2(A)$, $\forall A \in I(X)$ and that there exists a $B \in I(X)$, such that $\rho_2(B) < \rho_1(B)$. Let $\rho_2(B) < \varepsilon < \rho_1(B)$. If $t = 1$, then $t\rho_1(B) = \rho_1(B) > \varepsilon$, and so, $P_\varepsilon^1(B) = inf\{t > 0 : t\rho_1(B) > \varepsilon\} \leq 1$. Since ρ_2 is balanced, $P_\varepsilon^2(B) = inf\{t > 0 : t\rho_2(B) > \varepsilon\} \geq 1$. By Lemma 4.14, $P_\varepsilon^2(B) > 1$. Thus, $P_\varepsilon^2(B) > P_\varepsilon^1(B)$. This is a contradiction to our assumption that for every $\varepsilon \in (0,1)$, $P_\varepsilon^1(A) \geq P_\varepsilon^2(A)$, $\forall A \in I(X)$. Therefore, we must have $\rho_1(A) \leq \rho_2(A)$, $\forall A \in I(X)$. ■

We now define and prove an important fuzzy on ideal seminorm with the *-property. We begin with the following definition.

Definition 4.16. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Let M be also a linear subspace of X and $f{:}M \to$ be a linear functional. Furthermore, let $I(M)$ be an ideal of M, such that $M_0 \in I(M)$. We associate with f the function $\pi_f : I(M) \to \mathbb{R} \cup \{+\infty\}$ defined by $\pi_f(\emptyset) = 0$, and for $\emptyset \neq A \in I(M)$, $\pi_f(A) = \sup_{x \in A} |f(x)|$. Let $B_f = \{A \in I(M) : A \neq \emptyset$ $and\ \pi_f(A) \leq 1\}$. We define $\chi_{B_f} : I(M) \to [0,1]$ by $\chi_{Bf}(\emptyset) = 0$, and for $\emptyset \neq A \in I(X)$,

$$\chi_{B_f}(A) = \begin{cases} 1, & if\ A \in B_f; \\ 0, & otherwise. \end{cases}$$

Observe that, for $t > 0$, $\pi_f(tA) = t\pi_f(A)$. For convenience, whenever we have $\pi_f(A) = \sup_{x \in A} |f(x)| = +\infty$, we let the $lim_{t \to +\infty} \pi_f\left(\frac{A}{t}\right) = \frac{1}{t}\pi_f(A) = 1$ and say that the $sup_{t>0}\left(t\chi_{B_f}\right)(A) = sup_{t>0}\left(\chi_{B_f}\right)\left(\frac{A}{t}\right) = 1$.

Theorem 4.17. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Let M be also a linear subspace of X and $f{:}M \to$ be a linear functional. Furthermore, let $I(M)$ be an ideal of M, such that $M_0 \in I(M)$. Then, χ_{B_f} is a fuzzy on ideal seminorm with the *-property.

Proof. We first show that χ_{B_f} has the *-property. Let $\emptyset \neq A \in I(M)$. Suppose that $\chi_{B_f}(A) = 1$. Then, $\pi_f(A) = sup_{x \in A}|f(x)| \leq 1$. Then, for all $0 < t < 1$, $sup_{x \in A}|f(tx)| = t\,sup_{x \in A}|f(x)| \leq t < 1$. Hence, $\chi_{B_f}(tA) = 1$, for all $0 < t < 1$, and so, $inf\{\chi_{Bf}(tA) : 0 < t < 1\} = 1$. Suppose that $\chi_{Bf}(A) = 0$. Then, $\pi_f(A) = sup_{x \in A}|f(x)| > 1$. Assume $inf\{\chi_{Bf}(tA) | 0 < t < 1\} = 1$. It follows that, for all $0 < t < 1$, $\chi_{B_f}(tA) = 1$, implying that $\pi_f(tA) = sup_{x \in A}|f(tx)| = t\,sup_{x \in A}|f(x)| \leq 1$. Thus, for all $0 < t < 1$, $sup_{x \in A}|f(x)| \leq \frac{1}{t}$. Consequently, $sup_{x \in A}|f(x)| \leq 1 + \eta$

for all $\eta > 0$. Since η is arbitrary, $sup_{x\in A}|f(x)| \leq 1$, which contradicts to $\chi_{B_f}(A) = 0$. Hence, there exists $0 < t < 1$, such that $\chi_{B_f}(tA) = 0$, that is $inf\{\chi_{B_f}(tA): 0 < t < 1\} = 0$. Therefore, χ_{B_f} has the *-property.

