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Filter Representations of Normal Lattices

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خلاصة : يحوي هذا البحث حلاً لبعض المسائل في التبولوجيا وذلك باستعمال نظرية القياس . تركز الدراسة على خاصية العادية (Normality) في فضاء انفصالي (Separated Lattice) حيث تستعمل القياسات الثنائية بالقيمة صفراً وواحداً . توزيع الكاتيون داخل صخور الكروم في عمان .

ABSTRACT: In this paper we will investigate the properties of normality lattices and their relationships to zero-one measures. We will establish necessary and sufficient conditions for lattices to be normal. These properties are then investigated in the case of separated lattices.

Let X be an arbitrary set and $\mathcal{L}$ a lattice of subsets of X. Let $\mathcal{A}(\mathcal{L})$ be the algebra generated by $\mathcal{L}$, and let $I(\mathcal{L})$ be the set of non-trivial zero-one valued finitely additive measures on $\mathcal{A}(\mathcal{L})$. $I_R(\mathcal{L})$ will denote those $\mu \in I(\mathcal{L})$ that are $\mathcal{L}$-regular, and $I^{\sigma}_R(\mathcal{L})$ consists of those $\mu \in I_R(\mathcal{L})$ which are countably additive.

We will consider a number of equivalent characterizations of $\mathcal{L}$ being a normal lattice. We then associate with $\mathcal{L}$, a lattice $W_\sigma(\mathcal{L})$ in $I^{\sigma}_R(\mathcal{L})$. Assuming $\mathcal{L}$ is disjunctive, $W_\sigma(\mathcal{L})$ is always a replete lattice. We will give necessary and sufficient condition for $W_\sigma(\mathcal{L})$ to be a prime complete lattice. Next, we consider the set $I^{\sigma}_R(\mathcal{L})$ with the topology of closed sets given by $\tau W_\sigma(\mathcal{L})$ consisting of arbitrary intersections of sets of $W_\sigma(\mathcal{L})$. We investigate this topological space to some extent giving necessary and sufficient conditions for it to be T_2; Lindelöf and normal.

The notations and terminology used in this paper are standard and are consistent with Alexandrov (1937), Bachman *et al.* (1983), Szeto (1979), Wallman (1938) and Yallaoui (1991). Our work on normal lattices is closely related to work done in Camacho (1991). We begin with a brief review of some notations and some definitions for the reader's convenience.

Definitions and Notations

Let X be an abstract set and let $\mathcal{L}$ be the lattice of subsets of X. We will always assume that $\emptyset$ and X are in $\mathcal{L}$. If A is a subset of X then we will denote the complement of A by A' i.e. $A' = X - A$ and if $\mathcal{L}$ is a lattice of subset of X then $\mathcal{L}'$ is defined as $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$

Lattice Terminology

DEFINITION 1: Let $\mathcal{L}$ be a lattice of subsets of X be a Lattice of subsets of X. We say that $\mathcal{L}$ is :

1. δ-lattice if it is closed under countable intersections,
2. separating or T_1 if for $x, y \in X$ and $x \neq y$ then there exists $L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$,
3. Hausdorff or T_2 if for $x, y \in X$ and $x \neq y$ then there exists $A, B \in \mathcal{L}$ such that $x \in A'$, $y \in B'$ and $A' \cap B' = \emptyset$,
4. disjunctive if for $x \in X$ and $L \in \mathcal{L}$ where $x \notin L$, $\exists A \in \mathcal{L}$ such that $x \in A$ and $A \cap L = \emptyset$,
5. normal if for $A, B \in \mathcal{L}$ where $A \cap B = \emptyset$, $\exists \tilde{A}', \tilde{B}' \in \mathcal{L}$ such that $A \subset \tilde{A}'$, $B \subset \tilde{B}'$, and $\tilde{A}' \cap \tilde{B}' = \emptyset$,
6. compact if any covering of X by $\mathcal{L}'$ sets has a finite subcovering,
7. countably compact if any countable covering of X by $\mathcal{L}'$ sets has a finite subcovering.
8. $\mathcal{L}$ is Linderlöf if any covering of X by $\mathcal{L}'$ sets has a countable subcovering.