Next we show that it is a fuzzy on ideal set. It is enough to show the reverse inequality. Let $\varnothing \neq A,B \in I(M)$, such that $A \subseteq B$. If $\chi\chi_{B_f}(B) = 0$, then the reverse inequality automatically follows. Suppose that $\chi_{B_f}(B) = 1$. Then, $\pi_f(B) = sup_{x\in B}|f(x)| \leq 1$. Now, $\pi_f(A) = sup_{x\in A}|f(x)| \leq sup_{x\in B}|f(x)| \leq 1$. Hence, $\chi_{B_f}(A) = 1$. Thus, $\chi_{B_f}(B) \leq \chi_{B_f}(A)$.

We are left to show that χ_{B_f} is convex, balanced, and absorbing.

Let $\varnothing \neq A,B \in I(M)$. If $\chi_{B_f}(A) = 0$ or $\chi_{B_f}(B) = 0$, then the inequality for convex is clearly satisfied. Suppose that $A,B \in B_f$, that is, $\chi_{B_f}(A) = 1$ and $\chi_{B_f}(B) = 1$. Then, $sup_{x\in B}|f(x)| \leq 1$ and $sup_{x\in B}|f(x)| \leq 1$. It follows that, for each $0 \leq t \leq 1$, since f is linear, we have $sup_{x\in tA+(1-t)B}|f(x)| = sup_{a\in A,b\in B}|f(ta+(1-t)b)| = sup_{a\in A,b\in B}\{t|f(a)| + (1-t)|f(b)|\} \leq t + 1 - t = 1$.

Thus, $\chi_{B_f}(tA+(1-t)B) = 1$, and so χ_{B_f} is convex.

Let $\varnothing \neq A,B \in I(M)$ and $|t| \leq 1$. If $(t\chi\chi_{B_f})(A) = 0$, then the inequality for balanced is clearly satisfied. Suppose $(t\chi_{B_f})(A) = 1$. Then for each t with $|t| \leq 1$, we have $sup_{x\in A}\left|f\left(\frac{x}{t}\right)\right| = \frac{1}{t}sup_{x\in A}|f(x)| \leq 1$. Hence, $sup_{x\in A}|f(x)| \leq |f(x)| \leq |t| \leq 1$, and so, $\chi\chi_{B_f}(A) = 1$. Consequently, χ_{B_f} is balanced. Lastly, let $\varnothing \neq A \in I(M)$. If $\pi_f(A) = sup_{x\in A}|f(x)| < +\infty$, take $t_0 = sup_{x\in A}|f(x)| \leq |f(x)|$. Then, $sup_{x\in A}\left|f\left(\frac{x}{t_0}\right)\right| = \frac{1}{t_0}sup_{x\in A}|f(x)| = 1$. Thus, $(t_0\chi_{B_f})(A) = 1$. Therefore, $sup_{t>0}(t\chi_{B_f})(A) = 1$. If $\pi_f(A) = sup_{x\in A}|f(x)| \leq +\infty$, we have remarked prior to this theorem that $sup_{t>0}(t\chi_{B_f})(A) = 1$. Thus, χ_{B_f} is absorbing. ■

Finally, we have the fuzzy on ideal Hahn-Banach Theorem.

Theorem 4.18. Let X be a vector space over $\mathbb{R}$, and $I(X)$ be an ideal on X, such that X_0 is a linear subspace of X and $X_0 \in I(X)$. Let M be also a linear subspace of X, and $I(M)$ be an ideal of M, such that $M_0 \in I(M)$ and $M_0 \subseteq X_0$. Let $\rho \in I^{I(X)}$ be a fuzzy on ideal seminorm. If $f: M_0 \to \mathbb{R}$ is a linear functional, such that $\chi_{B_f}(A) \geq \rho(A)$ for all $A \in I(M)$, then there exists a linear functional $g: X_0 \to \mathbb{R}$, such that:

i. $f(x) = g(x),\ \forall x \in M_0$; and

ii. $\chi_{B_g}(A) \geq \rho(A)$ for all $A \in I(X)$.