We let

$\mathcal{A}(\mathcal{L})$ = the algebra generated by $\mathcal{L}$.

$\sigma(\mathcal{L})$ = the σ- algebra generated by $\mathcal{L}$.

$\delta(\mathcal{L})$ = the lattice of countable intersections of sets of $\mathcal{L}$.

$\tau(\mathcal{L})$ = the lattice of arbitrary intersections of sets of $\mathcal{L}$.

Measure Terminology

Let $\mathcal{L}$ be a lattice of subsets of X, then $M(\mathcal{L})$ will denote the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. Clearly since any measure in $M(\mathcal{L})$ can be written as a difference of two non-negative measures there is no loss of generality in assuming that the measures are non-negative, and we will assume so throughout this paper. We will say that a measure μ of $M(\mathcal{L})$ is regular if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup_{L \subset A,\, L \in \mathcal{L}} \mu(L)$.

$M_R(\mathcal{L})$ represents the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$.

DEFINITION 2:
1) A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{L}$, if for $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$; then $\mu(L_n) \to 0$.
2) A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{A}(\mathcal{L})$, if for $A_n \in \mathcal{A}(\mathcal{L})$ and $A_n \downarrow \emptyset$; then $\mu(A_n) \to 0$.

If $\mathcal{L}$ is a lattice of subsets of X, then

$M_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$
$M^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$
$M^\sigma_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$

DEFINITION 3: If $A \in \mathcal{A}(\mathcal{L})$ and if $x \in X$ then

$$\mu_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$ is the measure concentrated at x.

$I(\mathcal{L})$ is the subset of $M(\mathcal{L})$ which consist of non-trivial zero-one valued measures;

$I_R(\mathcal{L})$ is the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$
$I_\sigma(\mathcal{L})$ is the set of σ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$
$I^\sigma(\mathcal{L})$ is the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $I(\mathcal{L})$
$I^\sigma_R(\mathcal{L})$ is the set of $\mathcal{L}$-regular measures of $I^\sigma(\mathcal{L})$

DEFINITION 4: If $\mu \in M(\mathcal{L})$ then we define the support of μ to be: $S(\mu) = \cap\{L \in \mathcal{L} \mid \mu(L) = \mu(X)\}$.
Consequently if $\mu \in I(\mathcal{L})$ then $S(\mu) = \cap\{L \in \mathcal{L} \mid \mu(L) = 1\}$

DEFINITION 5: We say that the lattice is:

1) Replete if $S(\mu) \neq \emptyset$ for any $\mu \in I^\sigma_R(\mathcal{L})$.
2) Prime Complete if $S(\mu) \neq \emptyset$ for any $\mu \in I_\sigma(\mathcal{L})$.

We now list a few well known facts which will enable us to characterize some previously defined properties in a measure theoretic fashion. The lattice $\mathcal{L}$ is:

1. disjunctive if and only if $\mu_x \in I_R(\mathcal{L})$, $\forall x \in X$,
2. T_2 if and only if $S(\mu) = \emptyset$ or a singleton for any $\mu \in I(\mathcal{L})$,
3. compact if and only if $S(\mu) \neq \emptyset$ for any $\mu \in I_R(\mathcal{L})$,
4. countably compact if and only if $I_R(\mathcal{L}) = I^\sigma_R(\mathcal{L})$,
5. Lindelöf if and only if $S(\mu) \neq \emptyset$ for any $\mu \in I_\sigma(\mathcal{L})$,
6. normal if and only if for any $\mu \in I(\mathcal{L})$ there exists a unique $v \in I_R(\mathcal{L})$ such that $\mu \leq v$ on $\mathcal{L}$.

Filter and Measure Relationships

Let $\mathcal{L}$ be a lattice of subsets of X.

DEFINITION 6: We say that $\mathcal{F} \subset \mathcal{L}$ is an $\mathcal{L}$-filter if:

1) $\emptyset \notin \mathcal{F}$
2) If $L_1, L_2 \in \mathcal{F} \Rightarrow L_1 \cap L_2 \in \mathcal{F}$
3) If $L_1 \subset L_2$ and $L_1 \in \mathcal{F} \Rightarrow L_2 \in \mathcal{F}$

DEFINITION 7: $\mathcal{F}$ is said to be a prime $\mathcal{L}$-filter if:

1) $\mathcal{F}$ is an $\mathcal{L}$-filter, and
2) If $L_1, L_2 \in \mathcal{L}$ and $L_1 \cup L_2 \in \mathcal{F} \Rightarrow L_1 \in \mathcal{F}$ or $L_2 \in \mathcal{F}$

DEFINITION 8: If $\mathcal{F}$ is an $\mathcal{L}$-filter we say that $\mathcal{F}$ is an $\mathcal{L}$-ultrafilter if $\mathcal{F}$ is a maximal $\mathcal{L}$-filter.
If $\mu \in I(\mathcal{L})$ let $\mathcal{F}_\mu = \{L \in \mathcal{L} : \mu(L) = 1\}$.

PROPOSITION 9:
1) If $\mu \in I(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-prime filter and conversely any $\mathcal{L}$-prime filter determines an element $\mu \in I(\mathcal{L})$ and the correspondence is a bijection.

2) If $\mu \in I_R(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter and conversely any $\mathcal{L}$-ultrafilter determines an element $\mu \in I_R(\mathcal{L})$. This correspondence is also a bijection. $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter if and only if $\mu \in I_R(\mathcal{L})$.

Separation of Lattices

We are going to state a few known facts about the separation of lattices. We will use these results later on in the paper.

DEFINITION 10: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Lattices of subsets of X. We say that $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for any $A_2, B_2 \in \mathcal{L}_2$ and $A_2 \cap B_2 = \emptyset$ then there exists $A_1, B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1$, $B_2 \subset B_1$ and $A_1 \cap B_1 = \emptyset$.

PROPOSITION 11: Let $\mathcal{L}$ be a lattice of subset of X. Then $\mathcal{L}$ is compact if and only if $\tau\mathcal{L}$ is compact, in which case $\mathcal{L}$ separates $\tau\mathcal{L}$

PROPOSITION 12: $\mathcal{L}$ is Lindelöf if and only if $\tau\mathcal{L}$ is Lindelöf and if we further assume that $\mathcal{L}$ is also δ then $\mathcal{L}$ separates $\tau\mathcal{L}$

The proofs for these propositions are easy and will be omitted.

THEOREM 13: Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$. Then $\mathcal{L}_1$ is normal if and only if $\mathcal{L}_2$ is normal.

Proof:

1. Suppose that $\mathcal{L}_1$ is normal and let $A_2, B_2 \in \mathcal{L}_2$; $A_2 \cap B_2 = \emptyset$. Since $\mathcal{L}_1$ separates $\mathcal{L}_2$ then there exist $A_1, B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1$, $B_2 \subset A_2$ and $A_1 \cap B_1 = \emptyset$. Now since $\mathcal{L}_1$ is normal there exist $A, B \in \mathcal{L}_1 \subset \mathcal{L}_2$ such that $A_1 \subset A'$, $B_1 \subset B'$ and $A' \cap B' = \emptyset$. Therefore $A_2 \subset A_1 \subset A'$, $B_2 \subset B_1 \subset B'$ and $A' \cap B' = \emptyset$ i.e. $\mathcal{L}_2$ is normal.