Proof. Let f be a linear functional on M_0, such that $\chi_{B_f}(A) \geq \rho(A)$ for all $A \in I(M)$. Note first that χ_{B_f} is a fuzzy on ideal seminorm with the *-property. In Theorem 4.11, let $\chi_{B_f} = \rho_2$ and $\rho = \rho_1$. Then, the corresponding P_ε^2 is given by: for any $\varnothing \neq A \in I(M)$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
P_\varepsilon^2(A) &= inf\{t>0: t\rho_2(A) > \varepsilon\} \\
&= inf\{t>0: \rho_2(A/t) > \varepsilon\} \\
&= inf\{t>0: \rho_2(A/t) = 1\}, \text{ since } \rho_2 \text{ is a characteristic function} \\
&= inf\{t>0: \pi_f(A/t) \leq 1\}, \text{ that is } (A/t) \in B_f \\
&= inf\{t>0: \pi_f(A) \leq t\}, \text{ since } f \text{ is linear} \\
&= \pi_f(A).
\end{aligned}
$$

Thus, by Theorem 4.11, for all $\varepsilon \in (0,1)$, $P_\varepsilon^2(A) = \pi_f(A) \leq P_\varepsilon^1(A)$ for all $\varnothing \neq A \in I(M)$, where $P_\varepsilon^1(A) = inf\{t>0: t\rho(A) > \varepsilon\}$ for all $\varnothing \neq A \in I(X)$. Observe that we are considering here $A \in I(X)$, instead of just $I(M)$. This can be done because $\rho \in I^{I(X)}$. By the last inequality, $\pi_f(A) = sup_{x \in A}|f(x)| \leq P(A) = inf\{P_\varepsilon^1(A): \varepsilon \in (0,1)\}$ for all $\varnothing \neq A \in I(M)$. In particular, $|f(x)| \leq P(\{x\})$, for all $x \in M_0$.

Note that by Theorem 4.10, P restricted to the singletons can be seen as a sublinear functional on X_0. Hence, by applying the classical Hahn-Banach Theorem, there exists a linear functional $g: X_0 \rightarrow$ such that:

i. $f(x) = g(x)$, $\forall x \in M_0$; and

ii. $|g(x)| \leq P(\{x\})$, for all $x \in X_0$.

Now, let $\varnothing \neq A \in I(X)$. Note first that, by the definition of P and the reverse inequality satisfied by ρ, if $x \in A$, then $P(\{x\}) \leq P(A)$. Hence by (ii), $\forall x \in A$, $|g(x)| \leq P(A)$. Thus, we have $sup_{x \in A}|g(x)| = \pi_g(A) \leq P(A)$ for all $\varnothing \neq A \in I(X)$. Let $\chi_{Bg} = \rho_2$ in Theorem 4.15. Then, the corresponding P_ε^2 is given by: for any $\varnothing \neq A \in I(X)$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
P_\varepsilon^2(A) &= inf\{t>0: t\rho_2(A) > \varepsilon\} \\
&= inf\{t>0: \rho_2(A/t) > \varepsilon\} \\
&= inf\{t>0: \rho_2(A/t) = 1\}, \text{ since } \rho_2 \text{ is a characteristic function} \\
&= inf\{t>0: \pi_g(A/t) \leq 1\}, \text{ that is } (A/t) \in B_g \\
&= inf\{t>0: \pi_g(A) \leq t\}, \text{ since } g \text{ is linear} \\
&= \pi_g(A).
\end{aligned}
$$

Thus, for all $\varepsilon \in (0,1)$, $P_\varepsilon^2(A) \leq P(A) = inf\{P_\varepsilon^1(A): \varepsilon \in (0,1)\}$ for all $\emptyset \neq A \in I(X)$. Hence, for all $\varepsilon \in (0,1)$, $P_\varepsilon^2(A) \leq P_\varepsilon^1(A) = inf\{t>0:(t\rho)(A)>\varepsilon\}$ for all $\emptyset \neq A \in I(X)$. Since χ_{Bg} has the *-property, by Theorem 4.15, we have $\chi_{Bg}(A) \geq \rho(A)$ for all $A \in I(X)$. ■

ACKNOWLEDGEMENT

The first author was supported by the Commission on Higher Education (CHED) of the Philippines and the second author by the Premier Research Institute of Science and Mathematics (PRISM) of MSU-IIT.

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