2. Suppose that $\mathcal{L}_2$ is normal. Let $\mu_1 \in I(\mathcal{L}_1)$ and assume that there exist two measures $v_1, \tau_1 \in I_R(\mathcal{L}_1)$ and $\mu_1 \le v_1$, $\mu_1 \le \tau_1$ on $\mathcal{L}_1$. Let μ_2, v_2 and τ_2 be the respective extensions of the previous measures. Note that the latter two extensions are unique and belong to $I_R(\mathcal{L}_2)$. Furthermore it can be seen that $\mathcal{L}_1$ separates $\mathcal{L}_2$ then $\mu_2 \le v_2$ and $\mu_2 \le \tau_2$ on $\mathcal{L}_2$. However since $\mathcal{L}_2$ is normal then $v_2 = \tau_2$. Therefore $v_1 = \tau_1$ and thence $\mathcal{L}_1$ is normal.

The Wallman Space

If $\mathcal{L}$ is a disjunctive lattice of subsets of an abstract set X then there is a Wallman space associated with it. We will briefly review the fundamental properties of this Wallman space.

For any A in $\mathcal{A}(\mathcal{L})$, define $W(A)$ to be $W(A) = \{\mu \in I_R(\mathcal{L}) ; \mu(A) = 1\}$. If A, B $\in A(\mathcal{L})$ then:

1) $W(A \cup B) = W(A) \cup W(B)$,
2) $W(A \cap B) = W(A) \cap W(B)$,
3) $W(A') = W(A)'$,
4) $W(A) \subset W(B)$ if and only if $A \subset B$,
5) $W(A) = W(B)$ if and only if $A = B$,
6) $W[\mathcal{A}(\mathcal{L})] = \mathcal{A}[W(\mathcal{L})]$.

Let $W(\mathcal{L}) = \{W(L), L \in \mathcal{L}\}$.

$W(\mathcal{L})$ is a compact lattice, and the topological space $I_R(\mathcal{L})$ with closed sets $\tau W(\mathcal{L})$ is a compact T_1 space called the Wallman space associated with X and $\mathcal{L}$. Since $\mathcal{L}$ is disjunctive, it will be T_2 if and only if $\mathcal{L}$ is normal.

In addition to each $\mu \in M(\mathcal{L})$ there corresponds a unique $\hat{\mu} \in M(W(\mathcal{L}))$, where $\hat{\mu}(W(A)) = \mu(A)$ for $A \in \mathcal{A}(\mathcal{L})$ and conversely. Also, $\mu \in M_R(\mathcal{L})$ if and only if $\hat{\mu} \in M_R(W(\mathcal{L}))$.

Since $W(\mathcal{L})$ is compact so is $\tau W(\mathcal{L})$, and $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$ (see proposition 11). Furthermore $\hat{\mu} \in M_R(W(\mathcal{L})$ has a unique extension to $\tilde{\mu} \in M_R(\tau W(\mathcal{L}))$. Next we consider the space $I^{\sigma}_R(\mathcal{L})$ and its topology.

DEFINITION 14: Let $\mathcal{L}$ be a disjunctive lattice of subsets of X, $L \in \mathcal{L}$ *and* $A \in \mathcal{A}(\mathcal{L})$.

1) $W_\sigma(L) = \{\mu \in I^\sigma_R(\mathcal{L}) \mid \mu(L) = 1\}$.
2) $W_\sigma(A) = \{\mu \in I^\sigma_R(\mathcal{L}) \mid \mu(A) = 1\}$.
3) $W_\sigma(\mathcal{L}) = \{W_\sigma(L), L \in \mathcal{L}\} = W(\mathcal{L}) \cap I^\sigma_R(\mathcal{L})$.

The following properties hold and are not difficult to prove.

PROPOSITION 15: Let $\mathcal{L}$ be a disjunctive lattice. Then for $A, B \in A(\mathcal{L})$:

1) $W_\sigma(A \cup B) = W_\sigma(A) \cup W_\sigma(B)$,
2) $W_\sigma(A \cap B) = W_\sigma(A) \cap W_\sigma(B)$,
3) $W_\sigma(A') = W_\sigma(A)'$,
4) $W_\sigma(A) \subset W_\sigma(B)$ if and only if $A \subset B$,
5) $\mathcal{A}[\mathcal{W}_\sigma(\mathcal{L})] = \mathcal{W}_\sigma[\mathcal{A}(\mathcal{L})]$.
6) $\sigma[W_\sigma(\mathcal{L})] = W_\sigma[\sigma(\mathcal{L})]$.

For each $\mu \in M(\mathcal{L})$ there corresponds a unique $\mu' \in M(W_\sigma(\mathcal{L}))$, where $\mu'(W_\sigma(A)) = \mu(A)$ for $A \in \mathcal{A}(\mathcal{L})$ and conversely. Furthermore $\mu \in M_R(\mathcal{L})$ if and only if $\mu' \in M_R(W_\sigma(\mathcal{L}))$, and $\mu \in M_\sigma(\mathcal{L})$ if and only if μ' $M_\sigma(W_\sigma(\mathcal{L}))$

The lattice $W_\sigma(\mathcal{L})$ is replete and $I^\sigma_R(\mathcal{L})$ with $\tau W_\sigma(\mathcal{L})$ as the topology of closed sets is disjunctive and T_1. It will be T_2 if we further assume that for each $\mu \in I(\mathcal{L})$ there exists at most one $v \in I^\sigma_R(\mathcal{L})$ such that $\mu \le v$ on $\mathcal{L}$. A proof of the last statement can be found in Yallaoui (1991).

Normal Lattices

PROPOSITION 16: $\mathcal{L}$ is normal if and only if for each $L \in \mathcal{L}$ where $L \subset L'_1 \cup L'_2$ and $L_1, L_2 \in \mathcal{L}$; then there exists $A_1, A_2 \in \mathcal{L}$ such that $A_1 \subset L'_1$ and $A_2 \subset L'_2$ and L $= A_1 \cup A_2$

Proof:

1. Assume that $\mathcal{L}$ is normal and the $L \subset L'_1 \cup L'_2$. Then $L \cap L_1 \cap L_2 = \emptyset$ or equivalently $(L \cap L_1) \cap (L \cap L_2) = \emptyset$. Since $\mathcal{L}$ is normal there exist $\tilde{A}_1, \tilde{A}_2 \in \mathcal{L}$ such that $L \cap L_1 \subset \tilde{A}'_1$, $L \cap L_2 \subset \tilde{A}'_2$ and $\tilde{A}'_1 \cap \tilde{A}_2 = \emptyset$. Let $A_1 = L \cap \tilde{A}_1$ and $A_2 = L \cap \tilde{A}_2$. Clearly $A_1 \subset L'_1$ and $A_2 \subset L'_2$. Now $A_1 \cup A_2 = (L_1 \cap \tilde{A}_1) \cup (L_2 \cap \tilde{A}_2) = L \cap (\tilde{A}_1 \cup \tilde{A}_2) = L \cap X = L$.

2. Let $L_1 \cap L_2 = \emptyset$ and $L_1, L_2 \in \mathcal{L}$. Then $X = L'_1 \cup L'_2$ and by the condition there exist $A_1, A_2 \in \mathcal{L}$ such that $A_1 \subset L'_1$, $A_2 \subset L'_2$ and $A_1 \cup A_2 = X$, clearly $L_1 \subset A'_1$, $L_2 \subset A'_2$ and $A'_1 \cap A'_2 = \emptyset$ and thence $\mathcal{L}$ is normal.

DEFINITION 17: Let $\pi : \mathcal{L} \mapsto \{0,1\}$; π will be called a premeasure on $\mathcal{L}$ if $\pi(X) = 1$, and π is monotonic and multiplicative i.e. $\pi(L_1 \cap L_2) = \pi(L_1)$. $\pi(L_2)$ for $L_1, L_2 \in \mathcal{L}$. $\Pi(\mathcal{L})$ denotes all such premeasures defined on $\mathcal{L}$. It can be easily shown that there is a one to one correspondence between elements of $\Pi(\mathcal{L})$ and $\mathcal{L}$-filters.

DEFINITION 18: Let $\bar{I}(\mathcal{L}) = \{\pi \in \Pi((\mathcal{L}) ; \text{if } L_1 \cup L_2 = X \text{ then } \pi(L_1) = 1 \text{ or } \pi(L_2). = 1\}$. Clearly $I_R(\mathcal{L}) \subset I(\mathcal{L}) \subset \bar{I}(\mathcal{L}) \subset \Pi(\mathcal{L})$.

Let $\mathcal{F} = \{L \in \mathcal{L} : L \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\pi(A) =$

$1, \pi \in \tilde{\mathrm{I}}(\mathcal{L})\}$

THEOREM 19: $\mathcal{L}$ is normal if and only if $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter.

Proof:

1. Assuming that $\mathcal{L}$ is normal we have to show that:
   (a) $\emptyset \notin \mathcal{T}$; which is obvious
   (b) $L_1 \subset L_2, L_1 \in \mathcal{T} \Rightarrow L_2 \in \mathcal{T}$
   (c) $L_1, L_2, \in \mathcal{T} \Rightarrow L_1 \cap L_2 \in \mathcal{T}$

   We have to show that $L_1 \cap L_2 \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\pi(A) = 1$. Assume otherwise i.e. $L_1 \cap L_2 \cap A = \emptyset$ for some $A \in \mathcal{L}$ and $\pi(A) = 1$ where $\pi \in \tilde{\mathrm{I}}(\mathcal{L})$ then $(L_1 \cap A) \cap (L_2 \cap A) = \emptyset$. Since $\mathcal{L}$ is normal, there exist $A_1, A_2 \in \mathcal{L}$ such that $L_1 \cap A \subset A'_1, L_2 \cap A \subset A'_2$ and $A'_1 \cap A'_2 = \emptyset$. Clearly $A_1 \cup A_2 = X \Rightarrow \pi(A_1 \cup A_2) = 1 \Rightarrow \pi(A_1) = 1$ or $\pi(A_2) = 1$. If for instance $\pi(A_1) = 1$ then $\pi(A \cap A_1) = 1$ and $L_1 \cap A_1 \cap A =$ which is a contradiction since $L_1 \in \mathcal{T}$

(d) Now assume that $\mathcal{T} \subset G$ where G is an $\mathcal{L}$ ultrafilter. Assume there exists $L \in G$ but $L \notin \mathcal{T}$ hence there exists $A \in \mathcal{L}$ such that $\pi(A) = 1$ but $A \cap L = \emptyset$. However since $\pi(A) = 1$ then $L \cap A \neq \emptyset$ for all $L \in G \supset \mathcal{T} \supset \{A \in \mathcal{L} : \pi(A) = \infty\}$ which is a contradiction. Therefore $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter.

2. Now assume that $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter. We have to show that $\mathcal{L}$ is normal i.e. if $\mu \in I(\mathcal{L})$ there exists a unique $v \in I_R(\mathcal{L})$ and $\mu \leq v$ on $\mathcal{L}$.

Suppose that there exist $v_1, v_2 \in I_R(\mathcal{L})$ and $\mu \leq v_1, \mu \leq v_2$ on $\mathcal{L}$.

Let $F_\mu = \{L \in \mathcal{L} : \mu(L) = 1\}$ and $\mathcal{T}_\mu = \{L \in \mathcal{L} : L \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\mu(A) = 1\}$

$\mathcal{T}_\mu$ and $F v_i$ are ultrafilters and we have $\mu \leq v_i \Rightarrow F_\mu \subset F v_i \Rightarrow \mathcal{T}_\mu \subset \mathcal{T} v_i \Rightarrow \mathcal{T}_\mu = \mathcal{T} v_i$ for $i = 1,2$. Therefore $\mathcal{T}_\mu = \mathcal{T} v_1 = \mathcal{T} v_2$

Furthermore we have that $F v_i \subset \mathcal{T} v_i$ and hence $\mathcal{T}_\mu = F v_1 = F v_2$. Finally since all the ultrafilters are equal we get that $v_1 = v_2$ which proves that $\mathcal{L}$ is normal.

Let $\mu \in I_R(\mathcal{L})$. Define for any $E \in X$, $\overline{\mu}((E) = \inf_{E \subset L' \, L \in \mathcal{L}} \mu(L')$.

Then it is easily seen that $\overline{\mu}$ is a finitely subadditive outer measure.

PROPOSITION 20: $\mathcal{L}$ is normal if and only if $\mathrm{K} = \{L \in \mathcal{L} : \overline{\mu}(L) = 1\}$ is a prime $\mathcal{L}$-filter.

Proof:

Suppose $\mathcal{L}$ is normal. If $L_1, L_2 \in \mathrm{K}$ then $\overline{\mu}(L_1) = \overline{\mu}(L_2) = 1$.

Now if $\overline{\mu}(L_1 \cap L_2) = 0$ then there exists $A \in \mathcal{L}$ such that $L_1 \cap L_2 \subset A'$ and $\mu(A') = 0$. But then $L_1 \subset L_2 \subset A'$, and since $\mathcal{L}$ is normal, by proposition 16, we have $A = A_1 \cup A_2$ where $A_1, A_2 \in \mathcal{L}, A_1 \subset L'_1$, and $A_2 \subset L'_2$. Now $\mu(A) = 1$ then $\mu(A_1) = 1$ or $\mu(A_2) = 1$. If for instance $\mu(A_1) = 1$, then $\mu(A'_1) = 0$ which is a contradiction since $\mathcal{L}_1 \subset A'_1$ and $\overline{\mu}(L_1) = 1$

Thus $\mathcal{L}_1, L_2 \in \mathrm{K}$ implies $L_1 \cap L_2 \in \mathrm{K}$.

The converse of the proof is easy and left to reader.

THEOREM 21. Let $\pi \in \mathrm{II}(\mathcal{L})$. Then $\pi \in \tilde{\mathrm{I}}(\mathcal{L})$ if and only if there exists $v \in I(\mathcal{L})$ such that $v \underset{\mathcal{L}}{\leq} \pi$

Proof:

Suppose $\pi \in \tilde{\mathrm{I}}(\mathcal{L})$ and let $\mathcal{M} = \{L' \in \mathcal{L}' : \pi(L) = 0\}$

$\emptyset \notin \mathcal{M}$ and $\mathcal{M}$ has the finite intersection property. The intersection of elements of $\mathcal{M}$ form an $\mathcal{L}'$-filter base. Now assume that $\mathcal{M} \subset G$ and G is $\mathcal{L}'$-ultrafilter. Then $G \leftrightarrow \rho \in I_R(\mathcal{L}')$ and therefore $v \in I(\mathcal{L})$, and if $\pi(L) = 0$ then $L' \in \mathcal{M} \subset G \rightarrow \rho(L') = 1 \Rightarrow \rho(L) = 0$. Hence $\rho \leq \pi$ on $\mathcal{L}$ and therefore $\exists v \in I(\mathcal{L})$ such that $v = \rho \leq \pi$ on $\mathcal{L}$

The second part of the proof is easy and is omitted.

Let $\mathcal{H} \subset \mathcal{L}$, $X \notin \mathcal{H}$ and if $L_1, L_2 \in \mathcal{H}$ then $L_1 \cup L_2 \in \mathcal{H}$. Consider the set of all $\mathcal{L}$-filters $G\alpha$ that exclude $\mathcal{H}$, (i.e. $G\alpha \cap \mathcal{H} = \emptyset$). We partially order G by set inclusion. Since $\{X\}$ is an $\mathcal{L}$-filter that exclude $\mathcal{H}$, then there exists at least one $G\alpha$. Furthermore, since $\{G\alpha, \subseteq\}$ is a partial ordering, which is an inductive ordering then by Zorn's lemma there must exist a maximal element. Let G be this maximal element so that $G = \max\{G\alpha$: where $G\alpha$ are $\mathcal{L}$-filters that exclude $\mathcal{H}\}$ and $G \neq \emptyset$.

THEOREM 22. G is a prime $\mathcal{L}$-filter.

Proof:

G is certainly an $\mathcal{L}$ filter.

Let $A \cup B \in G$ where $A, B \in \mathcal{L}$. We have to show that $A \in G$ or $B \in G$. Assume otherwise; that is $A, B \notin G$. Suppose that $\exists F_0 \in G$ such that $A \cap F_0 = \emptyset$ then $F_0 \cap (A \cup B) \in G \rightarrow (F_0 \cap A) \cup (F_0 \cap B) = F_0 \cap B \in G \Rightarrow B \in G$ which is a contradiction and we may now assume that $A \cap F \neq \emptyset$ and $B \cap F \neq \emptyset$ for all $F \in G$.

Let $\mathcal{T}_1$ be the filter generated by all $\{A \cap F \mid F \in G\}$. Since $A \in \mathcal{T}_1$ and $A \notin G \rightarrow G \subset \mathcal{T}_1$, similarly let $\mathcal{T}_2$ be the filter generated by all $\{B \cap F \mid F \in G\}$, $G \subset \mathcal{T}_2$. So there exists $H_1 \in \mathcal{H}$, $H_1 \in \mathcal{T}_1$ such that $A \cap F_1 \cap H_1$ for some $F_1 \in G$. Similarly there exists $H_2 \in \mathcal{H}$, $H_2 \in \mathcal{T}_2$ such that $A \cap F_2 \subset H_2$ for some $F_2 \in G$.

Let $F_1 \cap F_2 = F_3$ then $H_1 \cup H_2 \supset (A \cap F_1) \cup (B \cap F_2) \supset (A$

$\cap F_3) \cup (B \cap F_3) \to H_1 \cup H_2 \supset (A \cup B) \cap F_3 \in G$. However since $H_1 \cup H_2 \in \mathcal{H}$, we have a contradiction. Thus $A \in G$ or $B \in G$ or equivalently G is a prime filter and so $G \leftrightarrow \mu \in I(\mathcal{L})$.

COROLLARY 23: Let $\pi \in \Pi(\mathcal{L})$ then $\pi = \bigwedge_{\pi \le \mu_\alpha \mu_\alpha \in I(\mathcal{L})} \mu_\alpha$

Proof:

Let $\mathcal{F}$ be the $\mathcal{L}$-filter representing π i.e. $\mathcal{F} = \{L \in \mathcal{L} : \pi(L) = 1\}$. Let G_α be the prime $\mathcal{L}$-filter representing μ_α, so that $G_\alpha = \{L \in \mathcal{L} : \mu_\alpha(L) = 1\}$.

Clearly $\mathcal{F} \subseteq \bigcap_{\mathcal{F} \subset G_\alpha} G_\alpha$. We have to show that $\mathcal{F} \supseteq \bigcap_{\mathcal{F} \subset G_\alpha} G_\alpha$.

Assume that there exists $A \in \mathcal{L}$ and $A \in G\alpha$ for all α but $A \notin \mathcal{F}$. Let $\mathcal{H}$ = A then A $\neq X$ i.e. $X \notin \mathcal{H}$.

Let G be a maximal $\mathcal{L}$-filter containing $\mathcal{F}$ and excluding $\mathcal{H}$.

From the previous theorem, G is a prime $\mathcal{L}$-filter and $A \notin G$, which is a contradiction; since A belongs to all prime $\mathcal{L}$ filters that contain $\mathcal{F}$. Therefore $\mathcal{F} = \subseteq \bigcap_{\mathcal{F} \subset G_\alpha} G_\alpha$, and hence $\pi \in \Pi$ $(\mathcal{L})$. Thus $\pi = \bigwedge_{\pi \le \mu_\alpha \mu_\alpha \in I(\mathcal{L})} \mu\alpha$

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